EL COLEGIO DE MÉXICO

CENTRO DE ESTUDIOS ECONÓMICOS

# MAESTRÍA EN ECONOMÍA

TRABAJO DE INVESTIGACIÓN PARA OBTENER EL GRADO DE MAESTRO EN ECONOMÍA

# ENDOGENOUS COMPATIBILITY AND TIMING IN DUOPOLIES WITH NETWORK EFFECTS

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**PROMOCIÓN 2023-2025**

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AGOSTO 2025

# Acknowledgments

To my thesis advisor, **Dra. Adriana Gama Velázquez**, for her motivating and generous guidance. Your clarity, vision, and unwavering commitment encouraged me to keep improving and to reach further than I thought possible. For your perceptive comments and your constant belief in my potential, even during uncertain times, I am deeply grateful.

To my parents, **Ana María Ruiz Hernández** and **Augusto Hernández Alcántara**, for instilling in me the values of perseverance, honesty, and education through their example. You were always patient, wise, and loving throughout each step of the journey.

To my sisters, **Janai Esperanza Hernández Morales**, **Salma Jacqueline Hernández Ruiz**, and **Aylin Fernanda Hernández Ruiz**, for being my joy and my sanctuary. You were always my source of strength, and you encouraged me to move forward even during the most difficult moments.

To my friends, who reminded me along the way that the academic journey may not be a solitary one — through words of advice, cups of coffee, or study groups. Thanks for the solidarity, the emotional support, and all the laughter that made it all possible.

To **El Colegio de México**, for providing not only the academic training but also the intellectual community and environment that shaped this work. This institution offered me a space of rigor, inspiration, and commitment to knowledge that will remain an essential part of my life.

To everyone who stood with me along the course of all these years in some way: you are appreciated for standing by, for hearing, and for believing.

# Abstract

We characterize equilibrium quantities, profits, consumer surplus, and total welfare across seven canonical games, and embed these outcomes into a bimatrix game in which firms endogenously select both network type and timing. The analysis reveals that full compatibility combined with sequential leadership yields the highest welfare but may not be privately optimal. In contrast, strategic isolation can increase individual profits while reducing aggregate efficiency.

The results highlight how decentralized decision-making in network industries may lead to misaligned incentives and suboptimal outcomes. These findings contribute to the understanding of coordination problems in markets with network effects and offer a theoretical basis for evaluating the role of interoperability policies and institutional design.

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# Chapter 1

# Introduction

Markets with network externalities tend to exhibit misalignment among firm-level incentives and socially optimal results. With each increase in the number of users increasing a product's value, firms must decide whether to create their products in a compatible form so as to take advantage of a wider effective network relative to making them in an incompatible form so as to maintain control over an exclusive set of users. Such a choice goes beyond affecting individual profitability to influence aggregate welfare as well when paired with whether and when firms enter markets.

This paper analyzes how competitive structure and network compatibility interact to determine welfare implications in a duopoly. It is presumed that firms decide their network structure (compatible or not) as well as their role (leader or follower) endogenously. The model illustrates that firms simultaneously decide their technological and strategic decisions, and that their interaction dictates efficiency in market results.

Compatibility affects whether network effects get internalized. With full compatibility, both firms gain from a common user base and greater network externalities; with incompatibility, network effects become firm-specific. As illustrated by Katz and Shapiro (1985), compatibility can increase total welfare by increasing effective demand but is not always privately optimal as it lessens firms' capacity to differentiate and extract rents.

To analyze the role played by timing, we take up the endogenous sequencing approach of

Hamilton and Slutsky (1990), with firms deciding whether to act first or second and thus achieve Stackelberg- or Cournot-type equilibria. This enables us to investigate how leadership is combined with compatibility choices. Our model also takes up an approach by Amir et al. (2021) in using the notion of fulfilled expectations Cournot equilibrium (FECE), providing consistency between firm choice and consumer anticipation about network size. That model is a good point of comparison for understanding the two polar extremes: full compatibility and full incompatibility under Cournot competition—both of which we replicate and build upon.

A related strand of the literature explicitly incorporates interoperability costs into firms' strategic decisions. Buccella et al. (2023) develop a two-stage game in which firms first choose whether to make their products compatible, and subsequently compete in prices. Compatibility entails either a fixed or marginal cost, capturing technological, organizational, or regulatory burdens associated with interoperability. This framework allows for a detailed examination of how cost structures affect equilibrium outcomes and the degree to which firms internalize the benefits of network externalities.

Their findings show that when interoperability costs are sufficiently high relative to the strength of the network effect, firms may rationally opt for incompatibility, even if mutual compatibility would improve total welfare. This result illustrates how equilibrium coordination failures can emerge, as individually optimal strategies lead to collectively inefficient outcomes.

By contrast, the approach constructed here abstracts from compatibility costs in an effort to single out the strategic effects of network externalities per se. Nevertheless, the consequences of doing so are set out and compared with those in cost-based models specifically in relation to the question of equilibrium selection and the role of inefficiencies.

We further take inspiration from Requate and Quaas (2024), who consider network formation in an uncertain setting. They demonstrate that permitting coexistence among networks is optimal in dynamic settings even if static optimality is seemingly achieved otherwise. While our model is deterministic, their observation underlines the institutional design role when coordination over network formation is needed. The dynamics examined here apply to a number of actual markets. Visa and Mastercard credit card systems are instances of compatible systems in which interoperability at both merchant and issuer level raises user engagement and general welfare. Video game consoles like PlayStation and Xbox provide examples of platforms with a closed system, network fragmentation and forcing customers to rebuy platform-specific material.

Messaging platforms such as WhatsApp and Telegram offer a clear illustration of incompatibility in digital services. While both applications provide similar functionalities and are often installed on the same devices, they do not permit cross-platform communication: a message sent via one app cannot be received by users of the other. This absence of interoperability effectively creates isolated communication networks, fragmenting the user base and imposing coordination costs on users—such as the need to maintain multiple accounts or replicate social connections across platforms. While such cases could be interpreted as exhibiting partial or functional compatibility, the model developed in this thesis adopts a binary structure, distinguishing only between full compatibility and complete incompatibility.

An instructive example of asymmetric compatibility can be found in operating systems. As noted by Kim and Choi (2015), Windows can be installed and run on Apple computers through Boot Camp or virtualization software, whereas macOS cannot legally or technically operate on non-Apple hardware. This results in a one-way compatibility configuration in which Apple users may access Windows environments, but not vice versa. Such asymmetry reflects deliberate strategic positioning by dominant firms, and shapes the distribution of network benefits by reinforcing control over platform access.

The institutional structure is also involved. With Cournot competition in which firms move simultaneously, coordination on compatibility is likely to fail because firms do not want to be strategically exploited—for example, by choosing compatibility while the rival does not, thereby losing strategic advantage. With Stackelberg competition, however, a pioneering firm with compatibility induces imitative behavior from a follower and leads to a more optimal result. These tensions among timing, compatibility and network effects lie at the heart of the framework set out in this thesis.

The primary contribution of this work is to combine endogenous compatibility and endogenous timing in a single framework. We identify equilibrium results—quantities, profits, consumer surplus, and aggregate welfare—throughout seven subgame benchmarks and then nest these results in an extended bimatrix game with firms deciding network type and timing. This framework enables us to analyze when individual incentives result in socially optimal results and when misalignments persist. We establish that compatibility, when coupled with strategic leadership, leads to maximum welfare but is not necessarily individually rational. In such configurations, decentralized decisions do not account for network externalities, resulting in inefficient allocations.. These results provoke calls for coordination devices or policy intervention in network industry operations that synchronize private incentives with collective efficiency.

# Chapter 2

# The model

Unlike earlier contributions such as Katz and Shapiro (1985), where product compatibility was treated as an exogenous condition, in this model firms strategically choose whether their products will be compatible or not. This decision directly affects the effective size of the network, influencing both demand and firms payoffs. One assumption made by the model is compatibility being costless. This is as opposed to the method used by Buccella et al. (2023), who include marginal or fixed costs of interoperability. By abstracting from those costs, analysis focuses on the strategic effect of network externalities without the interference of actual financial disincentives on the incentives of the firm for adopting compatibility.

Product compatibility is modeled as an endogenous decision with consequences not only for firms individual profits but also for overall social welfare. In addition, following the approach of Hamilton and Slutsky (1990) on games with endogenous sequencing, the model endogenizes the order of moves by allowing firms to choose whether to act as leaders or followers. This introduces the possibility of different competitive structures such as Stackelberg or Cournot, depending on the timing choices made by the firms.

The game unfolds in two stages:

- In the first stage, both firms simultaneously decide whether their products will be compatible or incompatible effectively choosing the type of network they want to join.

- In the second stage, each firm selects whether to move first or second, following a model of endogenous timing.

Then, the firms compete in quantities. If both choose to move at the same time, the outcome corresponds to a Cournot equilibrium; if one firm moves first and the other waits, a Stackelberg equilibrium results.

The interaction between the compatibility decision and the timing structure allows for a comparison of different strategic configurations, with the aim of identifying the most efficient outcome both from the firms' perspective and from that of a social planner.

## 2.1 Consumers and Firms

We consider a duopoly in which two identical firms, 1 and 2, produce homogeneous goods subject to consumption-side network effects. As discussed in previous sections, these products can represent digital platforms, telecommunications services, or interoperable technologies such as payment systems (e.g., Visa and Mastercard) or Bluetooth-enabled devices.. Network effects imply that a consumer's marginal utility depends not only on the quantity consumed but also on the number of users participating in the same network, creating interdependence in demand.

The network effect is modeled as a direct consumption externality: the value of a product increases (or decreases) with the number of consumers using it, generating either a bandwagon or snob effect, respectively. Following Katz and Shapiro (1985) and Buccella et al. (2023), this effect is assumed to be positive in the present model and is captured by the parameter $d > 0$, which measures the strength of the network externality.

### Inverse Demand with Network Effects

The two products are substitutes: consumers perceive them as offering similar functionalities, and an increase in the output of one firm reduces the perceived value of the other. The inverse demand function assumed in the model is common to all scenarios. It is given by:

$$P(q_1, q_2, s) = a - b(q_1 + q_2) + ds,$$

where:

- $a > 0$ is the demand intercept;
- $b > 0$ reflects the sensitivity of price to total market output;
- $s$ is the expected size of the network;
- $q_1$, $q_2$ denote the quantities produced by firms 1 and 2, respectively.

The value of $s$ depends on the degree of product compatibility:

- Under full compatibility, both products connect to a common network, so $s = q_1 + q_2$.
- Under incompatibility, each product builds its own network, and network effects are internalized separately by each firm: $s_1 = q_1$ and $s_2 = q_2$.

### Profit Function

We assume that both firms face identical and constant marginal costs, denoted by $c \geq 0$, such that the total cost for firm $i \in \{1, 2\}$ is given by $C(q_i) = cq_i$. The resulting profit function for firm $i$ is:

$$\pi_i = (a - b(q_1 + q_2) + ds - c)\, q_i,$$

where $s$ denotes the effective size of the network, which may depend on the compatibility configuration chosen by the firms. Once this configuration is determined, firms cannot influence the value of $s$ directly or commit to shaping it strategically.

In this framework, compatibility or incompatibility does not entail any additional cost component. Its influence is captured solely through its impact on perceived demand, as mediated by the network effect term $ds$. This abstraction enables a focused analysis on the strategic implications of network externalities without conflating them with cost-side considerations.

Our approach differs from that of Buccella et al. (2023), who incorporate explicit interoperability costs into firms' payoff functions. In their model, adopting compatibility involves either a marginal or fixed cost, reflecting potential technological, organizational, or regulatory burdens. As they demonstrate, such costs may discourage compatibility even in environments where interoperability enhances aggregate demand and welfare.

By omitting these cost components, we isolate the role of network effects as a strategic variable in shaping equilibrium outcomes. This simplification allows for a direct comparison of Cournot and Stackelberg equilibria under varying compatibility structures, and facilitates the identification of conditions under which compatibility or incompatibility becomes strategically optimal.

The equilibrium concept adopted throughout the analysis is that of *fulfilled expectations Cournot equilibrium* (FECE), as formalized by Katz and Shapiro (1985). Under this notion, firms choose output levels based on a expected size of the network $s$, which must be validated in equilibrium: each firm's actual output must match the level consumers anticipated. This ensures internal consistency between firms' strategic behavior and consumers' expectations—a critical feature in markets where demand is shaped by adoption externalities.

Amir et al. (2021) extend this framework to show that FECE provides a natural foundation for assessing the viability of network industries, particularly when firms operate over proprietary and incompatible platforms. In such contexts, equilibrium outcomes must not only be profit-maximizing but also expectation-consistent, especially in environments with pure network goods where demand collapses absent an installed base.

We follow this approach to maintain consistency between market outcomes and expectations, enabling a coherent comparison across compatibility regimes and timing structures. It also allows us to model endogenous market size in a static setting without invoking more complex dynamic or informational mechanisms.

*Critical remark.* While the exclusion of interoperability costs offers analytical clarity and focuses attention on the strategic role of network externalities, it may also limit the external validity of the results. In real-world settings, achieving compatibility often entails non-trivial investments—be they technological (e.g., platform standardization), legal (e.g., licensing agreements), or institutional (e.g., coordination through regulators). Ignoring such frictions can overstate the attractiveness of compatibility and underrepresent the strategic trade-offs faced by firms, particularly in industries where standardization is contested or path-dependent. As shown by Buccella et al. (2023), the inclusion of these costs may invert firms' incentives, making incompatibility a rational equilibrium even in the presence of strong positive network effects. Therefore, the insights derived from our model should be interpreted as conditional on the absence of compatibility costs, which may serve as a useful benchmark but not a general rule.

## 2.2 Equilibrium Feasibility Conditions

To ensure the economic relevance of the solutions derived in each strategic scenario, we impose the following regularity conditions:

- $a > c$: ensures that the market price exceeds marginal cost, making production profitable;
- $b > 2d > 0$: guarantees that the inverse demand curve remains downward sloping, even when network externalities are present. This condition also guarantees the existence of interior solutions and the positivity of equilibrium quantities, prices, and profits across all games.

These feasibility conditions are assumed throughout the analysis. They are necessary for the existence of non-trivial interior equilibria under both Cournot and Stackelberg competition.

## 2.3 Notation and Scenario Structure

The analysis presented in the previous sections involves multiple strategic environments, characterized by different timing structures and compatibility choices. To ensure clarity and consistency in the comparison of outcomes, we introduce the following notation system:

- The first letter denotes the competitive structure:
  - $S$: sequential competition (Stackelberg)
  - $C$: simultaneous competition (Cournot)
- The next two letters indicate the compatibility decisions of each firm:
  - $C$: compatible network
  - $I$: incompatible network

For example, the expression $W^{S,CI}$ refers to the level of social welfare in a Stackelberg game where the leader adopts a compatible network and the follower an incompatible one.

To distinguish individual outcomes within these configurations, we use the following subscript conventions:

- $\pi_L^{S,\cdot}$: profit of the leader in a Stackelberg setting.
- $\pi_F^{S,\cdot}$: profit of the follower in a Stackelberg setting.

- $\pi_1^{C,CI}$: profit of firm 1 in a Cournot game when it chooses a compatible network and its rival an incompatible one.

This labeling framework provides a structured basis for comparing strategic outcomes across all games. It facilitates the evaluation of each configuration in terms of individual payoffs, consumer surplus, and total welfare.

# Chapter 3

# Equilibrium Analysis under Different Compatibility Structures

## 3.1 Stackelberg Equilibrium with Full Compatibility ($S, CC$)

We consider the benchmark scenario in which both firms operate over a fully compatible network and compete sequentially in quantities. Firm L acts as the leader and firm F as the follower, conforming to a standard Stackelberg structure. Under full compatibility, network effects are shared symmetrically: each firm benefits from the total market size, rather than from its own consumer base alone.

This configuration represents the optimal environment for internalizing positive consumption externalities. The effective network size is given by $s = q_L + q_F$, and both firms face the same inverse demand function, which captures the aggregate value generated through interoperability. The sequential nature of competition allows the leader to anticipate the follower's reaction and to adjust its output accordingly.

**Lemma 1 (Full Compatibility and Stackelberg Competition)** *In the Stackelberg game with full compatibility, the leader produces twice the quantity of the follower. Profits are distributed asymmetrically, with the leader earning twice as much as the follower. The configuration yields a strictly positive consumer surplus and total welfare. Specifically, the equilibrium quantities,* *profits, consumer surplus, and total welfare are given by:*

$$
\begin{aligned}
q_L^{S,CC} &= \frac{2(a-c)}{4b-3d}, \quad q_F^{S,CC} = \frac{a-c}{4b-3d} \\
\pi_L^{S,CC} &= \frac{2b(a-c)^2}{(4b-3d)^2}, \quad \pi_F^{S,CC} = \frac{b(a-c)^2}{(4b-3d)^2} \\
CS^{S,CC} &= \frac{9b(a-c)^2}{2(4b-3d)^2}, \quad W^{S,CC} = \frac{15b(a-c)^2}{2(4b-3d)^2}
\end{aligned}
$$

The proof of Lemma 1, including the follower's reaction function, the leader's optimization problem, and the internalization of network size, is provided in the Appendix. Similar derivations for the remaining Lemmas are presented in their corresponding sections of the Appendix.

The outcomes in this game illustrate the strategic complementarity between sequential market leadership and product interoperability. The leader captures a greater share of market surplus due to its timing advantage, while the follower benefits equally from the joint network without incurring any coordination loss. This symmetric treatment of compatibility, combined with asymmetric market positioning, generates the highest level of total welfare across all configurations analyzed.

When the network externality parameter $d = 0$, the model reduces to the standard quantity competition framework without network effects. In this limiting case, the effective network size becomes irrelevant, and strategic outcomes depend solely on the timing of moves and the structure of competition. This simplification applies not only to the Stackelberg configuration, but also to all Cournot games analyzed, confirming that the general model encompasses classical quantity competition as a special case nested within the broader framework with network externalities.

The $(S, CC)$ equilibrium serves as a reference point for evaluating the welfare implications of alternative network and competition structures. Deviations from this configuration—either by introducing asymmetry in network choices or by altering the timing of decisions—lead to measurable losses in consumer surplus and production efficiency, as explored in the following sections.

## 3.2 Stackelberg Equilibrium with Full Incompatibility ($S,II$)

**Lemma 2 (Full Incompatibility and Stackelberg Competition)** *In the Stackelberg game with full incompatibility, firm L (the leader) produces twice as much as firm F (the follower), reflecting the strategic advantage of early commitment in quantity-setting. Despite the lack of interoperability, each firm internalizes its own network effects individually. Profits are asymmetric, and consumer surplus remains strictly positive, although below the levels observed under full compatibility. The equilibrium quantities, profits, and welfare components are:*

$$
\begin{aligned}
q_L^{S,II} &= \frac{2(a-c)(b-d)}{4b^2-7bd+2d^2}, & q_F^{S,II} &= \frac{(a-c)(b-2d)}{4b^2-7bd+2d^2},\\
\pi_L^{S,II} &= \frac{2b(a-c)^2(b-d)^2}{(4b^2-7bd+2d^2)^2}, & \pi_F^{S,II} &= \frac{b(a-c)^2(b-2d)^2}{(4b^2-7bd+2d^2)^2},\\
CS^{S,II} &= \frac{b(a-c)^2(3b-4d)^2}{2(4b^2-7bd+2d^2)^2}, & W^{S,II} &= \frac{b(a-c)^2(15b^2-40bd+28d^2)}{2(4b^2-7bd+2d^2)^2}.
\end{aligned}
$$

The equilibrium under symmetric incompatibility and Stackelberg structure highlights the negative effects of network fragmentation. While the leader may benefit strategically, the lack of compatibility reduces the effective network size and, consequently, lowers social welfare compared to the compatible case ($CC$).

## 3.3 Stackelberg Equilibrium with a Compatible Leader and an Incompatible Follower ($S,CI$)

A concrete example of asymmetric compatibility can be found in the relationship between operating systems such as Windows and macOS. As noted by Kim and Choi (2015), Windows can be installed and run on Apple hardware through Boot Camp or virtualization, while macOS cannot be legally or technically operated on non-Apple devices. This unidirectional interoperability allows users within the Apple ecosystem to access a broader platform, whereas users of other systems are denied reciprocal access. This form of asymmetry mirrors the ($S,CI$) configuration analyzed in the model, where the leader benefits from compatibility and captures broader network externalities, while the follower remains confined to an isolated network environment.

**Lemma 3 (Asymmetric Compatibility** $(S,CI)$**)** *In the Stackelberg game where the leader operates on a compatible network and the follower on an incompatible one, firms perceive network effects asymmetrically. The leader internalizes a broader network size, while the follower captures only private benefits from its own output. This asymmetry leads to differentiated demand responses and uneven strategic advantages. Specifically, the leader secures a larger market share and higher profits than the follower, though the overall efficiency remains below the full compatibility benchmark. The equilibrium outcomes are:*

$$q_L^{S,CI} = \frac{2b(a-c)}{4b^2-5bd+2d^2}, \quad q_F^{S,CI} = \frac{(a-c)(b-2d)}{4b^2-5bd+2d^2},$$
$$\pi_L^{S,CI} = \frac{2b^3(a-c)^2}{(4b^2-5bd+2d^2)^2}, \quad \pi_F^{S,CI} = \frac{b(a-c)^2(b-2d)^2}{(4b^2-5bd+2d^2)^2},$$
$$CS^{S,CI} = \frac{b(a-c)^2(3b-2d)^2}{2(4b^2-5bd+2d^2)^2}, \quad W^{S,CI} = \frac{b(a-c)^2(15b^2-20bd+12d^2)}{2(4b^2-5bd+2d^2)^2}.$$

This intermediate case reveals how asymmetry in compatibility decisions alters strategic incentives. The leader benefits from a broader network, while the follower faces a narrower one. Although total welfare is lower than in the fully compatible case, this configuration may still yield superior outcomes compared to symmetric incompatibility.

## 3.4 Stackelberg Equilibrium with an Incompatible Leader and a Compatible Follower $(S,IC)$

**Lemma 4 (Incompatible Leader and Compatible Follower** $(S,IC)$**)** *In the Stackelberg game where the leader operates on an incompatible network and the follower on a compatible one.*

*The equilibrium quantities, profits, and welfare components are:*

$$
\begin{aligned}
q_L^{S,IC} &= \frac{2(a-c)(b-d)}{4b^2-5bd+2d^2}, \quad q_F^{S,IC} = \frac{b(a-c)}{4b^2-5bd+2d^2},\\
\pi_L^{S,IC} &= \frac{2b^2(a-c)^2(b-d)}{(4b^2-5bd+2d^2)^2}, \quad \pi_F^{S,IC} = \frac{b(a-c)^2(b^2-2bd+2d^2)}{(4b^2-5bd+2d^2)^2},\\
CS^{S,IC} &= \frac{b(a-c)^2(3b-2d)^2}{2(4b^2-5bd+2d^2)^2}, \quad W^{S,IC} = \frac{b(a-c)^2(15b^2-20bd+8d^2)}{2(4b^2-5bd+2d^2)^2}.
\end{aligned}
$$

In this subgame, network externalities are asymmetrically distributed in favor of the follower. The leader, operating on an incompatible network, internalizes only the value generated by its own output ($s_1 = q_1$), whereas the follower, by choosing compatibility, benefits from an expanded network that includes both outputs ($s_2 = q_1 + q_2$). Despite moving second, the follower captures a broader externality, which partially offsets its disadvantage in timing. Relative to the fully compatible benchmark, the leader supplies a smaller quantity and obtains lower profits, reflecting the limitations of operating within an isolated network.

This configuration mirrors the structure of the $(S,CI)$ case, with reversed roles. In both settings, the firm that adopts compatibility captures a larger share of the network externality and secures a relative advantage, regardless of whether it leads or follows. These results suggest that compatibility can act as a strategic substitute for early commitment. Although total welfare in $(S,IC)$ is lower than in the symmetric compatibility case $(S,CC)$, it coincides with that of $(S,CI)$, indicating that the extent of network access—rather than the timing of moves—plays a more decisive role in determining equilibrium efficiency.

This game represents the mirror image of the previous case, in which the leader fails to benefit from compatibility, while the follower internalizes a stronger network effect. Despite acting first, the incompatible leader finds itself at a strategic disadvantage. In terms of total welfare, this configuration yields results analogous to the $(S,CI)$ case.

## 3.5 Cournot Equilibrium with Full Compatibility $(C,CC)$

We now consider the case in which both firms operate on a fully compatible network and choose quantities simultaneously, following the standard Cournot framework. In this setting, network externalities are symmetrically internalized through a shared effective size $s = q_1 + q_2$, and the firms face identical strategic environments. This configuration and and the $(C,II)$ game is analytically equivalent to the baseline case studied in Amir et al. (2021). It is nonetheless included here to ensure that the analysis remains self-contained and allows for direct comparison with the other subgames examined in this framework.

**Lemma 5 (Cournot Competition with Full Compatibility** $(C,CC)$**)** *Under simultaneous competition and full compatibility, the equilibrium outcomes are given by:*

$$q_1^{C,CC} = q_2^{C,CC} = \frac{a-c}{3b-2d},$$
$$\pi_1^{C,CC} = \pi_2^{C,CC} = \frac{b(a-c)^2}{(3b-2d)^2},$$
$$CS^{C,CC} = \frac{2(a-c)^2}{(3b-2d)^2},\ W^{C,CC} = \frac{2(a-c)^2(b+1)}{(3b-2d)^2}.$$

In this configuration, both firms benefit equally from the network externality, resulting in symmetric outcomes in quantities, profits, and welfare. Although total surplus is lower than under Stackelberg leadership with full compatibility, the $(C,CC)$ scenario consistently yields higher consumer surplus and social welfare than any configuration involving incompatibility. Its symmetric structure and the joint internalization of network effects support a more efficient allocation of demand across firms, reducing fragmentation and enhancing network value for consumers. These properties make it a robust benchmark within simultaneous-move settings.

## 3.6 Cournot Equilibrium with Full Incompatibility $(C,II)$

We now turn to the case in which both firms operate over incompatible networks and choose quantities simultaneously. This configuration represents a fully fragmented market in which each firm internalizes network effects solely from its own users, with no interoperability.

**Lemma 6 (Cournot Competition with Full Incompatibility** $(C, II)$**)** *In the Cournot game with full incompatibility, both firms operate independently and internalize only firm specific network externalities* $s_1 = q_1$, $s_2 = q_2$. *The absence of compatibility limits the benefits from demand-side externalities, despite the symmetric structure of the game. The equilibrium outcomes are:*

$$q_1^{C,II} = q_2^{C,II} = \frac{a-c}{3b-d},$$
$$\pi_1^{C,II} = \pi_2^{C,II} = \frac{b(a-c)^2}{(3b-d)^2},$$
$$CS^{C,II} = \frac{2(a-c)^2}{(3b-d)^2},\ W^{C,II} = \frac{2(a-c)^2(b+1)}{(3b-d)^2}.$$

Although the strategic environment is symmetric, the lack of interoperability prevents firms from capturing the full value of network externalities. As a result, this configuration yields the lowest levels of consumer surplus and total welfare across all scenarios considered. The $(C, II)$ case highlights the efficiency losses associated with technological fragmentation, even in the absence of timing asymmetries.

## 3.7 Cournot Equilibrium with Asymmetric Compatibility $(C, CI)$

We now examine the asymmetric case in which firm 1 adopts compatibility and benefits from the total network size, while firm 2 remains incompatible and internalizes only its own user base. Although competition unfolds simultaneously, the asymmetry in network access creates differentiated incentives and outcomes.

Due to the simultaneous nature of Cournot competition, the configuration $(C, CI)$ is symmetric to the reverse case $(C, IC)$, where the roles of compatibility are interchanged. In both settings, one firm benefits from the aggregate network effect, while the other remains isolated. Since firms move at the same time and face structurally identical payoff functions—differing only in the assignment of roles—the resulting equilibria are symmetric in form, with outcomes mirroring each other depending on which firm adopts compatibility.

**Lemma 7 (Cournot Competition with Asymmetric Compatibility** $(C, CI)$**)** *In the Cournot game with asymmetric compatibility, firm 1 internalizes a joint network effect (*$s_1 = q_1 + q_2$*), whereas firm 2 only benefits from its own demand externality (*$s_2 = q_2$*). This asymmetry leads to differences in output, profits, and strategic positioning. The equilibrium outcomes are:*

$$
\begin{aligned}
q_1^{C,CI} &= \frac{b(a-c)}{3b^2-3bd+d^2}, \quad q_2^{C,CI} = \frac{(a-c)(b-d)}{3b^2-3bd+d^2},\\
\pi_1^{C,CI} &= \frac{b^3(a-c)^2}{(3b^2-3bd+d^2)^2}, \quad \pi_2^{C,CI} = \frac{b(a-c)^2(b-d)^2}{(3b^2-3bd+d^2)^2},\\
CS^{C,CI} &= \frac{b(a-c)^2(2b-d)^2}{2(3b^2-3bd+d^2)^2}, \quad W^{C,CI} = \frac{b(a-c)^2(8b^2-8bd+3d^2)}{2(3b^2-3bd+d^2)^2}.
\end{aligned}
$$

This equilibrium represents an intermediate case between full compatibility and complete fragmentation. The compatible firm partially capitalizes on the shared network effect, while the incompatible firm internalizes only its own network. Social welfare improves relative to the symmetric incompatibility case but does not reach the levels achieved under full compatibility.

# Chapter 4

# Strategic Comparison and Equilibrium Selection

Based on the results derived from the different games, we can establish a ranking of outcomes in terms of social welfare, consumer surplus, and individual profits. These rankings reflect how various combinations of network compatibility and competitive structures (sequential or simultaneous) affect market performance.

**Proposition 1 (Profit Ordering by Role and Structure)** *When comparing individual firm profits, the Stackelberg leader under full incompatibility generally secures the highest payoff, while followers under asymmetric or simultaneous settings earn the least. The strict ranking is as follows, with two conditional reversals depending on the relative value of d and b:*

***(i)*** *If $d < b\left(1 - \frac{b}{\sqrt{2}}\right)$, then:*

$$\pi_L^{S,II} > \pi_L^{S,CC} > \pi_L^{S,IC} > \pi_1^{C,CI} > \pi_L^{S,CI} > \pi^{C,CC} > \pi_F^{S,IC} > \pi^{C,II} > \pi_2^{C,CI} > \pi_F^{S,CC} > \pi_F^{S,II} > \pi_F^{S,CI}.$$

***(ii)*** *If $d > b\left(1 - \frac{b}{\sqrt{2}}\right)$, then:*

$$\pi_L^{S,II} > \pi_L^{S,CC} > \pi_1^{C,CI} > \pi_L^{S,IC} > \pi_L^{S,CI} > \pi^{C,CC} > \pi_F^{S,IC} > \pi^{C,II} > \pi_F^{S,CC} > \pi_2^{C,CI} > \pi_F^{S,II} > \pi_F^{S,CI}.$$

Note: In the cases of Cournot competition with full compatibility ($C,CC$) and full incompatibility $(C, II)$, profits are symmetric across firms. For this reason, the subscripts denoting firm 1 or 2 are omitted.

**Proposition 2 (Complete Consumer Surplus Ranking Conditional on** $d$**)** *Let the model parameters satisfy the assumptions stated in Section 2. Then, the complete ordering of consumer surplus across all timing–compatibility subgames satisfies the following strict inequalities, depending on the value of* $d$*:*

***Case 1.*** *If* $d < 0.435$*, then:*

$$CS^{S,CC} > CS^{S,IC} = CS^{S,CI} > CS^{S,II} > CS^{C,CC} > CS^{C,CI} > CS^{C,II}.$$

***Case 2.*** *If* $0.435 < d < 0.680$*, then:*

$$CS^{S,CC} > CS^{S,IC} = CS^{S,CI} > CS^{S,II} > CS^{C,CC} > CS^{C,II} > CS^{C,CI}.$$

***Case 3.*** *If* $d > 0.680$*, then:*

$$CS^{S,CC} > CS^{S,IC} = CS^{S,CI} > CS^{S,II} > CS^{C,CI} > CS^{C,CC} > CS^{C,II}.$$

**Proposition 3 (Welfare Ranking Across Games)** *Let the model parameters satisfy the restrictions defined in Section 2. Then, total social welfare across all compatibility–timing configurations satisfies the following complete and strict orderings, depending on the relative values of* $b$ *and* $d$*:*

***Case 1.*** *If* $b > 2.365d$ *and* $d < 0.351$*, then:*

$$W^{S,CC} > W^{S,CI} > W^{S,IC} > W^{S,II} > W^{C,CC} > W^{C,II} > W^{C,CI}.$$

***Case 2.*** *If* $b > 2.365d$ *and* $0.351 < d < 0.907$*, then:*

$$W^{S,CC} > W^{S,CI} > W^{S,IC} > W^{S,II} > W^{C,CI} > W^{C,CC} > W^{C,II}.$$

***Case 3.*** *If* $b < 2.365d$ *and* $d < 0.351$*, then:*

$$W^{S,CC} > W^{S,II} > W^{S,IC} > W^{S,CI} > W^{C,CC} > W^{C,II} > W^{C,CI}.$$

***Case 4.*** *If* $b < 2.365d$ *and* $d > 0.907$*, then:*

$$W^{C,CI} > W^{S,CC} > W^{S,II} > W^{S,IC} > W^{S,CI} > W^{C,CC} > W^{C,II}.$$

In the absence of network effects ($d = 0$), the ranking of welfare and consumer surplus across games aligns with standard results in oligopoly theory: Stackelberg configurations systematically outperform Cournot, and compatibility yields greater efficiency than incompatibility. When network effects are introduced, this general structure remains intact, but the advantage associated with compatible networks becomes more pronounced. In particular, the configurations $(S, CI)$ and $(S, IC)$—which coincide with $(S, II)$ when $d = 0$—begin to dominate the symmetric incompatibility case once $d > 0$. This shift underscores the reinforcing effect of network externalities: compatibility enhances the value of market participation and can compensate for the disadvantages of moving second or lacking a leadership role.

The outcomes associated with Cournot competition under full compatibility ($CC$) and full incompatibility ($II$) reproduce, in structure and implication, the theoretical results of Amir et al. (2021). Within their comparative framework, these two scenarios represent polar market architectures: a unified network environment versus one composed of firm-specific and mutually incompatible platforms. Under standard assumptions—such as concavity of demand and the presence of strategic substitutes—they show that compatibility strictly improves both total welfare and consumer surplus, i.e., $W^{C,CC} > W^{C,II}$ and $CS^{C,CC} > CS^{C,II}$.

The results obtained here are fully consistent with those findings. Compatibility not only ensures market viability by expanding the effective demand base, but also enhances allocative efficiency. This alignment reinforces the theoretical robustness of the model and highlights how endogenous network formation and consumer expectations mediate the relationship between market design and equilibrium outcomes. In particular, the Cournot setting—by eliminating strategic timing asymmetries—serves as a clean benchmark for isolating the welfare effects of interoperability alone.

Taken together, the rankings reveal a clear tension between private incentives and socially efficient outcomes in markets with network effects. Firms may have a strategic interest in operating within incompatible environments, particularly when doing so allows them to isolate demand and increase individual profits. However, these configurations consistently lead to lower consumer surplus and total welfare. In contrast, full compatibility under sequential competition achieves the highest efficiency, even if it reduces the leading firm's relative payoff. This gap between private and collective outcomes highlights the potential role of coordination or policy mechanisms aimed at promoting interoperability—especially in settings where expectations and network size shape market performance.

## 4.1 Strategic Equilibrium Selection: Extended Game of Compatibility and Leadership

The results from the various games can be embedded into a strategic matrix game in which firms non-cooperatively choose both the type of network (compatible or incompatible) and the timing of their market entry (leader or follower). This extended bimatrix game—where strategies are ordered pairs such as $1C$, $1I$, $2C$, $2I$—enables the identification of Nash equilibrium under endogenous sequencing and compatibility decisions.

Each firm selects whether to act as a leader (1) or follower (2), and whether to operate on a compatible ($C$) or incompatible ($I$) network. The payoff symetric matrix below summarizes the resulting individual profits for all strategy profiles:

**Firm 2**

|  |  | $1C$ | $2C$ | $1I$ | $2I$ |
|---|---|---|---|---|---|
| **Firm 1** | $1C$ | $\underline{\pi^{C,CC}}, \underline{\pi^{C,CC}}$ | $\underline{\pi_L^{S,CC}}, \pi_F^{S,CC}$ | $\underline{\pi_1^{C,CI}}, \pi_2^{C,CI}$ | $\underline{\pi_L^{S,CI}}, \pi_F^{S,CI}$ |
|  | $2C$ | $\pi_F^{S,CC}, \underline{\pi_L^{S,CC}}$ | $\pi^{C,CC}, \pi^{C,CC}$ | $\pi_F^{S,IC}, \pi_L^{S,IC}$ | $\pi_1^{C,CI}, \pi_2^{C,CI}$ |
|  | $1I$ | $\pi_2^{C,CI}, \underline{\pi_1^{C,CI}}$ | $\pi_L^{S,IC}, \pi_F^{S,IC}$ | $\pi^{C,II}, \pi^{C,II}$ | $\pi_L^{S,II}, \pi_F^{S,II}$ |
|  | $2I$ | $\pi_F^{S,CI}, \underline{\pi_L^{S,CI}}$ | $\pi_1^{C,IC}, \pi_2^{C,IC}$ | $\pi_F^{S,II}, \pi_L^{S,II}$ | $\pi^{C,II}, \pi^{C,II}$ |

### Equilibrium Identification

Based on the matrix above, it can be shown that the strategic profile $(1C, 1C)$—in which both firms choose to operate on compatible networks and opt to move as early as possible-strategy Nash equilibrium. Neither firm has an incentive to unilaterally deviate, as any deviation reduces its profit without improving the collective outcome.

This result contrasts with the findings of (Buccella et al., 2023), where compatibility is modeled as a costly strategic choice. In their compatibility decision game (CDG), firms incur a quasi-fixed cost $Z$ when opting for interoperability. As a result, equilibrium outcomes depend critically on the magnitude of this cost relative to the strength of the network externality. Their analysis identifies multiple possible subgame perfect equilibria, including symmetric incompatibility, asymmetric configurations, and cases of coordination failure. In particular, when $Z$ exceeds a certain threshold—defined in relation to the parameter $n$ capturing the intensity of network effects—the unique equilibrium involves full incompatibility, even though compatibility would raise overall welfare.

In contrast, the model developed here assumes that compatibility is costless. Under this assumption, firms have clear incentives to coordinate on mutual compatibility and early market entry. The resulting equilibrium $(1C, 1C)$ emerges as a stable outcome in which both firms internalize the benefits of interoperability and strategic leadership. This configuration can be interpreted as a self-enforcing solution to a coordination problem, where no external enforcement is needed to achieve the efficient allocation.

This reasoning aligns with the framework of Hamilton and Slutsky (1990), in which firms compete not only in quantities but also in timing. In environments where both players prefer to lead, symmetric equilibria may emerge when leadership is mutually beneficial and rivals lack credible commitment mechanisms.

The equilibrium $(1C, 1C)$ can be interpreted as a self-organized solution to a coordination game in which both firms correctly internalize the positive effects of compatibility and leadership. As argued by Buccella et al. (2023), when interoperability is costless or when its benefits outweigh any marginal cost, mutual compatibility tends to arise as an efficient solution.

Furthermore, the strategic symmetry observed in $(1C, 1C)$ suggests that even in the absence of commitment mechanisms or external regulation, firms may converge to a cooperatively desirable configuration through purely individual incentives. Shared leadership in a compatible environment functions as an implicit form of coordination that prevents market fragmentation—without requiring standardization mandates or collusion.

Importantly, even a small interoperability cost $Z > 0$—as introduced in (Buccella et al., 2023) —can significantly affect the equilibrium outcome. Their analysis shows that when $Z$ exceeds certain thresholds, firms may find it optimal to coordinate on incompatible strategies, such as $(1I, 1I)$ or $(2I, 2I)$, resulting in symmetric incompatibility. This illustrates how fragile compatibility-based coordination can be when interoperability is not costless, and underscores the risk of inefficient outcomes in the absence of external incentives or policy intervention.

As shown previously, configurations that combine compatibility with strategic leadership tend to yield the highest levels of both social welfare and consumer surplus. Specifically:

- $W^{S,CC}$ represents the maximum level of social welfare.

- $CS^{S,CC}$ dominates in all configurations under the stated conditions.

- Although $\pi_L^{S,II}$ is the highest individual profit, this outcome is not sustainable as a strategic equilibrium.

# Chapter 5

# Conclusion

This thesis has examined how the interaction between product compatibility and market structure affects equilibrium outcomes in network industries. By allowing firms to choose both their network configuration—compatible or incompatible—and the timing of their strategic moves—leader or follower—the model captures the essential trade-offs between private incentives and collective efficiency.

Using a static duopoly framework with fulfilled expectations equilibrium, we characterized equilibrium quantities, profits, consumer surplus, and total welfare across seven canonical games. The results show that full compatibility combined with sequential competition (i.e., Stackelberg leadership) consistently yields the highest levels of consumer surplus and social welfare. However, this outcome is not always privately optimal. In several configurations, particularly those involving incompatibility or simultaneous competition, firms may secure higher individual profits while reducing overall efficiency.

These findings reflect a structural misalignment between what is profitable for firms and what is desirable from a social perspective. The strategic value of incompatibility—in terms of market power, exclusivity, or differentiation—can lead to fragmented networks, even when a shared system would generate higher aggregate surplus. Moreover, the endogenous timing of competition plays a crucial role in shaping incentives. While Stackelberg leadership can help coordinate on efficient outcomes, its success depends on the ability of firms to internalize the benefits of compatibility and anticipate their rival's response.

The extended bimatrix game developed in the final section integrates all subgame results into a unified framework. The analysis shows that when compatibility is costless, mutual compatibility and shared leadership emerge as a stable Nash equilibrium. This equilibrium, $(1C, 1C)$, represents a self-enforcing solution to the coordination problem, aligning private and social incentives without external intervention. In contrast, introducing even small interoperability costs—as shown by Buccella et al. (2023)—can destabilize this outcome, leading to inefficient equilibria marked by symmetric incompatibility.

The model also highlights the importance of expectations. Since network value depends on anticipated adoption, coordination failures can arise even in the absence of direct costs. This reinforces the relevance of institutional design and regulatory policy in markets where compatibility decisions are strategic. Policies that reduce the cost of interoperability or facilitate standardization can help shift firms toward efficient configurations.

Overall, this thesis contributes to the understanding of strategic compatibility in network industries by linking endogenous product design and endogenous market structure. The framework developed here can be extended in several directions. Future work may incorporate dynamic elements such as innovation, learning, or platform entry over time, or consider heterogeneous firms with asymmetric costs or demand parameters. Another avenue is to endogenize the cost of compatibility, modeling trade-offs between investment in interoperability and market reach.

The key insight remains: in the presence of network effects, decentralized decisions do not always produce efficient outcomes. Identifying when firm incentives support welfare-enhancing strategies—and when they do not—is essential for the design of markets where compatibility matters.

# Chapter 6

# Appendix

### Proof of lemma 1

We now consider the case in which both firms produce fully compatible products and interact in a Stackelberg structure, with firm 1 as the leader and firm F as the follower.

Since the products are compatible, consumers form rational expectations and perceive the effective network size as $s = q_L + q_F$.

The follower maximizes:

$$\pi_F(q_L, q_F, s) = (a - b(q_L + q_F) + ds - c)q_F,$$

which yields the first-order condition:

$$\frac{\partial \pi_F}{\partial q_F} = a - bq_L - 2bq_2 + ds - c = 0,$$

and the reaction function:

$$q_F(q_L, s) = \frac{a - c + ds - bq_L}{2b}.$$

Anticipating this response, the leader maximizes:

$$\pi_L(q_L, q_F, s) = (a - b(q_L + q_F) + ds - c)q_L.$$

Substituting $q_F(q_L, s)$ into the expression and applying the first-order condition gives:

$$q_L(s) = \frac{a - c + ds}{2b}.$$

Plugging this into the follower's reaction function yields:

$$q_F(s) = \frac{a - c + ds}{4b}.$$

The total effective network size is then:

$$s = q_L(s) + q_F(s) = \frac{a - c + ds}{2b} + \frac{a - c + ds}{4b} = \frac{3(a - c) + 3ds}{4b}.$$

Solving for $s$:

$$s = \frac{3(a - c)}{4b - 3d}.$$

Substituting back, we obtain the equilibrium quantities:

$$q_L^{S,CC} = \frac{2(a - c)}{4b - 3d}, \quad q_F^{S,CC} = \frac{a - c}{4b - 3d}.$$

The corresponding profits are:

$$\pi_L^{S,CC} = \frac{2b(a - c)^2}{(4b - 3d)^2}, \quad \pi_F^{S,CC} = \frac{b(a - c)^2}{(4b - 3d)^2}.$$

Consumer surplus is given by:

$$CS^{S,CC} = \frac{9b(a - c)^2}{2(4b - 3d)^2},$$

and total market welfare is:

$$W^{S,CC} = \pi_L^{S,CC} + \pi_F^{S,CC} + CS^{S,CC} = \frac{15b(a-c)^2}{2(4b-3d)^2}.$$

### Proof of Lemma 2

We consider the scenario in which both firms operate over incompatible networks and compete sequentially, with firm L acting as the Stackelberg leader and firm F as the follower.

Since network effects are not shared, each firm internalizes only the benefit from its own user base. Therefore, in equation we have $s_L = q_L$ and $s_F = q_F$. The follower maximizes the profit function:

$$\pi_F(q_L, q_F, s_F) = (a - b(q_L + q_F) + ds_F - c)q_F,$$

with first-order condition:

$$\frac{\partial \pi_F}{\partial q_F} = a - bq_L - 2bq_F + ds_F - c = 0,$$

which yields the follower's reaction function:

$$q_F(q_L, s_F) = \frac{a - c + ds_F - bq_L}{2b}.$$

Anticipating this response, the leader maximizes:

$$\pi_L(q_L, q_F, s_L) = (a - b(q_L + q_F) + ds_L - c)q_L,$$

substituting the follower's best response and applying the first-order condition gives:

$$q_L(s_L, s_F) = \frac{a - c + 2ds_L - ds_F}{2b}.$$

Substituting this into the follower's reaction function results in:

$$q_F(s_L, s_F) = \frac{a - c - 2ds_L + 3ds_F}{4b}.$$

Using the rational expectations conditions $s_L = q_L$ and $s_F = q_F$, we solve the system:

$$q_L = \frac{a - c + 2dq_L - dq_F}{2b}, q_F = \frac{a - c - 2dq_L + 3dq_F}{4b}.$$

Solving this system yields the equilibrium quantities:

$$\begin{aligned} q_L^{S,II} &= \frac{2(a-c)}{4b-3d}, \\ q_F^{S,II} &= \frac{a-c}{4b-3d}. \end{aligned}$$

Substituting into the profit functions, we obtain:

$$\begin{aligned} \pi_L^{S,II} &= \frac{2b(a-c)^2}{(4b-3d)^2}, \\ \pi_F^{S,II} &= \frac{b(a-c)^2}{(4b-3d)^2}. \end{aligned}$$

Consumer surplus is:

$$CS^{S,II} = \frac{9b(a-c)^2}{2(4b-3d)^2},$$

and total welfare is:

$$W^{S,II} = \pi_L^{S,II} + \pi_F^{S,II} + CS^{S,II} = \frac{15b(a-c)^2}{2(4b-3d)^2}.$$

## Proof of Lemma 3

We examine the Stackelberg game in which firm L, the leader, adopts a compatible network, while firm F, the follower, remains incompatible. Due to this asymmetry, network effects are perceived differently: the leader internalizes $s_L = q_L$, while the follower benefits only from $s_F = q_F$.

The follower maximizes its profit given the leader's output, with the profit function defined as:

$$\pi_F(q_L, q_F, s_F) = (a - b(q_L + q_F) + ds_F - c)q_F.$$

The first-order condition is:

$$\frac{\partial \pi_F}{\partial q_F} = a - bq_L - 2bq_F + ds_F - c = 0,$$

from which we obtain the follower's reaction function:

$$q_F(q_L, s_F) = \frac{a - c + ds_F - bq_L}{2b}.$$

Anticipating this, the leader maximizes:

$$\pi_L(q_L, q_F, s_L, s_F) = (a - b(q_L + q_F) + ds_L + ds_F - c)q_L,$$

leading to its first-order condition:

$$\frac{\partial \pi_L}{\partial q_L} = a - 2bq_L - bq_F + 2ds_L + ds_F - c = 0,$$

which simplifies (under $s_L = q_L$ and $s_F = q_F$) to:

$$q_L = \frac{a - c + 2dq_L + dq_F}{2b}.$$

We thus obtain the system of equations:

$$\begin{cases} q_L = \frac{a-c+2dq_L+dq_F}{2b}, \\ q_F = \frac{a-c-2dq_L+3dq_F}{4b}. \end{cases}$$

Solving yields the equilibrium quantities:

$$q_L^{S,CI} = \frac{2b(a-c)}{4b^2-5bd+2d^2}, \quad q_F^{S,CI} = \frac{(a-c)(b-2d)}{4b^2-5bd+2d^2}.$$

The corresponding profits are:

$$\pi_L^{S,CI} = \frac{2b^3(a-c)^3}{(4b^2-5bd+2d^2)^2}, \quad \pi_F^{S,CI} = \frac{b(a-c)^2(b-2d)^2}{(4b^2-5bd+2d^2)^2}.$$

Consumer surplus is given by:

$$CS^{S,CI} = \frac{b(a-c)^2(3b-2d)^2}{2(4b^2-5bd+2d^2)^2},$$

and total welfare is:

$$W^{S,CI} = \pi_L^{S,CI} + \pi_F^{S,CI} + CS^{S,CI} = \frac{b(a-c)^2(15b^2-20bd+12d^2)}{2(4b^2-5bd+2d^2)^2}.$$

### Proof of Lemma 4

We consider the Stackelberg game where firm L, the leader, operates over an incompatible network, while firm F, the follower, chooses compatibility. The leader internalizes a firm-specific network, $s_L = q_L$, while the follower benefits from the aggregate network, $s_F = q_L + q_F$.

The follower's profit function is:

$$\pi_F(q_L, q_F, s_L, s_F) = (a - b(q_L + q_F) + ds_L + ds_F - c)\, q_F.$$

The first-order condition is:

$$\frac{\partial \pi_F}{\partial q_F} = a - bq_L - 2bq_F + ds_L + ds_F - c = 0,$$

which yields the reaction function:

$$q_F(q_L, s_L, s_F) = \frac{a - c + ds_L + ds_F - bq_L}{2b}.$$

Anticipating this, the leader maximizes:

$$\pi_L(q_L, q_F, s_L) = (a - b(q_L + q_F) + ds_L - c)\, q_L,$$

with first-order condition:

$$\frac{\partial \pi_L}{\partial q_L} = a - 2bq_L - bq_F + ds_L - c = 0.$$

Substituting $s_L = q_L$ and $s_F = q_L + q_F$, the system becomes:

$$\begin{cases} q_L = \frac{a-c+dq_L-d(q_L+q_F)}{2b}, \\ q_F = \frac{a-c-dq_L+3d(q_L+q_F)}{4b}. \end{cases}$$

Solving this system yields the equilibrium quantities:

$$q_L^{S,IC} = \frac{2(a-c)(b-d)}{4b^2 - 5bd + 2d^2}, \quad q_F^{S,IC} = \frac{b(a-c)}{4b^2 - 5bd + 2d^2}.$$

The resulting profits are:

$$\pi_L^{S,IC} = \frac{2b^2(a-c)^2(b-d)}{(4b^2 - 5bd + 2d^2)^2}, \quad \pi_F^{S,IC} = \frac{b(a-c)^2(b-d)}{(4b^2 - 5bd + 2d^2)^2}.$$

Consumer surplus is:

$$CS^{S,IC} = \frac{b(a-c)^2(3b-2d)^2}{2(4b^2-5bd+2d^2)^2},$$

and total welfare is:

$$W^{S,IC} = \pi_L^{S,IC} + \pi_F^{S,IC} + CS^{S,IC} = \frac{b(a-c)^2(15b^2-20bd+8d^2)}{2(4b^2-5bd+2d^2)^2}.$$

### Proof of Lemma 5

We analyze the Cournot game in which both firms choose to operate over a fully compatible network and select quantities simultaneously. Because products are interoperable, the network effect is shared: each firm benefits from the total output in the market, so the effective network size is $s = q_L + q_F$.

Each firm maximizes the profit function:

$$\pi_i(q_L, q_F, s) = (a - b(q_L + q_F) + ds - c)q_i,$$

where $i \in \{L, F\}$ and $s = q_L + q_F$. The first-order condition for each firm is:

$$\frac{\partial \pi_i}{\partial q_i} = a - 2bq_i - bq_j + d(q_L + q_F) - c = 0.$$

By symmetry, in equilibrium both firms choose the same quantity. Letting $q_L = q_F = q$, we solve:

$$\frac{\partial \pi_i}{\partial q} = a - 3bq + 2dq - c = 0,$$

which yields:

$$q^* = \frac{a-c}{3b-2d}.$$

Thus, the equilibrium quantities are:

$$q_L^{C,CC} = q_F^{C,CC} = \frac{a-c}{3b-2d}.$$

Substituting into the profit function, the individual profits are:

$$\pi_L^{C,CC} = \pi_F^{C,CC} = \frac{b(a-c)^2}{(3b-2d)^2}.$$

Consumer surplus is given by:

$$CS^{C,CC} = \frac{2(a-c)^2}{(3b-2d)^2},$$

and total welfare is:

$$W^{C,CC} = 2\pi_L^{C,CC} + CS^{C,CC} = \frac{2(a-c)^2(b+1)}{(3b-2d)^2}.$$

#### Proof of Lemma 6

We consider the Cournot game in which both firms choose to operate on incompatible networks and select their output levels simultaneously. Since the networks are firm-specific, each firm internalizes only the effect of its own quantity: $s_L = q_L$ and $s_F = q_F$.

Each firm maximizes the profit function:

$$\pi_i(q_L, q_F, s_i) = (a - b(q_L + q_F) + ds_i - c)q_i,$$

where $i \in \{L, F\}$. The first-order condition for each firm is:

$$\frac{\partial \pi_i}{\partial q_i} = a - 2bq_i - bq_j + dq_i - c = 0.$$

By symmetry, the equilibrium satisfies $q_L = q_F = q$. Substituting, we solve:

$$a - 3bq + dq - c = 0 \quad \Rightarrow \quad q^* = \frac{a-c}{3b-d}.$$

Thus, the equilibrium quantities are:

$$q_L^* = q_F^* = \frac{a-c}{3b-d}.$$

Substituting into the profit function, we obtain:

$$\pi_L^{C,II} = \pi_F^{C,II} = \frac{b(a-c)^2}{(3b-d)^2}.$$

Consumer surplus is given by:

$$CS^{C,II} = \frac{2(a-c)^2}{(3b-d)^2},$$

and total welfare is:

$$W^{C,II} = 2\pi_L^{C,II} + CS^{C,II} = \frac{2(a-c)^2(b+1)}{(3b-d)^2}.$$

### Proof of Lemma 7

We consider the Cournot game in which firm L operates on a compatible network and firm F on an incompatible one. Both firms choose quantities simultaneously, but due to the asymmetry in compatibility, network effects are perceived differently: firm L internalizes the full network size, $s_L = q_L + q_F$, while firm F only benefits from its own output, $s_F = q_F$.

Each firm maximizes its profit given these network effects. The resulting first-order conditions yield the following system under rational expectations:

$$q_L = \frac{a-c+2d(q_L+q_F)+dq_F}{3b}, \quad q_F = \frac{a-c+d(q_L+q_F)+dq_F}{3b}.$$

Solving this system leads to the equilibrium quantities:

$$q_L^{C,CI} = \frac{b(a-c)}{3b^2-3bd+d^2}, \quad q_F^{C,CI} = \frac{(a-c)(b-d)}{3b^2-3bd+d^2}.$$

The corresponding profits are:

$$\pi_L^{C,CI} = \frac{b^3(a-c)^2}{(3b^2-3bd+d^2)^2}, \quad \pi_F^{C,CI} = \frac{b(a-c)^2(b-d)^2}{(3b^2-3bd+d^2)^2}.$$

Consumer surplus is given by:

$$CS^{C,CI} = \frac{b(a-c)^2(2b-d)^2}{2(3b^2-3bd+d^2)^2},$$

and total welfare is:

$$W^{C,CI} = \pi_L^{C,CI} + \pi_F^{C,CI} + CS^{C,CI} = \frac{b(a-c)^2(8b^2-8bd+3d^2)}{2(3b^2-3bd+d^2)^2}.$$

**Proof of Proposition 1**

- $\pi_L^{S,CI} > \pi_L^{S,II}$:

  We aim to prove that, under the conditions $a > c > 0$ and $b > 2d > 0$, the following inequality holds:

  $$\pi_L^{S,CI} = \frac{2b^3(a-c)^2}{(4b^2-5bd+2d^2)^2} > \frac{2b(a-c)^2(b-d)^2}{(4b^2-7bd+2d^2)^2} = \pi_L^{S,II}.$$

  To analyze this, we define the difference between both sides as:

  $$\Delta(b) := \frac{2b^3(a-c)^2}{(4b^2-5bd+2d^2)^2} - \frac{2b(a-c)^2(b-d)^2}{(4b^2-7bd+2d^2)^2}.$$

  We factor out the positive common term $2b(a-c)^2$, valid under the assumptions $a > c > 0$ and $b > 0$, obtaining:

  $$\Delta(b) = 2b(a-c)^2\left(\frac{b^2}{(4b^2-5bd+2d^2)^2} - \frac{(b-d)^2}{(4b^2-7bd+2d^2)^2}\right).$$

Let $\delta(b)$ denote the expression in parentheses:

$$\delta(b) = \frac{b^2}{(4b^2-5bd+2d^2)^2} - \frac{(b-d)^2}{(4b^2-7bd+2d^2)^2}.$$

To assess its sign, we reduce it to a single rational expression:

$$\delta(b) = \frac{b^2(4b^2-7bd+2d^2)^2-(b-d)^2(4b^2-5bd+2d^2)^2}{(4b^2-5bd+2d^2)^2(4b^2-7bd+2d^2)^2}.$$

Denoting the numerator by $N(b,d)$, we expand it as:

$$N(b,d) = b^2(4b^2-7bd+2d^2)^2-(b-d)^2(4b^2-5bd+2d^2)^2.$$

After performing the algebraic simplifications, we obtain:

$$N(b,d) = 16b^5d-72b^4d^2+114b^3d^3-81b^2d^4+28bd^5-4d^6.$$

We begin by evaluating this expression at $b = 2d$, which yields:

$$N(2d,d) = 0.$$

This confirms that the difference vanishes at this point: $\Delta(2d) = 0$. We then study the behavior of $N(b,d)$ in a neighborhood of this threshold to determine whether it becomes positive when $b > 2d$.

Computing the derivative of $N$ with respect to $b$ yields:

$$\frac{\partial N}{\partial b} = 80b^4d-288b^3d^2+342b^2d^3-162bd^4+28d^5.$$

Solving the equation $\frac{\partial N}{\partial b} = 0$ yields two real roots. The first is $b = d$, and the second is given by the expression:

$$b = \frac{1}{30}\left(26d + \frac{347d^2}{2^{1/3}\left(10814d^3 + 75\sqrt{5934}d^3\right)^{1/3}} + \frac{\left(10814d^3 + 75\sqrt{5934}d^3\right)^{1/3}}{2^{2/3}}\right).$$

This expression defines the largest real root of the derivative of $N(b,d)$. To determine the behavior of $N$ beyond this point, we evaluate the sign of the derivative at a strictly larger value, namely $b = 2d$. Substituting this into the derivative, we obtain:

$$\left.\frac{\partial N}{\partial b}\right|_{b=2d} = 48d^5 > 0.$$

This confirms that $N(b,d)$ is increasing at the point $b = 2d$. Since we also verified that $N(2d,d) = 0$, it follows that $N(b,d) > 0$ for all $b > 2d$.

Consequently, $\delta(b) > 0$ and hence $\Delta(b) > 0$ in that same domain. This establishes the desired result:

$$\frac{2b^3(a-c)^2}{(4b^2-5bd+2d^2)^2} > \frac{2b(a-c)^2(b-d)^2}{(4b^2-7bd+2d^2)^2} \quad \text{for all } b > 2d,$$

as claimed.

- $\pi_L^{S,II} > \pi_L^{S,CC}$.

We aim to prove that, under the conditions $a > c > 0$ and $b > 2d > 0$, the following inequality holds:

$$\pi_L^{S,II} = \frac{2b(a-c)^2(b-d)^2}{(4b^2-7bd+2d^2)^2} > \frac{2b(a-c)^2}{(4b-3d)^2} = \pi_L^{S,CC}.$$

We cross-multiply, noting that the denominators are strictly positive under the assumption $b > 2d$. In particular, we have:

$$4b - 3d > 0 \quad \text{and} \quad 4b^2 - 7bd + 2d^2 > 0,$$

which implies that the inequality remains directionally consistent after multiplication. The resulting expression is:

$$b(a-c)^2(b-d)^2(4b-3d)^2 > b(a-c)^2(4b^2-7bd+2d^2)^2.$$

Bringing all terms to one side, we obtain an equivalent condition:

$$b(a-c)^2(b-d)^2(4b-3d)^2 - b(a-c)^2(4b^2-7bd+2d^2)^2 > 0.$$

Expanding both sides and simplifying yields the following factored form:

$$b(a-c)^2(4b-5d)(4b-3d)^2(2b-d)d^2(4b^2-7bd+2d^2)^2 > 0.$$

We proceed to verify the sign of each multiplicative component. All are strictly positive under the given parameter restrictions:

- $b > 0$ and $(a-c)^2 > 0$ by assumption,
- $4b - 5d > 0$ follows from $b > 2d \Rightarrow 4b > 8d > 5d$,
- $(4b-3d)^2 > 0$ as a squared term,
- $2b - d > 0$ since $b > 2d \Rightarrow 2b > 4d > d$,
- $d^2 > 0$ for all $d > 0$,

- $(4b^2 - 7bd + 2d^2)^2 > 0$ by construction.

As all terms are strictly positive, the inequality is satisfied. Therefore, we conclude that:

$$\frac{2b(a-c)^2(b-d)^2}{(4b^2-7bd+2d^2)^2} > \frac{2b(a-c)^2}{(4b-3d)^2},$$

as claimed.

- $\pi_L^{S,CC} > \pi_1^{C,CI}$:

We aim to prove that the inequality

$$\pi_L^{S,CC} = \frac{(2b(a-c)^2)}{(4b-3d)^2} > \frac{(b^3(a-c)^2)}{3b^2-3bd+d^2)^2} = \pi_1^{C,CI}.$$

The difference betweeen both sides as:

$$\frac{2b^4 - 12b^3d + 21b^2d^2 - 12bd^3 + 2d^4}{(4b-3d)^2(3b^2-3bd+d^2)^2} > 0,$$

holds for certain values of $d$ as a function of $b$.

Since the denominator is a product of strictly positive quadratic expressions whenever $b > 0$ and $d > 0$, the sign of the entire expression depends solely on the numerator:

$$g(b,d) := 2b^4 - 12b^3d + 21b^2d^2 - 12bd^3 + 2d^4.$$

We thus aim to determine when $g(b,d) > 0$.

To identify the threshold where the sign changes, we analyze the roots of $g(b,d)$ as a polynomial in $d$, for a fixed $b > 0$. It can be verified that:

$$g\left(b, b\left(1 - \frac{1}{\sqrt{2}}\right)\right) = 0,$$

which indicates that the expression vanishes when $d = b\left(1 - \frac{1}{\sqrt{2}}\right)$.

To understand the behavior of $g(b,d)$ around this root, we evaluate the polynomial at reference values. First, if $d = 0$, then:

$$g(b,0) = 2b^4 > 0\,.$$

Second, if $d = b$, then:

$$g(b,b) = 2b^4 - 12b^4 + 21b^4 - 12b^4 + 2b^4 = -b^4 < 0\,.$$

Therefore, the function changes sign as $d$ increases, and we conclude by continuity that:

$$g(b,d) > 0 \quad \text{if and only if} \quad d \in \left(0,\, b\left(1 - \frac{1}{\sqrt{2}}\right)\right).$$

This implies that the original inequality holds precisely when:

$$\frac{2b^4 - 12b^3d + 21b^2d^2 - 12bd^3 + 2d^4}{(4b-3d)^2(3b^2-3bd+d^2)^2} > 0 \quad \text{if and only if} \quad d < b\left(1 - \frac{1}{\sqrt{2}}\right).$$

- $\pi_L^{S,II} > \pi_L^{S,IC}$:

We aim to demonstrate that the inequality

$$\pi_L^{S,II} = \frac{2b(a-c)^2(b-d)^2}{(4b^2-7bd+2d^2)^2} > \frac{2b^2(a-c)^2(b-d)}{(4b^2-5bd+2d^2)^2} = \pi_L^{S,IC}$$

holds for all $b > 2d > 0$.

This is equivalent to showing that

$$2b(a-c)^2(b-d)\left[\frac{(b-d)}{(4b^2-7bd+2d^2)^2} - \frac{b}{(4b^2-5bd+2d^2)^2}\right] > 0\,,$$

Let us define the function

$$f(b,d) := \frac{b-d}{(4b^2-7bd+2d^2)^2} - \frac{b}{(3b-d)^2}\,.$$

Writing this as a single rational expression and simplifying the numerator yields

$$f(b,d) = \frac{16b^3d^2 - 33b^2d^3 + 20bd^4 - 4d^5}{(4b^2-7bd+2d^2)^2(3b-d)^2}\,.$$

Since the denominator is strictly positive for all $b,d > 0$, it suffices to study the sign of the numerator:

$$g(b) := 16b^3d^2 - 33b^2d^3 + 20bd^4 - 4d^5\,.$$

To assess its behavior for $b > 2d$, we compute its derivative:

$$g'(b) = 48b^2d^2 - 66bd^3 + 20d^4\,.$$

Solving $g'(b) = 0$, we obtain the roots of the quadratic equation:

$$24b^2 - 33bd + 10d^2 = 0\,,$$

whose largest root is

$$b^* = \frac{1}{48}(33 + \sqrt{129})d \approx 1.86d\,.$$

Thus, for all $b > 2d$, it follows that $b > b^*$ and consequently $g'(b) > 0$. Therefore, $g(b)$ is strictly increasing on the interval $b > 2d$.

Evaluating $g$ at $b = 2d$ yields

$$g(2d) = 128d^5 - 132d^5 + 40d^5 - 4d^5 = 32d^5 > 0\,.$$

Since $g$ is increasing for $b > 2d$, we conclude that

$$g(b) > 0 \quad \text{for all } b > 2d\,.$$

It follows that

$$\frac{2b(a-c)^2(b-d)^2}{(4b^2-7bd+2d^2)^2} > \frac{2b^2(a-c)^2(b-d)}{(4b^2-5bd+2d^2)^2}$$

holds for all $b > 2d > 0$, as desired.

- $\pi_1^{C,CI} > \pi_L^{S,IC}$:

We want to prove that, under the conditions $a > c > 0$ and $b > 2d > 0$, the following inequality holds:

$$\pi_1^{C,CI} = \frac{b^3(a-c)^2}{(3b^2-3bd+d^2)^2} > \frac{2b(a-c)^2(b-d)}{(4b^2-5bd+2d^2)^2} = \pi_L^{S,IC}.$$

We begin by rewriting the inequality as a difference:

$$\Delta(b) := \frac{b^3(a-c)^2}{(3b^2-3bd+d^2)^2} - \frac{2b(a-c)^2(b-d)}{(4b^2-5bd+2d^2)^2}.$$

We factor out the positive common term $(a-c)^2 > 0$, reducing the proof to verifying:

$$\delta(b) := \frac{b^3}{(3b^2-3bd+d^2)^2} - \frac{2b(b-d)}{(4b^2-5bd+2d^2)^2} > 0.$$

This expression can be written as a single rational function:

$$\delta(b) = \frac{N(b,d)}{(3b^2-3bd+d^2)^2(4b^2-5bd+2d^2)^2},$$

where the numerator is:

$$N(b,d) = -2b^5 + 14b^4 d - 25b^3 d^2 + 22b^2 d^3 - 10bd^4 + 2d^5.$$

We evaluate this at $b = 2d$:

$$N(2d,d) = -2(32d^5) + 14(16d^5) - 25(8d^5) + 22(4d^5) - 10(2d^5) + 2d^5 = 30d^5 > 0,$$

next, we compute the derivative of the numerator:

$$\frac{\partial N}{\partial b} = -10b^4 + 56b^3 d - 75b^2 d^2 + 44bd^3 - 10d^4.$$

Solving $\frac{\partial N}{\partial b} = 0$, the largest real root is:

$$b = \frac{7d}{5} + \frac{\sqrt{142d^2 - \frac{315d^4}{(185d^6 - 4\sqrt{17767}d^6)^{1/3}} + 5(185d^6 - 4\sqrt{17767}d^6)^{1/3}}}{10\sqrt{2}} + \cdots$$

We denote this root as $b^*$, and numerical evaluation confirms that $b^* < 2d$.

Now we check the sign of the derivative at $b = 2d$:

$$\left.\frac{\partial N}{\partial b}\right|_{b=2d} = 66d^4 > 0.$$

Since $N(2d,d) > 0$, $\frac{\partial N}{\partial b}(2d) > 0$, and the only real root of the derivative occurs for $b < 2d$, we conclude that:

$$N(b,d) > 0 \quad \text{for all } b > 2d.$$

Hence, the expression $\delta(b)$ is strictly positive for $b > 2d$, and we obtain:

$$\Delta(b) > 0 \quad \text{for all } b > 2d.$$

This proves the original inequality:

$$\frac{b^3(a-c)^2}{(3b^2-3bd+d^2)^2} > \frac{2b(a-c)^2(b-d)}{(4b^2-5bd+2d^2)^2}.$$

- $\pi_L^{S,IC} > \pi_1^{C,CC}$:

We want to prove that, under the conditions $a > c > 0$ and $b > 2d > 0$, the following inequality holds:

$$\frac{2b^2(a-c)^2(b-d)}{(4b^2-5bd+2d^2)^2} > \frac{b(a-c)^2}{(3b-2d)^2}.$$

We start by factoring out the common positive term $b(a-c)^2 > 0$. This allows us to rewrite the inequality as:

$$\delta(b) := \frac{2b(b-d)}{(4b^2-5bd+2d^2)^2} - \frac{1}{(3b-2d)^2} > 0,$$

to express this as a single rational function, we write:

$$\delta(b) = \frac{N(b,d)}{(3b-2d)^2(4b^2-5bd+2d^2)^2},$$

where the numerator is given by:

$$N(b,d) = 2b^4 - 2b^3d - 9b^2d^2 + 12bd^3 - 4d^4.$$

We evaluate this numerator at $b = 2d$:

$$N(2d,d) = 2(16d^4) - 2(8d^4) - 9(4d^4) + 12(2d^4) - 4d^4 = 0.$$

This shows that $\delta(2d) = 0$. To determine the sign of $\delta(b)$ for $b > 2d$, we analyze the growth of the numerator.

We compute the derivative of $N$ with respect to $b$:

$$\frac{\partial N}{\partial b} = 8b^3 - 6b^2d - 18bd^2 + 12d^3,$$

solving $\frac{\partial N}{\partial b} = 0$, we find that the only real root is:

$$b = \frac{1}{4}\left( d + \frac{13d^2}{\left(-29d^3 + 2i\sqrt{339}d^3\right)^{1/3}} + \left(-29d^3 + 2i\sqrt{339}d^3\right)^{1/3} \right).$$

This real root is strictly less than $2d$. We now evaluate the derivative at $b = 2d$:

$$\left.\frac{\partial N}{\partial b}\right|_{b=2d} = 8(8d^3) - 6(4d^2)d - 18(2d)d^2 + 12d^3 = 64d^3 - 24d^3 - 36d^3 + 12d^3 = 16d^3 > 0.$$

Since the function $N(b,d)$ vanishes at $b = 2d$ and is strictly increasing for $b > 2d$, we conclude that:

$$N(b,d) > 0 \quad \text{for all } b > 2d.$$

As the denominator of $\delta(b)$ is always positive (being the square of real-valued expressions), it follows that:

$$\delta(b) > 0 \quad \text{for all } b > 2d,$$

thus, the original inequality holds:

$$\frac{2b^2(a-c)^2(b-d)}{(4b^2-5bd+2d^2)^2} > \frac{b(a-c)^2}{(3b-2d)^2}.$$

- $\pi_1^{C,CC} > \pi_1^{C,II}$:

We aim to prove the following inequality:

$$\pi_1^{C,CC} = \frac{b(a-c)^2}{(3b-2d)^2} > \frac{b(a-c)^2}{(3b-d)^2} = \pi_1^{C,II},$$

under the assumptions:

$$a > c > 0, \quad b > 2d > 0.$$

Since both sides share the strictly positive term $b(a-c)^2$, the inequality reduces to:

$$\frac{1}{(3b-2d)^2} > \frac{1}{(3b-d)^2}.$$

Multiplying both sides by the positive product $(3b-2d)^2(3b-d)^2$ preserves the inequality:

$$(3b-d)^2 > (3b-2d)^2.$$

Expanding both expressions yields:

$$(3b-d)^2-(3b-2d)^2 = 9b^2-6bd+d^2-(9b^2-12bd+4d^2) = 6bd-3d^2.$$

Factoring the result, we obtain:

$$6bd-3d^2 = 3d(2b-d).$$

Given the assumption $b > 2d > 0$, we know that both $d > 0$ and $2b-d > 0$, so it follows that:

$$3d(2b-d) > 0.$$

Therefore, the initial inequality is valid:

$$\frac{b(a-c)^2}{(3b-2d)^2} > \frac{b(a-c)^2}{(3b-d)^2}.$$

- $\pi_1^{C,II} > \pi_2^{C,CI}$:

We aim to prove the following inequality:

$$\frac{b(a-c)^2}{(3b-d)^2} > \frac{b(a-c)^2(b-d)^2}{(3b^2-3bd+d^2)^2}$$

under the assumptions:

$$a > c > 0, \quad b > 2d > 0.$$

To simplify, we multiply both sides by the positive expression $(3b-d)^2(3b^2-3bd+d^2)^2$, which is valid under the stated assumptions. This yields:

$$b(a-c)^2(3b^2-3bd+d^2)^2 > b(a-c)^2(b-d)^2(3b-d)^2.$$

Bringing all terms to one side for analysis:

$$b(a-c)^2(3b^2-3bd+d^2)^2 - b(a-c)^2(b-d)^2(3b-d)^2 > 0.$$

Factoring the left-hand side yields:

$$b^2(a-c)^2(3b-2d)(2b-d)d(3b-d)^2(3b^2-3bd+d^2)^2 > 0.$$

We now examine the sign of each factor under the assumptions $a > c > 0$ and $b > 2d > 0$:

- $b^2 > 0$ since $b > 0$,
- $(a-c)^2 > 0$ because $a > c$,
- $3b-2d > 0$, given that $b > 2d \Rightarrow 3b > 6d > 2d$,
- $2b-d > 0$, since $2b > 4d > d$,
- $d > 0$ by assumption,
- $(3b-d)^2 > 0$, as a perfect square,

  - $(3b^2 - 3bd + d^2)^2 > 0$, also a perfect square.

All factors are strictly positive under the given conditions. Therefore, the original inequality holds:

$$\frac{b(a-c)^2}{(3b-d)^2} > \frac{b(a-c)^2(b-d)^2}{(3b^2-3bd+d^2)^2}.$$

- $\pi_2^{C,CI} > \pi_F^{S,CC}$:

We aim to prove that the following inequality holds under certain conditions on $b$ and $d$:

$$\pi_2^{C,CI} = \frac{b(a-c)^2(b-d)^2}{(3b^2-3bd+d^2)^2} > \frac{(b(a-c)^2)}{(4b-3d)^2} = \pi_F^{S,CC},.$$

Since all denominators are strictly positive for $b > d > 0$, we multiply both sides by these positive terms to obtain the equivalent inequality:

$$(b-d)^2(4b-3d)^2 < (3b^2-3bd+d^2)^2\,.$$

We define the auxiliary function:

$$f(b,d) := (3b^2-3bd+d^2)^2 - (b-d)^2(4b-3d)^2\,.$$

Expanding both terms and simplifying leads to:

$$f(b,d) = 7b^4 - 38b^3d + 58b^2d^2 - 36bd^3 + 8d^4\,.$$

We are interested in determining the values of $b$ for which $f(b,d) > 0$. Solving the equation $f(b,d) = 0$, we find a real root given by:

$$b^* = d(2-\sqrt{2})\,.$$

To analyze the sign of $f(b,d)$ around this critical value, we compute the first derivative with respect to $b$:

$$\frac{\partial f}{\partial b} = 28b^3 - 114b^2 d + 116bd^2 - 36d^3\,.$$

Evaluating at $b = b^*$, we obtain:

$$\left.\frac{\partial f}{\partial b}\right|_{b=b^*} = (72 - 52\sqrt{2})d^3 > 0\,,$$

which indicates that the function $f(b,d)$ changes sign from negative to positive at $b = d(2-\sqrt{2})$. Therefore, we conclude:

$$f(b,d) > 0 \quad \Leftrightarrow \quad b > d(2-\sqrt{2})\,.$$

Hence, the original inequality holds if and only if:

$$\frac{(b-d)^2}{(3b^2 - 3bd + d^2)^2} < \frac{1}{(4b-3d)^2} \quad \text{if and only if} \quad b > d(2-\sqrt{2})\,,$$

or, equivalently,

$$d \in \left(0,\, b\left(1 - \frac{1}{\sqrt{2}}\right)\right).$$

- $\pi_1^{C,II} > \pi_F^{S,IC}$:

We aim to prove that the following inequality holds under the conditions $a > c > 0$ and $b > 2d > 0$:

$$\pi_1^{C,II} = \frac{(b(a-c))^2}{(3b-d)^2} > \frac{(a-c)^2(b^2 - 2bd + 2d^2)}{(4b^2 - 5bd + 2d^2)^2} = \pi_F^{S,IC}\,.$$

Factoring out the common positive term $(a-c)^2$, the inequality reduces to verifying that:

$$\frac{b^2}{(3b-d)^2} - \frac{b^2 - 2bd + 2d^2}{(4b^2 - 5bd + 2d^2)^2} > 0\,.$$

Combining the two terms over a common denominator and simplifying using `Mathematica`, we obtain:

$$\frac{7b^4 - 16b^3d + 10b^2d^2 - 6bd^3 + 2d^4}{(3b-d)^2(4b^2 - 5bd + 2d^2)^2} > 0\,.$$

Let us define the numerator as the auxiliary function:

$$f(b) = 7b^4 - 16b^3d + 10b^2d^2 - 6bd^3 + 2d^4\,.$$

Evaluating at $b = 2d$, we find:

$$f(2d) = 14d^4 > 0\,,$$

which shows that the function is positive at this point.

To confirm monotonicity beyond this point, we compute the first derivative:

$$f'(b) = 28b^3 - 48b^2d + 20bd^2 - 6d^3\,.$$

Solving the equation $f'(b) = 0$, the largest real root is given by:

$$b^* = \frac{4d}{7} + \frac{13 \cdot 2^{2/3} d^2}{3^{1/3}(1107d^3 + 7\sqrt{22857}d^3)^{1/3}} + \frac{(1107d^3 + 7\sqrt{22857}d^3)^{1/3}}{6^{2/3}} \approx 1.29d\,,$$

which satisfies $b^* < 2d$.

Evaluating the derivative at $b = 2d$, we get:

$$f'(2d) = 66d^3 > 0\,,$$

proving that the function $f(b)$ is increasing for all $b > 2d$. Since $f(2d) > 0$, and $f$ is increasing in that region, we conclude that:

$$f(b) > 0 \quad \text{for all } b > 2d\,.$$

Therefore, the original inequality holds for all $b > 2d > 0$, as both the numerator and denominators are strictly positive in this domain:

$$\frac{(b(a-c))^2}{(3b-d)^2} > \frac{(a-c)^2(b^2-2bd+2d^2)}{(4b^2-5bd+2d^2)^2}.$$

- $\pi_F^{S,CC} > \pi_F^{S,IC}$

We aim to prove that, under the conditions $a > c > 0$ and $b > 2d > 0$, the following inequality holds:

$$\pi_F^{S,CC} = \frac{b(a-c)^2}{(4b-3d)^2} > \frac{b(a-c)^2(b^2-2bd+2d^2)}{(4b^2-5bd+2d^2)^2} = \pi_F^{S,IC}.$$

We begin by canceling the common positive factor $(a-c)^2$ from both sides. Thus, we consider:

$$\delta(b) := \frac{1}{(4b-3d)^2} - \frac{b^2-2bd+2d^2}{(4b^2-5bd+2d^2)^2} > 0,$$

we write this as a single rational expression:

$$\delta(b) = \frac{2N(b,d)}{(4b-3d)^2(4b^2-5bd+2d^2)^2},$$

where the numerator is given by:

$$N(b,d) = 8b^3d - 24b^2d^2 + 23bd^3 - 7d^4.$$

We evaluate this expression at $b = 2d$:

$$N(2d,d) = 8(8d^4) - 24(4d^4) + 23(2d^4) - 7d^4 = 64d^4 - 96d^4 + 46d^4 - 7d^4 = 7d^4 > 0.$$

To establish that $\delta(b) > 0$ for all $b > 2d$, we analyze the growth of the numerator. We compute the derivative:

$$\frac{\partial N}{\partial b} = 24b^2d - 48bd^2 + 23d^3,$$

solving $\frac{\partial N}{\partial b} = 0$, we find the real roots:

$$b = \frac{1}{12}(12d \pm \sqrt{6}d).$$

The largest real root is:

$$b = \frac{1}{12}(12d + \sqrt{6}d) < 2d.$$

We then evaluate the derivative at $b = 2d$:

$$\left.\frac{\partial N}{\partial b}\right|_{b=2d} = 96d^3 - 96d^3 + 23d^3 = 23d^3 > 0.$$

This confirms that $N(b,d)$ is strictly increasing for $b > 2d$. Since $N(2d,d) > 0$, it follows that:

$$N(b,d) > 0 \quad \text{for all } b > 2d,$$

and as the denominator is always positive (being a square), we conclude that:

$$\delta(b) > 0 \quad \text{for all } b > 2d.$$

Therefore, the inequality:

$$\frac{b(a-c)^2}{(4b-3d)^2} > \frac{b(a-c)^2(b^2-2bd+2d^2)}{(4b^2-5bd+2d^2)^2},$$

is valid for all $a > c > 0$ and $b > 2d > 0$.

- $\pi_F^{S,IC} > \pi_F^{S,II}$

We aim to prove that, under the conditions $a > c > 0$ and $b > 2d > 0$, the following inequality holds:

$$\pi_F^{S,IC} = \frac{b(a-c)^2(b^2-2bd+2d^2)}{(4b^2-5bd+2d^2)^2} > \frac{(b(a-c)^2(b-2d)^2)}{(4b^2-7bd+2d^2)^2} = \pi_F^{S,II}.$$

To verify this inequality, we define the expression:

$$\delta(b) := \frac{b(a-c)^2(b^2-2bd+2d^2)}{(4b^2-5bd+2d^2)^2} - \frac{(b(a-c)^2(b-2d)^2)}{(4b^2-7bd+2d^2)^2}.$$

The common factor $b(a-c)^2$ appears in the first and third terms and is strictly positive by assumption. Therefore, we can analyze the sign of the following rational expression:

$$\delta(b) = \frac{2(8b^5d-28b^4d^2+37b^3d^3-29b^2d^4+16bd^5-4d^6)}{(4b^2-7bd+2d^2)^2(4b^2-5bd+2d^2)^2}.$$

Let us denote the numerator of this expression as:

$$N(b,d) = 8b^5 d - 28b^4 d^2 + 37b^3 d^3 - 29b^2 d^4 + 16bd^5 - 4d^6,$$

we first evaluate $N(b,d)$ at $b = 2d$:

$$N(2d,d) = 8(32)d^6 - 28(16)d^6 + 37(8)d^6 - 29(4)d^6 + 16(2)d^6 - 4d^6 = 16d^6 > 0.$$

This confirms that the numerator is positive at the threshold $b = 2d$. To determine whether this positivity holds for all $b > 2d$, we study the behavior of $N(b,d)$ using the first derivative test.

We compute the derivative:

$$\frac{\partial N}{\partial b} = 40b^4 d - 112b^3 d^2 + 111b^2 d^3 - 58bd^4 + 16d^5.$$

Solving the equation $\frac{\partial N}{\partial b} = 0$, we find that the largest real root is:

$$b = \frac{7d}{10} + \frac{1}{20\sqrt{2}} \sqrt{22d^2 + \frac{285d^2}{(3389 + 4\sqrt{706258})^{1/3}} + 5(3389 + 4\sqrt{706258})^{1/3} d^2 + \ldots}$$

Numerical evaluation confirms that this root is strictly less than $2d$. Hence, we evaluate the derivative at $b = 2d$:

$$\left.\frac{\partial N}{\partial b}\right|_{b=2d} = 88d^5 > 0.$$

This implies that $N(b,d)$ is increasing at $b = 2d$, and since $N(2d,d) > 0$, it follows that:

$$N(b,d) > 0 \quad \text{for all } b > 2d.$$

Because the denominator of $\delta(b)$ is strictly positive (being a product of squared expressions), the full expression satisfies:

$$\delta(b) > 0 \quad \text{for all } b > 2d.$$

Therefore, we conclude that the original inequality holds for all parameter values such that $a > c > 0$ and $b > 2d > 0$:

$$\frac{b(a-c)^2(b^2-2bd+2d^2)}{(4b^2-5bd+2d^2)^2} > \frac{(b(a-c)^2(b-2d)^2)}{(4b^2-7bd+2d^2)^2}.$$

- $\pi_F^{S,II} > \pi_F^{S,CI}$:

We aim to demonstrate that the following inequality holds under the assumptions $a > c > 0$ and $b > 2d > 0$:

$$\pi_F^{S,II} = \frac{b(a-c)^2(b-2d)^2}{(4b^2-7bd+2d^2)^2} > \frac{b(a-c)^2(b-2d)^2}{(4b^2-5bd+2d^2)^2} = \pi_F^{S,CI}.$$

Bringing all terms to one side:

$$b^2(a-c)^2(b-2d)^2(b-d)(2b-d)d(4b^2-7bd+2d^2)^2(4b^2-5bd+2d^2)^2 > 0.$$

To determine the sign of this expression, we analyze each factor under the assumed parameter conditions. The terms $b^2$, $(a-c)^2$, and $(b-2d)^2$ are positive due to the assumptions $b > 0$, $a > c$, and $b > 2d$.

Similarly, $b-d > 0$, $2b-d > 0$, and $d > 0$. Both quadratic expressions in the denominator, being perfect squares, are strictly positive for all $b, d > 0$.

Since every factor in the inequality is strictly positive under the assumed conditions, the inequality is satisfied:

$$\frac{b(a-c)^2(b-2d)^2}{(4b^2-7bd+2d^2)^2} > \frac{b(a-c)^2(b-2d)^2}{(4b^2-5bd+2d^2)^2}.$$

### Proof of Proposition 2

- $CS^{S,CC} > CS^{S,CI}$:

  We aim to demonstrate that, under the conditions $a > c > 0$ and $b > 2d > 0$, the following inequality holds:

  $$CS^{S,CC} = \frac{9b(a-c)^2}{2(4b-3d)^2} > \frac{b(a-c)^2(3b-2d)^2}{2(4b^2-5bd+2d^2)^2} = CS^{S,CI}.$$

  Bringing all terms to one side, we obtain a single expression whose positivity must be verified. After algebraic simplification, the inequality can be rewritten as:

  $$b^2(a-c)^2(4b-3d)^2 d(4b^2-5bd+2d^2)^2(6b^2-8bd+3d^2) > 0.$$

  We now analyze the sign of each factor under the assumptions $a > c > 0$ and $b > 2d > 0$. The terms $b^2$, $(a-c)^2$, and $(4b-3d)^2$ are strictly positive. The variable $d$ is positive by hypothesis, and both squared terms $(4b^2-5bd+2d^2)^2$ and $(4b-3d)^2$ are always positive.

To assess the last factor, we consider the quadratic expression $6b^2 - 8bd + 3d^2$. Substituting $b = 2d$ gives:

$$6(4d^2) - 8(2d)(d) + 3d^2 = 24d^2 - 16d^2 + 3d^2 = 11d^2 > 0,$$

which confirms that the expression is strictly positive for all $b > 2d$.

Since all terms in the factored expression are strictly positive under the assumed conditions, the inequality is satisfied:

$$\frac{9b(a-c)^2}{2(4b-3d)^2} > \frac{b(a-c)^2(3b-2d)^2}{2(4b^2-5bd+2d^2)^2}.$$

- $CS^{S,CI} > CS^{S,II}$:

We aim to prove that, under the conditions $a > c > 0$ and $b > 2d > 0$, the following inequality holds:

$$CS^{S,CI} = \frac{b(a-c)^2(3b-2d)^2}{2(4b^2-5bd+2d^2)^2} > \frac{b(a-c)^2(3b-4d)^2}{2(4b^2-7bd+2d^2)^2} = CS^{S,II}.$$

We begin by factoring out the common positive term $\frac{1}{2}b(a-c)^2$, which is strictly positive under the assumed conditions. The inequality becomes:

$$\frac{(3b-2d)^2}{(4b^2-5bd+2d^2)^2} > \frac{(3b-4d)^2}{(4b^2-7bd+2d^2)^2}.$$

Denoting the difference between both sides as:

$$\delta(b) := \frac{(3b-2d)^2}{(4b^2-5bd+2d^2)^2} - \frac{(3b-4d)^2}{(4b^2-7bd+2d^2)^2},$$

we analyze the sign of this expression. Bringing both terms to a common denominator, we obtain:

$$\delta(b) = \frac{P(b,d)}{(4b^2-5bd+2d^2)^2(4b^2-7bd+2d^2)^2},$$

where the numerator is the polynomial:

$$P(b,d) = 12b^5 - 66b^4d + 137b^3d^2 - 135b^2d^3 + 64bd^4 - 12d^5.$$

We evaluate the sign of $P(b,d)$ at $b = 2d$:

$$P(2d,d) = 12(32)d^5 - 66(16)d^5 + 137(8)d^5 - 135(4)d^5 + 64(2)d^5 - 12d^5 = 0,$$

To verify whether $P(b,d)$ remains positive for all $b > 2d$, we compute its partial derivative with respect to $b$:

$$\frac{\partial P}{\partial b} = 60b^4d - 264b^3d^2 + 411b^2d^3 - 270bd^4 + 64d^5.$$

Solving $\frac{\partial P}{\partial b} = 0$ yields four roots in total, two of which are real. The largest real root is given by the following expression:

$$\begin{aligned} b^\star = &\frac{11d}{10} + \frac{1}{20\sqrt{3}}\sqrt{82d^2 + \frac{645d^2}{(1625 + 4\sqrt{30871})^{1/3}} + 5(1625 + 4\sqrt{30871})^{1/3}d^2} \\ &+ \frac{1}{2}\sqrt{\frac{41d^2}{75} - \frac{43d^2}{20(1625 + 4\sqrt{30871})^{1/3}} - \frac{(1625 + 4\sqrt{30871})^{1/3}d^2}{60} + \cdots} \end{aligned}$$

This root is strictly less than $2d$, and therefore lies below the threshold of interest. To determine the sign of the derivative in the relevant domain, we evaluate it at $b = 2d$:

$$\left.\frac{\partial P}{\partial b}\right|_{b=2d} = 16d^5 > 0.$$

The positivity of the derivative at $b = 2d$, together with the fact that there are no larger real roots, confirms that $P(b,d) > 0$ for all $b > 2d$. Hence, $\delta(b) > 0$ in this domain, and we conclude:

$$\frac{b(a-c)^2(3b-2d)^2}{2(4b^2-5bd+2d^2)^2} > \frac{b(a-c)^2(3b-4d)^2}{2(4b^2-7bd+2d^2)^2},$$

as was to be shown.

- $CS^{S,II} > CS^{C,CI}$:

We aim to prove that, under the conditions $a > c > 0$ and $b > 2d > 0$, the following inequality holds:

$$CS^{S,II} = \frac{b(a-c)^2(3b-4d)^2}{2(4b^2-7bd+2d^2)^2} > \frac{b(a-c)^2(2b-d)^2}{2(3b^2-3bd+d^2)^2} = CS^{C,CI}.$$

To proceed, let us consider the difference between both sides:

$$\Delta(b) := \frac{b(a-c)^2(3b-4d)^2}{2(4b^2-7bd+2d^2)^2} - \frac{b(a-c)^2(2b-d)^2}{2(3b^2-3bd+d^2)^2}.$$

Factoring out the strictly positive term $\frac{1}{2}b(a-c)^2$, which does not affect the sign of the expression, we obtain:

$$\Delta(b) = \frac{1}{2}b(a-c)^2\left(\frac{(3b-4d)^2}{(4b^2-7bd+2d^2)^2} - \frac{(3b-2d)^2}{(3b^2-3bd+d^2)^2}\right).$$

Denote the expression in parentheses as $\delta(b)$. After algebraic simplification, we can express $\delta(b)$ as a rational function:

$$\delta(b) = \frac{P(b,d)}{(3b^2-3bd+d^2)^2(4b^2-7bd+2d^2)^2},$$

where the numerator is given by the polynomial

$$P(b,d) = 17b^6 - 90b^5d + 211b^4d^2 - 274b^3d^3 + 200b^2d^4 - 76bd^5 + 12d^6.$$

To determine the sign of $\Delta(b)$, we evaluate $P$ at $b = 2d$:

$$P(2d,d) = 52d^6 > 0.$$

Next, we analyze whether this positivity extends to the entire region $b > 2d$. We compute the first derivative of $P$ with respect to $b$:

$$\frac{\partial P}{\partial b} = 102b^5 - 450b^4d + 844b^3d^2 - 822b^2d^3 + 400bd^4 - 76d^5.$$

Solving $\frac{\partial P}{\partial b} = 0$ yields three real positive roots. The largest one is approximately $b^\star \approx 1.22d$, which lies strictly below the threshold $b = 2d$. We then evaluate the derivative at $b = 2d$:

$$\left.\frac{\partial P}{\partial b}\right|_{b=2d} = 252d^5 > 0.$$

Since the derivative is positive beyond the last turning point, and the function is already positive at $b = 2d$, it follows that $P(b,d) > 0$ for all $b > 2d$. Consequently, the numerator of $\delta(b)$ remains strictly positive in that range, and so does the full expression $\Delta(b)$.

We therefore conclude that

$$\Delta(b) > 0 \quad \text{for all } b > 2d,$$

which establishes the inequality

$$\frac{b(a-c)^2(3b-4d)^2}{2(4b^2-7bd+2d^2)^2} > \frac{b(a-c)^2(2b-d)^2}{2(3b^2-3bd+d^2)^2}.$$

- $CS^{C,CC} > CS^{C,II}$:

We aim to show that, under the assumptions $a > c > 0$ and $b > 2d > 0$, the following inequality holds:

$$CS^{C,CC} = \frac{2(a-c)^2}{(3b-2d)^2} > \frac{2(a-c)^2}{(3b-d)^2} = CS^{C,II}.$$

To proceed, we divide both sides by the strictly positive factor $2(a-c)^2$, yielding:

$$\frac{1}{(3b-2d)^2} > \frac{1}{(3b-d)^2}.$$

We now multiply both sides by the denominators, which are positive due to the assumption $b > 2d$, and obtain:

$$(3b-d)^2 > (3b-2d)^2.$$

Expanding both sides:

$$(3b-d)^2 = 9b^2 - 6bd + d^2,$$

$$(3b-2d)^2 = 9b^2 - 12bd + 4d^2.$$

Subtracting one side from the other:

$$(9b^2 - 6bd + d^2) - (9b^2 - 12bd + 4d^2) = 6bd - 3d^2 = 3d(2b-d).$$

Thus, the inequality reduces to:

$$3d(2b-d) > 0.$$

This inequality clearly holds under the assumption $b > 2d$.

It follows that all terms involved are strictly positive, and hence:

$$\frac{2(a-c)^2}{(3b-2d)^2} > \frac{2(a-c)^2}{(3b-d)^2},$$

as was to be shown.

- $CS^{C,CI} > CS^{C,CC}$:

We aim to demonstrate that, under the conditions $a > c > 0$, $b > 2d > 0$, the following inequality holds:

$$\frac{b(a-c)^2(2b-d)^2}{2(3b^2-3bd+d^2)^2} > \frac{2(a-c)^2}{(3b-2d)^2}.$$

Extracting the common positive factor $\frac{1}{2}(a-c)^2$, the inequality reduces to:

$$\frac{4}{(3b-2d)^2} + \frac{b(-2b+d)^2}{(3b^2-3bd+d^2)^2} > 0.$$

This expression is strictly positive if and only if the following polynomial numerator is positive:

$$-36b^4 + 36b^5 + 72b^3d - 48b^4d - 60b^2d^2 + 23b^3d^2 + 24bd^3 - 28b^2d^3 - 4d^4 + 4bd^4 > 0.$$

We define the function:

$$f(b) = -36b^4 + 36b^5 + 72b^3d - 48b^4d - 60b^2d^2 + 23b^3d^2 + 24bd^3 - 28b^2d^3 - 4d^4 + 4bd^4,$$

and evaluate it at $b = 2d$:

$$f(2d) = -196d^4 + 288d^5.$$

To ensure this expression is positive, it suffices to verify:

$$-196d^4 + 288d^5 > 0 \quad \Leftrightarrow \quad d > \frac{49}{72}.$$

Since the condition $b > 2d$ implies $d < \frac{b}{2}$, it follows that if $d > \frac{49}{72} \approx 0.68$, then the inequality is satisfied for all $b > 2d$, which also implies $b > 1$.

To strengthen the conclusion, we analyze the monotonicity of $f(b)$ using the first derivative:

$$f'(b) = -144b^3 + 180b^4 + 216b^2d - 336b^3d - 120bd^2 + 219b^2d^2 + 24d^3 - 56bd^3 + 4d^4 .$$

The real root of this derivative corresponds to a critical point:

$$b^* = \frac{1}{90}(24 + 41d + \cdots) ,$$

which can be shown to satisfy $b^* < 2d$.

Evaluating the derivative at $b = 2d$, we obtain:

$$f'(2d) = 24d^3(-21 + 40d),$$

which is strictly positive whenever $d > \frac{21}{40} = 0.525$.

Therefore, if $d > \frac{49}{72}$, then:

- $f(2d) > 0$ ;
- $f'(2d) > 0$, so the function $f(b)$ is strictly increasing for all $b > 2d$ .

Under the condition $d > \frac{49}{72}$, the inequality

$$\frac{b(a-c)^2(2b-d)^2}{2(3b^2-3bd+d^2)^2} > \frac{2(a-c)^2}{(3b-2d)^2}$$

holds for all $d > \frac{49}{72}$ .

- $CS^{C,CI} > CS^{C,II}$

We aim to prove that the following inequality holds for certain values of $b$ and $d$ satisfying

$b > 2d > 0$ and $a > c > 0$:

$$\frac{b(a-c)^2(2b-d)^2}{2(3b^2-3bd+d^2)^2} > \frac{2(a-c)^2}{(3b-2d)^2}\,.$$

Factoring out the positive term $\frac{(a-c)^2}{2}$, this inequality becomes equivalent to:

$$\frac{4}{(3b-2d)^2} + \frac{b(-2b+d)^2}{(3b^2-3bd+d^2)^2} > 0\,.$$

We define the polynomial numerator of the simplified rational expression as the auxiliary function:

$$f(b) = -36b^4 + 36b^5 + 72b^3d - 60b^4d - 60b^2d^2 + 37b^3d^2 + 24bd^3 - 10b^2d^3 - 4d^4 + bd^4\,.$$

Substituting $b = 2d$, we obtain:

$$f(2d) = -196d^4 + 450d^5\,,$$

which is strictly positive if and only if $450d^5 > 196d^4$, that is,

$$d > \frac{98}{225} \quad \Rightarrow \quad b = 2d > \frac{196}{225} \approx 0.87\,.$$

Hence, $f(2d) > 0$ under this condition.

To ensure that the function $f(b)$ remains positive for all $b > 2d$, we study its monotonicity via the first derivative:

$$f'(b) = -144b^3 + 180b^4 + 216b^2d - 240b^3d - 120bd^2 + 111b^2d^2 + 24d^3 - 20bd^3 + d^4\,.$$

Evaluating this derivative at $b = 2d$, we find:

$$f'(2d) = 21d^3(-24 + 65d),$$

which is strictly positive whenever $d > \frac{24}{65} \approx 0.369$, a weaker condition than the one previously established.

Therefore, under the assumption $d > \frac{98}{225}$, we conclude that:

- $f(2d) > 0$,
- $f'(2d) > 0$, hence $f(b)$ is increasing for $b > 2d$ .

Since the function is increasing for all $b > 2d$, and since $f(2d) > 0$ whenever $d > \frac{98}{225}$, we conclude that the original inequality holds for all $b > 2d$ under this condition.

### Proof of Proposition 3

- $W^{S,CC} > W^{S,CI}$:

We aim to demonstrate that, under the conditions $a > c > 0$, $b > 2d > 0$, the following inequality holds:

$$\frac{15b(a-c)^2}{2(4b-3d)^2} > \frac{b(a-c)^2(15b^2 - 20bd + 12d^2)}{2(4b^2 - 5bd + 2d^2)^2}.$$

To this end, we define the difference function:

$$\delta(b) := \frac{15b(a-c)^2}{2(4b-3d)^2} - \frac{b(a-c)^2(15b^2 - 20bd + 12d^2)}{2(4b^2 - 5bd + 2d^2)^2}.$$

The expression can be factored as:

$$\delta(b) = \frac{b(a-c)^2(4b-3d)^2 d(4b^2-5bd+2d^2)^2 \cdot \underbrace{\left(10b^3-24b^2d+21bd^2-6d^3\right)}_{f(b,d)}}{2(4b-3d)^2(4b^2-5bd+2d^2)^2.}$$

All the terms in the numerator are strictly positive under the assumption $b > 2d > 0$, $a > c > 0$, except for the last factor. Therefore, we analyze the sign of:

$$f(b,d) := 10b^3 - 24b^2d + 21bd^2 - 6d^3.$$

Evaluating this function at $b = 2d$:

$$f(2d,d) = 10(2d)^3 - 24(2d)^2d + 21(2d)d^2 - 6d^3 = 80d^3 - 96d^3 + 42d^3 - 6d^3 = 20d^3 > 0.$$

Next, we compute the derivative:

$$\frac{\partial f}{\partial b} = 30b^2 - 48bd + 21d^2,$$

solving $\frac{\partial f}{\partial b} = 0$, we obtain complex roots:

$$b = \frac{1}{10}(8d \pm i\sqrt{6}d).$$

Since the roots are not real, the derivative does not vanish on $\mathbb{R}$. We evaluate it at $b = 2d$:

$$\left.\frac{\partial f}{\partial b}\right|_{b=2d} = 30(2d)^2 - 48(2d)d + 21d^2 = 120d^2 - 96d^2 + 21d^2 = 45d^2 > 0.$$

This shows that $f(b,d)$ is strictly increasing in $b$ on the real line. Since it is positive at $b = 2d$ and increasing, we conclude that:

$$f(b,d) > 0 \quad \text{for all } b > 2d,$$

consequently, the numerator of $\delta(b)$ is strictly positive in that interval, and we conclude that:

$$\delta(b) > 0 \quad \text{for all } b > 2d.$$

Therefore, it holds that:

$$\frac{15b(a-c)^2}{2(4b-3d)^2} > \frac{(a-c)^2(15b^2-20bd+12d^2)}{2(4b^2-5bd+2d^2)^2} \quad \text{whenever } b > 2d.$$

- $W^{S,CI} > W^{S,IC}$:

We want to prove that, under the conditions $a > c > 0$ and $b > 2d > 0$, the following inequality holds:

$$W^{S,CI} = \frac{b(a-c)^2(15b^2-20bd+12d^2)}{2(4b^2-5bd+2d^2)^2} > \frac{b(a-c)^2(15b^2-20bd+8d^2)}{2(4b^2-5bd+2d^2)^2} = W^{S,IC}.$$

To verify this, we subtract the right-hand side from the left-hand side and define the resulting expression as:

$$\Delta := \frac{b(a-c)^2\left[(15b^2-20bd+12d^2)-(15b^2-20bd+8d^2)\right]}{2(4b^2-5bd+2d^2)^2}.$$

Simplifying the numerator, we get:

$$(15b^2-20bd+12d^2)-(15b^2-20bd+8d^2)=4d^2,$$

so the expression becomes:

$$\Delta = \frac{2b(a-c)^2d^2}{(4b^2-5bd+2d^2)^2}.$$

Under the given assumptions, all terms in the numerator are strictly positive: $b>0$, $a-c>0$, and $d>0$ imply $(a-c)^2>0$, $d^2>0$, and $b>0$. The denominator is a square, hence strictly positive for all $b,d>0$. Thus, we conclude that $\Delta>0$, which proves that:

$$\frac{b(a-c)^2(15b^2-20bd+12d^2)}{2(4b^2-5bd+2d^2)^2} > \frac{b(a-c)^2(15b^2-20bd+8d^2)}{2(4b^2-5bd+2d^2)^2} \quad \text{for all } b>2d.$$

- $W^{S,IC} > W^{S,II}$:

We aim to prove that, under the conditions $a>c>0$ and $b>2d>0$, the following inequality holds:

$$W^{S,IC} = \frac{(b(a-c)^2)(15b^2-20bd+8d^2)}{2(4b^2-5bd+2d^2)^2} > \frac{(b(a-c)^2)(15b^2-40bd+28d^2)}{2(4b^2-7bd+2d^2)^2} = W^{S,II}.$$

To study this inequality, we define the difference between the right-hand side and the left-hand side as follows:

$$\Delta(b) := \frac{(b(a-c)^2)(15b^2-20bd+8d^2)}{2(4b^2-5bd+2d^2)^2} - \frac{(b(a-c)^2)(15b^2-40bd+28d^2)}{2(4b^2-7bd+2d^2)^2},$$

since $b(a-c)^2 > 0$, which holds under the stated assumptions, we can factor it out. Thus, the inequality reduces to showing:

$$\delta(b) := \frac{15b^2-20bd+8d^2}{(4b^2-5bd+2d^2)^2} - \frac{15b^2-40bd+28d^2}{(4b^2-7bd+2d^2)^2} > 0.$$

Combining both terms into a single fraction yields:

$$\delta(b) = \frac{N(b,d)}{(4b^2-5bd+2d^2)^2(4b^2-7bd+2d^2)^2}.$$

where the numerator is given by:

$$N(b,d) = 20b^4 - 90b^3d + 133b^2d^2 - 84bd^3 + 20d^4.$$

We aim to prove that $N(b,d) > 0$ for all $b > 2d$. We begin by solving the equation $N(b,d) = 0$ in order to identify the threshold at which the sign of the expression might change. Among the real roots obtained, the largest positive solution is found to be approximately $b \approx 2.365d$. We then evaluate the function at this point and obtain:

$$N(2.365d,d) = 0.404302d^4\, d^4 > 0.$$

Next, we analyze the monotonicity of the function by studying its first derivative:

$$\frac{\partial N}{\partial b} = 80b^3 - 270b^2d + 266bd^2 - 84d^3.$$

The largest real root of this derivative is approximately $b^* \approx 1.94d$, which is strictly less than $2d$. Therefore, we evaluate the derivative at $b = 2d$:

$$\left.\frac{\partial N}{\partial b}\right|_{b=2d} = 8d^3 > 0,$$

this implies that $N(b,d)$ is strictly increasing for all $b > 2d$. Since we previously verified that $N(b,d) > 0$ at $b = 2.365d$, and $N$ is increasing beyond this point, it follows that:

$$\forall b > 2.365d, \quad N(b,d) > 0 \quad \Rightarrow \quad \delta(b) > 0 \quad \Rightarrow \quad \Delta(b) > 0.$$

Therefore, the original inequality holds:

$$\frac{(b(a-c)^2)(15b^2-20bd+8d^2)}{2(4b^2-5bd+2d^2)^2} > \frac{(b(a-c)^2)(15b^2-40bd+28d^2)}{2(4b^2-7bd+2d^2)^2} \quad \text{for all } b > 2.365d.$$

- $W^{S,CI} > W^{S,II}$:

  We aim to prove that, under the assumptions $a > c > 0$ and $b > 2d > 0$, the following inequality holds:

$$W^{S,CI} = \frac{b(a-c)^2(3b-2d)^2}{2(4b^2-5bd+2d^2)^2} > \frac{b(a-c)^2(3b-4d)^2}{2(4b^2-7bd+2d^2)^2} = W^{S,II}.$$

  To demonstrate this, we define the difference between both sides:

$$\Delta(b) := \frac{b(a-c)^2(3b-2d)^2}{2(4b^2-5bd+2d^2)^2} - \frac{b(a-c)^2(3b-4d)^2}{2(4b^2-7bd+2d^2)^2}.$$

  We factor out the strictly positive term $\frac{1}{2}b(a-c)^2$, and define the remaining expression as:

$$\delta(b) := \frac{(3b-2d)^2}{(4b^2-5bd+2d^2)^2} - \frac{(2b-d)^2}{(4b^2-7bd+2d^2)^2}.$$

This expression simplifies algebraically to:

$$\delta(b) = \frac{4P(b,d)}{(4b^2 - 5bd + 2d^2)^2(4b^2 - 7bd + 2d^2)^2},$$

where the numerator is:

$$P(b,d) = 12b^5 - 66b^4d + 137b^3d^2 - 135b^2d^3 + 64bd^4 - 12d^5.$$

We evaluate this polynomial at $b = 2d$, obtaining:

$$P(2d,d) = 0.$$

This implies that $\Delta(2d) = 0$. To determine the sign of $\Delta(b)$ for $b > 2d$, we analyze the derivative of $P(b,d)$:

$$\frac{\partial P}{\partial b} = 60b^4d - 264b^3d^2 + 411b^2d^3 - 270bd^4 + 64d^5.$$

Solving $\partial P/\partial b = 0$, we find four roots, and the largest real positive root is:

$$b_0 = \frac{11d}{10} + \frac{1}{20\sqrt{3}}\sqrt{82d^2 + \frac{645d^2}{(1625 + 4\sqrt{30871})^{1/3}} + 5(1625 + 4\sqrt{30871})^{1/3}d^2} + \dots,$$

which satisfies $b_0 < 2d$.

We then evaluate the derivative at $b = 2d$:

$$\left.\frac{\partial P}{\partial b}\right|_{b=2d} = 16d^5 > 0.$$

Since $P(2d,d) = 0$ and the derivative is strictly positive for $b = 2d$, we conclude that $P(b,d) > 0$ for all $b > 2d$, which implies $\delta(b) > 0$, and hence $\Delta(b) > 0$ for all $b > 2d$.

This confirms the validity of the inequality:

$$\frac{b(a-c)^2(3b-2d)^2}{2(4b^2-5bd+2d^2)^2} > \frac{b(a-c)^2(3b-4d)^2}{2(4b^2-7bd+2d^2)^2}.$$

- $W^{S,II} > W^{C,CI}$:

  We aim to prove that, under the conditions $a > c > 0$ and $b > 2d > 0$, the following inequality holds:

$$W^{S,II} = \frac{b(a-c)^2(15b^2-40bd+28d^2)}{2(4b^2-7bd+2d^2)^2} > \frac{b(a-c)^2(8b^2-8bd+3d^2)}{2(3b^2-3bd+d^2)^2} = W^{C,CI}.$$

Factoring out the positive term $(a-c)^2$, the inequality reduces to comparing the following rational functions:

$$\Delta(b) := \frac{15b^2-40bd+28d^2}{(4b^2-7bd+2d^2)^2} - \frac{8b^2-8bd+3d^2}{(3b^2-3bd+d^2)^2} > 0.$$

We define the difference of numerators as:

$$N(b,d) := (15b^2-40bd+28d^2)(3b^2-3bd+d^2)^2 - (8b^2-8bd+3d^2)(4b^2-7bd+2d^2)^2.$$

Expanding and simplifying using `Mathematica`, we obtain:

$$N(b,d) = 7b^6 - 54b^5d + 181b^4d^2 - 282b^3d^3 + 224b^2d^4 - 92bd^5 + 16d^6.$$

Substituting $b = 2d$, we find:

$$N(2d,d) = 88d^6 > 0.$$

To analyze the sign of $N(b,d)$ for $b > 2d$, we compute the derivative with respect to $b$:

$$\frac{\partial N}{\partial b} = 42b^5 - 270b^4 d + 724b^3 d^2 - 846b^2 d^3 + 448bd^4 - 92d^5.$$

Solving $\frac{\partial N}{\partial b} = 0$, the largest real root is approximately:

$$b \approx 0.84d,$$

which satisfies $b^* < 2d$. Evaluating the derivative at $b = 2d$, we obtain:

$$\left.\frac{\partial N}{\partial b}\right|_{b=2d} = 236d^5 > 0.$$

Hence, by the first derivative test, the function $N(b,d)$ is strictly increasing for all $b > 2d$. Since $N(2d,d) > 0$, it follows that:

$$N(b,d) > 0 \quad \text{for all } b > 2d,$$

which implies:

$$\Delta(b) > 0 \quad \text{for all } b > 2d.$$

Therefore, the original inequality is valid under the stated assumptions:

$$\frac{b(a-c)^2(15b^2 - 40bd + 28d^2)}{2(4b^2 - 7bd + 2d^2)^2} > \frac{b(a-c)^2(8b^2 - 8bd + 3d^2)}{2(3b^2 - 3bd + d^2)^2},$$

for all $b > 2d$, with $a > c > 0$ and $d > 0$.

- $W^{S,CI} > W^{C,CI}$:

We aim to show that, under the conditions $a > c > 0$ and $b > 2d > 0$, the following inequality holds:

$$W^{S,CI} = \frac{(b(a-c))^2(15b^2-20bd+8d^2)}{2(4b^2-5bd+2d^2)^2} > \frac{(b(a-c))^2(15b^2-40bd+28d^2)}{2(4b^2-7bd+2d^2)^2} = W^{C,CI}.$$

Factoring out the strictly positive term $\frac{1}{2}b(a-c)^2$, we reduce the inequality to the analysis of the following expression:

$$\Delta(b,d) := \frac{15b^2-20bd+8d^2}{(4b^2-5bd+2d^2)^2} - \frac{15b^2-40bd+28d^2}{(4b^2-7bd+2d^2)^2}.$$

Combining the terms over a common denominator and simplifying, we obtain:

$$\Delta(b,d) = \frac{7b^6-2b^5d-39b^4d^2+74b^3d^3-60b^2d^4+24bd^5-4d^6}{(3b^2-3bd+d^2)^2(4b^2-5bd+2d^2)^2}.$$

Let us denote the numerator as:

$$N(b,d) := 7b^6-2b^5d-39b^4d^2+74b^3d^3-60b^2d^4+24bd^5-4d^6.$$

Evaluating this expression at $b = 2d$, we obtain:

$$N(2d,d) = 156d^6 > 0.$$

To verify that $N(b,d) > 0$ for all $b > 2d$, we study the first derivative with respect to $b$, which yields:

$$\frac{\partial N}{\partial b} = 42b^5 - 10b^4d - 156b^3d^2 + 222b^2d^3 - 120bd^4 + 24d^5.$$

The real roots of this derivative include:

$$b = \frac{1}{28}\left(-3d - \sqrt[3]{\cdots} + \cdots\right) \approx 0.848d,$$

which is strictly smaller than $2d$. Thus, evaluating the derivative at $b = 2d$ gives:

$$\left.\frac{\partial N}{\partial b}\right|_{b=2d} = 608d^5 > 0.$$

This confirms that the function $N(b,d)$ is increasing for all $b > 2d$. Since we already showed that

$$N(2d,d) = 156d^6 > 0,$$

and that the function increases beyond this point, we conclude that

$$N(b,d) > 0 \quad \text{for all } b > 2d.$$

It follows that

$$\Delta(b,d) > 0 \quad \Rightarrow \quad \frac{(b(a-c))^2(15b^2-20bd+8d^2)}{(4b^2-5bd+2d^2)^2} > \frac{(b(a-c))^2(15b^2-40bd+28d^2)}{(4b^2-7bd+2d^2)^2},$$

which completes the proof of the desired inequality.

- $W^{C,CC} > W^{C,II}$:

We aim to prove that, under the conditions $a > c > 0$ and $b > 2d > 0$, the following inequality holds:

$$W^{C,CC} = \frac{2(a-c)^2(b+1)}{(3b-2d)^2} > \frac{2(a-c)^2(b+1)}{(3b-d)^2} = W^{C,II}.$$

Since both sides share the positive factor $2(a-c)^2(b+1) > 0$, the inequality reduces to comparing the reciprocals of the denominators. Specifically, we must show:

$$\frac{1}{(3b-2d)^2} > \frac{1}{(3b-d)^2},$$

which is equivalent to:

$$(3b-d)^2 > (3b-2d)^2.$$

We compute the difference:

$$(3b-d)^2-(3b-2d)^2 = (9b^2-6bd+d^2)-(9b^2-12bd+4d^2) = 6bd-3d^2.$$

Factoring out $3d$, we obtain:

$$6bd-3d^2 = 3d(2b-d)$$

Under the assumption $b > 2d > 0$, it follows that both $d > 0$ and $2b-d > 0$, we conclude:

$$\frac{2(a-c)^2(b+1)}{(3b-2d)^2} > \frac{2(a-c)^2(b+1)}{(3b-d)^2},$$

as required, for all values such that $a > c > 0$ and $b > 2d > 0$.

- $W^{C,CI} > W^{C,CC}$:

We aim to show that, under the conditions $a > c > 0$, $d > 0$, and $b > 2d$, the following inequality holds:

$$W^{C,CI} = \frac{b(a-c)^2\left(8b^2-8bd+3d^2\right)}{2(3b^2-3bd+d^2)^2} > \frac{2(a-c)^2(1+b)}{(3b-2d)^2} = W^{C,CC}\,.$$

This is equivalent to proving that:

$$\Delta(b,d) := -\frac{4(1+b)}{(3b-2d)^2} + \frac{b(8b^2-8bd+3d^2)}{(3b^2-3bd+d^2)^2} > 0\,.$$

Bringing both terms to a common denominator and simplifying algebraically, we obtain:

$$\Delta(b,d) = \frac{-36b^4+36b^5+72b^3d-96b^4d-60b^2d^2+95b^3d^2+24bd^3-44b^2d^3-4d^4+8bd^4}{(3b-2d)^2(3b^2-3bd+d^2)^2}\,.$$

We denote the numerator as the polynomial:

$$f(b,d) := -36b^4+36b^5+72b^3d-96b^4d-60b^2d^2+95b^3d^2+24bd^3-44b^2d^3-4d^4+8bd^4\,.$$

Evaluating the function at $b = 2d$, we find:

$$f(2d,d) = -196d^4+216d^5\,.$$

To determine the sign of this expression, we solve the inequality:

$$-196d^4+216d^5 > 0 \quad \Longleftrightarrow \quad d > \frac{49}{54}\,.$$

then

$$f(2d,d) > 0 \quad \Rightarrow \quad \Delta(b,d) > 0\,.$$

Finally, since $\Delta(b,d)$ is continuous and increasing in $b$ for $b > 2d$, and since the evaluation at $b = 2d$ is positive when $d > \frac{49}{54}$, we conclude that:

$$\Delta(b,d) > 0 \quad \text{for all } d > \frac{49}{54}.$$

- $W^{C,CI} > W^{C,II}$:

  We aim to demonstrate that, under the conditions $a > c > 0$, $d > 0$, and $b > 2d$, the following inequality holds:

$$\frac{b(a-c)^2\left(8b^2-8bd+3d^2\right)}{2(3b^2-3bd+d^2)^2} > \frac{2(a-c)^2(1+b)}{(3b-2d)^2}.$$

  This is equivalent to showing that:

$$\Delta(b,d) := \frac{4(1+b)}{(3b-2d)^2} + \frac{b(8b^2-8bd+3d^2)}{(3b^2-3bd+d^2)^2} > 0.$$

  By bringing both terms to a common denominator and simplifying algebraically, we obtain:

$$\Delta(b,d) = \frac{-36b^4+36b^5+72b^3d-96b^4d-60b^2d^2+95b^3d^2+24bd^3-44b^2d^3-4d^4+8bd^4}{(3b-2d)^2(3b^2-3bd+d^2)^2}.$$

  Defining the numerator as a polynomial in $b$, we write:

$$f(b,d) := -36b^4+36b^5+72b^3d-96b^4d-60b^2d^2+95b^3d^2+24bd^3-44b^2d^3-4d^4+8bd^4.$$

  Evaluating the function at $b = 2d$, we find:

$$f(2d,d) = -196d^4+216d^5.$$

This expression is strictly positive if and only if:

$$-196d^4+216d^5>0 \quad\Longleftrightarrow\quad d>\frac{49}{54}.$$

Since the condition $b>2d$ implies $d<\frac{b}{2}$, it follows that if $b>\frac{98}{54}=\frac{49}{27}$, then the condition $d>\frac{49}{54}$ is compatible. Therefore,

$$f(2d,d)>0 \quad\Rightarrow\quad \Delta(b,d)>0\,.$$

Moreover, since $\Delta(b,d)$ is continuous and increasing in $b$ for $b>2d$, and its value at $b=2d$ is strictly positive when $b>\frac{49}{27}$, we conclude that:

$$\Delta(b,d)>0 \quad \text{for all } b>\frac{49}{27}.$$

We now consider the related inequality:

$$\frac{(b(a-c))^2(8b^2-8bd+3d^2)}{2(3b^2-3bd+d^2)^2}>\frac{2(a-c)^2(b+1)}{(3b-d)^2}.$$

All quadratic expressions in the numerator and denominator are strictly positive for $b>d>0$, and $(a-c)^2>0$. Hence, it suffices to show that the following rational expression is strictly positive:

$$f(b,d)=\frac{-36b^4+36b^5+72b^3d-48b^4d-60b^2d^2+23b^3d^2+24bd^3-2b^2d^3-4d^4-bd^4}{(3b-d)^2(3b^2-3bd+d^2)^2}.$$

Let us denote the numerator as:

$$N(b,d)=-36b^4+36b^5+72b^3d-48b^4d-60b^2d^2+23b^3d^2+24bd^3-2b^2d^3-4d^4-bd^4\,.$$

Evaluating at $b = 2d$, we obtain:

$$N(2d,d) = -196d^4 + 558d^5 .$$

This expression is strictly positive if and only if:

$$-196d^4 + 558d^5 > 0 \quad \Longleftrightarrow \quad d > \frac{98}{279} .$$

To ensure that $N(b,d) > 0$ for all $b > 2d$, we apply the first-derivative test. We compute the derivative with respect to $b$:

$$\frac{\partial N}{\partial b} = -144b^3 + 180b^4 + 216b^2 d - 192b^3 d - 120bd^2 + 69b^2 d^2 + 24d^3 - 4bd^3 - d^4 .$$

Evaluating at $b = 2d$, and noting that the largest real root of the derivative is strictly less than $2d$, we find:

$$\frac{\partial N}{\partial b}(2d,d) = 9d^3(-56 + 179d) .$$

This is positive if and only if:

$$-56 + 179d > 0 \quad \Longleftrightarrow \quad d > \frac{56}{179} .$$

We observe that $\frac{56}{179} \approx 0.312$ and $\frac{98}{279} \approx 0.351$, so the condition $d > \frac{98}{279}$ guarantees that both inequalities are satisfied simultaneously.

Hence, under the condition $d > \frac{98}{279}$, we have:

- $N(2d,d) > 0$ ;
- $\frac{\partial N}{\partial b}(2d,d) > 0$, which implies that $N(b,d)$ is strictly increasing for $b > 2d$ .

Therefore, $N(b,d) > 0$ for all $b > 2d$, which implies that the original inequality holds for all $b > 2d$, provided that $d > \frac{98}{279}$.

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