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Summary

In this dissertation paper I develop two models where I analyze how trade liberalization affects a country's environmental policies and social welfare.

My contribution to this literature is to explore the effects of asymmetrical information and asymmetrical market competition structure. In the first model, I use asymmetric information in a Cournot market structure, and in the second I use perfect information in a Stackelberg model.

In the first model, I assume that government cannot distinguish if a local firm is high or low efficient. Hence, it is optimal for government to provide an environmental policy contract menu that is composed of both a marginal and a lump sum tax in order to attain a separating equilibrium.

I show that there are two effects: the ``rent-shifting effect" and the ``separating effect". In the case of the high efficiency type, both effects under-internalize pollution externality. In the case of the low efficient firm type, the separating effect causes government to over-internalize pollution externality. Therefore, if the ``separating effect" is stronger than the ``rent-shifting effect", the environmental policy for low efficiency type firm may over-internalize pollution externality. I conclude that government does not always have an incentive to under-internalize its pollution externality.

In the second model, I develop a complete information game in a Stackelberg market structure. I analyze the strategic behavior of environmental policy under conditions of trade liberalization. I also compare the Stackelberg model results with the Autarky case and the Cournot competition model.

I find that government in the leader firm country has incentive to reduce pollution externality as its dominant effect. The dominant effect in the follower firm country, by contrast, gives an incentive to increase firm competitiveness and gain greater market share.

Additionally, the environmental policy of the leader firm country in the Stackelberg model is the highest compared with the policy of the follower firm country, but it is also higher compared to the Autarky case and in the Cournot model. The environmental policy of the

country with a firm that competes à la Cournot is the second highest compared to other market structures. The lowest environmental policies are found in the Autarky case and the follower firm country.

In addition, I analyze the behavior of the social welfare on the countries. I compare the effect of reducing commerce tariffs on social welfare in countries with different market structures in order to determine whether or not some countries have greater incentives to enter into trade agreements.

When there are low or medium pollution abatement efficiency levels, I find that the social welfare of the leader firm country increases less than in the case of the follower firm country under a Stackelberg model with trade liberalization. This result suggests that the follower firm country has greater incentives for a trade agreement in a Stackelberg model.

Finally, when I compare all market structures, I conclude that the countries that gain most from subscribing to trade agreements are those in a Cournot model.

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Chapter 1

Introduction

In this dissertation paper I develop two models where I analyze how trade liberalization affects a country's environmental policies and social welfare.

Previous literature demonstrates that government has an incentive to under-internalize pollution externality in imperfect competitive international markets so that local firm is more competitive relative to foreign firm, when they compete à la Cournot.

My contribution to this literature is to explore the effects of asymmetrical information and asymmetrical market competition structure for previous findings on market equilibrium. In the first model, I use asymmetric information in a Cournot model, and in the second I use perfect information in a Stackelberg model.

In chapter 2, I develop a model of asymmetric information, under a Cournot competition structure. I assume that government cannot distinguish if a local firm is high or low efficient. Hence, it is optimal for government to provide an environmental policy contract menu that is composed of both a marginal and a lump sum tax in order to attain a separating equilibrium.

I show that there are two effects: the "rent-shifting effect" and the "separating effect". In the case of the high efficiency type, both effects under-internalize pollution externality. In the case of the low efficient firm type, the separating effect causes government to over-internalize pollution externality. Therefore, if the "separating effect" is stronger than the "rent-shifting effect", the environmental policy for low efficiency type firm may over-internalize pollution externality. I conclude that government does not always have an incentive to under-internalize its pollution externality.

In chapter 3, I develop a complete information model in which firms compete à la Stackelberg. In this chapter, I analyze the strategic behavior of environmental policy under conditions of trade liberalization. I also compare the Stackelberg model results with the Autarky case and the Cournot competition model.

I find that government in the leader firm country has incentive to reduce pollution externality as its dominant effect. The dominant effect in the follower firm country, by contrast, gives an incentive to increase firm competitiveness and gain greater market share.

Additionally, the environmental policy of the leader firm country in the Stackelberg model is the highest compared with the policy of the follower firm country, but it is also higher compared to the Autarky case and in the Cournot model. The environmental policy of the country with a firm that competes à la Cournot is the second highest compared to other market structures. The lowest environmental policies are found in the Autarky case and the follower firm country.

Finally in chapter 4, I analyze the behavior of the social welfare on the countries under the second model. I compare the effect of reducing commerce tariffs on social welfare in countries with different market structures in order to determine whether or not some countries have greater incentives to enter into trade agreements.

When there are low or medium pollution abatement efficiency levels, I find that the social welfare of the leader firm country increases less than in the case of the follower firm country under a Stackelberg model with trade liberalization. This result suggests that the follower firm country has greater incentives for a trade agreement in a Stackelberg model.

In addition, when comparing all market structures, the countries that gain most from subscribing to trade agreements are those in a Cournot model. The reasons for this is that under Cournot conditions, as commerce tariffs decrease, countries gain more market with environmental policy that is not as low as in the follower firm country. Moreover, the production of the firm under Cournot Competition is lower than that of the leader firm. Therefore, its environmental externality is lower than that of the leader firm country that competes à la Stackelberg.

Chapter 2

Environmental Policy with Asymmetric Information in a model of international trade with competition à la Cournot

2.1 Introduction

There has been much concern about the strategic use of environmental policies to achieve competitive advantage in international trade. Brander and Spencer [3] [4] develop an oligopoly model in which they establish the existence of "rent-shifting" incentives in strategic trade policy. Conrad [5] demonstrates that optimal environmental policy under-internalizes pollution externality when international oligopoly competition is in play.

Some authors, however, questions these results under conditions of asymmetric information. If government still under-internalizes pollution externality, in order to make its domestic firm more competitive, although it does not know which type of firm is in the country, then such asymmetric information could decrease government commitment to supporting these domestic firm.

Assuming that free trade exists and the government cannot discern if its local firm is high or low efficient. I develop a screening model for government and domestic firm.

Environmental policy has two components, marginal and lump sum taxes, which seek to ensure the existence of a separating equilibrium. Marginal and lump sum taxes are different for firm with different type of efficiency. I also take into account consumer surplus in welfare function, in order to analyze government incentives when it chooses its environmental policy.

Under conditions of complete information, the result is that government has incentives to under-internalize pollution externality for firm of any level of efficiency, thus choosing an environmental policy weaker than in the Autarky case with complete information. Although this model differs from Conrad's model [5], it is consistent with it under conditions of complete information.

However, under conditions of asymmetric information, I find two effects: the "rent-shifting effect" and the "separating effect". In the case of high efficiency type firm, both effects under-internalize the pollution externality. In the case of low efficiency type firm, the separating effect causes government to over-internalize pollution externality. Therefore, if the "separating effect" is stronger than the "rent-shifting effect", the environmental policy for the low efficiency firm type may over-internalize pollution externality. The conclusion is that government may not always have an incentive to under-internalize its pollution externality.

Finally, this chapter helps to determine which incentives can help countries fulfill obligations to environmental agreements, such as the Kyoto protocol, under conditions of asymmetrical information. It could also explain why some countries under-internalize its environmental externality. As the incentive to have more efficient firm gains market share, "rent-shifting effect", is stronger than the "separating effect" that give incentives to distinguish between types of firm efficiency. This, therefore, it could explain US's decision to not ratify the Kyoto protocol, as it would harm its national firms given that the US could not under-internalized its environmental externality.

This model is closely related to Nannerup [9] who analyzes a unilateral commerce model under asymmetric information. The government establishes environmental policy with two components: an environmental standard and a lump sum tax. However, as Brainard and Martimort [1], he adds the social opportunity cost of subsidizing local firm. Nannerup [9] finds that, for all efficiency levels, the optimal environmental policy is stronger under conditions of asymmetric information than complete information.

Other literature examines situations where the government establishes an optimal commerce tariff. Qiu [10] develops a model where asymmetric information exists between government and domestic firm, but also between domestic and foreign firm. He finds that, in cases when firms are competing à la Cournot, there is separating equilibrium, but if they are competing à la Bertrand, there is pooling equilibrium. Creane [6] also finds that local firm has incentives to disclose their type if they compete à la Cournot, but have no such incentives if they compete à la Bertrand.

Brainard and Martimort [2] find that asymmetric information has the effect of diminishing government commitment to subsidize its local firm. But when asymmetric information occurs in a rival interventionist government, the commitment of local government is strong despite the asymmetric information. In contrast, Maggi [7] finds when capacity constraints exist, government always adopts the optimal policy of subsidizing local firm capacity. Therefore, informational constraints do not diminish government incentives to distort international competition.

In a screening model, Maggi [8] finds that asymmetric information increases government incentive to subsidize domestic firm. Additionally, he finds that with asymmetric information, the available equilibriums are strictly worse than with complete information. Brainard and Martimort [1] include the social opportunity cost of using funds in local firm instead of in other social programs. They discover that it is possible to obtain a Brander and Spencer equilibrium [4] with lump sum taxes.

The current chapter is organized as follows: in the following Section 2, the model is introduced. In Section 3, optimal policies are derived for Autarky case under conditions of complete and asymmetric information. In Section 4, optimal policies are derived for international oligopoly competition. Finally, conclusions are drawn in Section 5.

2.2 The Model

This model is based on Brander and Spencer [3]. I assume the existence of two countries: Home (1) and Foreign (2). There are two tradable goods. The first one is produced by two firms; Firm (1) is in Home and Firm (2) is in Foreign. Firm (1) only produces and sells in Home and Firm (2) exports to Home, but it only produces in Foreign country. Firms compete in quantities in Home's market. The price for the good is given by the inverse demand function, $P(Q)$.

The second good is a numeraire m and is produced in a perfect competitive market. Benefits of Firm (1) are used to consume numeraire good.

The consumer utility function has a quasi-lineal form $U(A) = u(A) + m$, where u(.) is a Bernoulli utility function for the first good. This utility function has been chosen because it does not have an income effect and it has no difficulties to be aggregated.

Firms (1) and (2) have a constant marginal cost. The marginal cost of Firm (1) is affected by environmental regulation. Firm (1) has an efficiency level, θ, that modifies the marginal cost and also the implementation of the environmental regulation. It can be a high efficiency type firm, θ_H, or low efficiency type firm, θ_L. Environmental externality is only local.

Government of Home chooses its environmental policy in order to maximize social welfare of Home. The environmental policy has two components, a marginal and a lump sum tax (t, T). There is a trade agreement in effect that does not allow commerce tariffs between the two countries.

Government cannot observe if the Home firm is low or high efficiency type. It offers two different environmental contracts: one for low type (t_L, T_L) and another for high type (t_H, T_H). The implementation of marginal environmental policy is affected by the efficiency level of Firm (1) $\tau(\theta)$ and it can be rewritten as $\tau(\theta) = t * \tau_t(\theta)$, where $\tau_t(\theta)$ is the derivative of $\tau(\theta)$ with respect to t. I am assuming that $\tau_t(\theta) > 0$, $\tau_{tt}(\theta) = 0$ and $\tau_{t\theta}(\theta) < 0$. Instead, because Firm (2) is exporting, it needs to give more information to the governments so that its efficiency level is observable.[1]

The model is a three stage game. In the first stage, Home government chooses its environmental policy and offers two different contracts, $(t_L, T_L), (t_H, T_H)$. In the second stage, after observing the environment contracts, Firm (1) chooses its contract. In the third stage, after observing the environmental policies contract that is chosen, Firm (1) and Firm (2) choose their output for Home market y and x, respectively. Firm (1) output depends on the firm type, there is high type firm output, y_H, and low type firm output, y_L. Let the total output for Home market be $A_\theta = x + y_\theta$. Notice that the output of firm 2 to Home Market are exports.

We define welfare as total social surplus minus environment externality. Environmental externality $h(y)$ depends on the local output production. I assume that $h_y > 0$ and $h_{yy}(y) > 0$.

Therefore, the Expected Welfare function is:

$$E(W) = q[u(A_L) - pA_L + py_L - (\tfrac{c}{\theta_L} + \tau_L(\theta_L))y_L - T_L - h(y_L)] + (1-q)[u(A_H) - pA_H + py_H - (\tfrac{c}{\theta_H} + \tau_H(\theta_H))y_H - T_H - h(y_H)]$$

where q is the probability of Firm (1) to be the low type and A_L and A_H are the third stage equilibrium output. The terms on the right-hand side are, respectively, consumer surplus, benefits of local firm, and environmental externality.

The function will be solved with a Subgame Perfect Equilibrium, so it will be calculated with backward induction. This problem has a unique separating equilibrium.[2]

[1] Since Home government cannot impose a separating environmental policy to Foreign firm, in order to know its type, because pollution is local, then Home government can only do an estimation of Foreign efficient type. The estimation could be the average efficiency level of Foreign countries, and that will be the type that home government and local firm will assume to Foreign firm, to maximize the welfare and to choose the optimal contract, respectively. In order to simplify the model, I assume that the Foreign efficiency level is observable.

[2] Go to Appendix to see the proof of Single Crossing Property.

Third stage: given the chosen environmental contract (t_θ, T_θ), Firm (1) and Firm (2) maximize their benefits by choosing its output for Home market y and x, respectively.

Second stage: given the environmental contracts menu offered by Government (t_θ, T_θ), Firm (1) chooses the contract that maximizes its benefits. Notice that the marginal cost of Firm (1) is the marginal cost of producing plus the marginal environmental policy $\frac{c}{\theta} + \tau(\theta)$, where θ is the efficiency level.

First stage: Home government maximizes its national Welfare by choosing the environmental policy contracts menu (t_θ, T_θ).

2.3 The Autarky case

In this section, I analyze the optimal environmental policies that the government chooses in order to maximize the social welfare, when there is no trade. First I present the case with complete information and later the case with asymmetric information.

2.3.1 Complete Information case

In the Autarky case with complete information the model is a two stage game. In the first stage, local government chooses its environmental policy with complete information, t_θ, in order to maximize its social welfare, which depends on the efficiency level of the local firm. In the second stage, after observing the environmental policy, the local monopoly chooses its output for Home market, y_θ.

In this case, the Welfare function is the following:

$$W^{AC} = u(y_\theta) - py_\theta + py_\theta - (\tfrac{c}{\theta} + \tau(\theta))y_\theta - T_\theta - h(y_\theta)$$

Where y_θ is the second stage equilibrium output.

It will be solved with a Subgame Perfect Equilibrium, so it will be calculated with backward induction.

Second stage: given the environmental policy t_θ, local monopoly maximizes its benefits by choosing its output for Home market y_θ.

First stage: Home government maximizes its national Welfare by choosing the environmental policy t_θ.

Proposition 1: The optimum environmental policy is equal to the environmental externality.

From First Order Condition of Government problem we obtain:

$t_\theta = \frac{1}{\tau_t(\theta)} \frac{\epsilon_\theta}{\epsilon_\theta + 1}(-h_y + p - \frac{c}{\theta})$

In the case of perfect competition, when price equals marginal cost, the marginal environmental policy, t_θ, would be equal to marginal externality, taking into account the production elasticity and the efficiency. However, because the firm is a monopoly, it is necessary to subtract the marginal benefit, because the firm produces less than the point where the marginal income is equal to zero, so the marginal externality is lower.

From the monopoly problem, we know that in equilibrium we have the following:

$p = \frac{c}{\theta} + t(\theta) - p_y y_\theta$

Substituting the equilibrium price, the environmental policy is as following:

$t_\theta = \frac{\epsilon_\theta}{\tau_t(\theta)}(-h_y - p_y y_\theta)$ ■

The optimum marginal environmental tax is equal to the marginal environmental externality minus the marginal benefit, taking into account the production elasticity and the efficiency level. This environmental tax internalizes the total pollution externality.

I am assuming that the externality of high type firm is higher than the externality of low type firm, taking account the supply elasticity. This means that $t_L \tau_t(\theta_L) < t_H \tau_t(\theta_H)$, so $\epsilon_L(-h_y - p_y y_L) < \epsilon_H(-h_y - p_y y_H)$. This is true if $| - h_y - p_y y_L| << | - h_y - p_y y_H|$, which is possible because function h is convex and even if $|\epsilon_H| \leq |\epsilon_L|$, I am assuming that the inequality holds.

I divide the optimum unique marginal environmental policy into a marginal tax and a lump sum tax.

A marginal tax will be charged until y^B, where $y^B \leq y^*_\theta$ and y^*_θ is the optimum production output for local firm with efficient level θ. If the firm produces more than y_B, then government charges the lump sum tax.

$t^{AC} = \frac{\epsilon_\theta}{\tau_t(\theta)}(-h_y - p_y y_\theta)$, until y^B

$T^{AC} = y^D_\theta \epsilon_\theta(-h_y - p_y y_\theta)$, where $y^D_\theta = y^*_\theta - y^B$

Note that y^B is the same for any efficient level, but $y^D_L < y^D_H$, since $y^*_L < y^*_H$.

2.3.2 Asymmetric Information case

In the Autarky case with asymmetric information the model is a three stage game. In the first stage, local government chooses its menu of environmental policy contracts and offers two different, $(t_L, T_L), (t_H, T_H)$. In the second stage, after observing the environment contracts, the local monopoly chooses its contract. In the third stage, after chose its contract, local monopoly chooses its output for Home market.

We define expected social welfare as expected total social surplus minus expected environment externality.

In this case, the expected Welfare function is the following:

$E(W^{AI}) = q[u(y_L) - py_L + py_L - (\frac{c}{\theta_L} + \tau_L(\theta_L))y_L - T_L - h(y_L)] + (1-q)[u(y_H) - py_H + py_H - (\frac{c}{\theta_H} + \tau_H(\theta_H))y_H - T_H - h(y_H)]$

Where y_L and y_H are the third stage equilibrium output.

It will be solved with a Subgame Perfect Equilibrium, so it will be calculated with backward induction.

Third stage: given the chosen environmental contract (t_θ, T_θ), local monopoly maximizes its benefits by choosing its output for Home market y_θ.

Second stage: given the environmental contracts menu offered by government (t_θ, T_θ), local monopoly chooses the contract that maximizes its benefits. Notice that the marginal cost of local firm is the marginal cost of producing plus the marginal environmental policy $\frac{c}{\theta} + \tau(\theta)$, where θ is the efficiency level.

First stage: Home government maximizes its national Welfare by choosing the environmental policy contracts menu (t_θ, T_θ), in order to differentiate the firm types.

Government cannot use the first best contract, the one that uses with complete information, because the high type firm has incentives to deviate.[3]

It is necessary that the individual rationality (IR) and the incentive constraint (IC) hold with asymmetric information, in order to achieve equilibrium.

$IR_H : 0 \leq py_H - (\frac{c}{\theta_H} + \tau_H(\theta_H))y_H - T_H$

$IR_L : 0 \leq py_L - (\frac{c}{\theta_L} + \tau_L(\theta_L))y_L - T_L$

$IC_H : py_H - (\frac{c}{\theta_H} + \tau_L(\theta_H))y_H - T_L \leq py_H - (\frac{c}{\theta_H} + \tau_H(\theta_H))y_H - T_H$

$IC_L : py_L - (\frac{c}{\theta_L} + \tau_H(\theta_L))y_L - T_H \leq py_L - (\frac{c}{\theta_L} + \tau_L(\theta_L))y_L - T_L$

Because the high type firm is the one that has incentives to deviate, the IC of high type is used to calculate the marginal tax for low type firm.

$py_H - \frac{c}{\theta_H} * y_H - t_L\tau_t(\theta_H)y^B - T_L = py_H - \frac{c}{\theta_H}y_H - t_H\tau_t(\theta_H)y^B - T_H$

Therefore, the condition for marginal tax for low type, in order to incentive constraint of high type holds, is the following:

$t_L^{AI} = t_H^{AI} + \frac{T_H^{AI} - T_L^{AI}}{\tau_t(\theta_H)y^B}$

[3] Go to Appendix to see the proof.

Taking into account the incentives constraint of high, the expected Welfare that government needs to maximize with asymmetric information is the following:

$E(W^{AI}) = q[u(y_L) - \frac{c}{\theta_L}y_L - t_H\tau_t(\theta_L)y_L - \frac{T_H - T_L}{\tau_t(\theta_H)y^B}\tau_t(\theta_L)y_L - T_L - h(y_L)] + (1-q)[u(y_H) - \frac{c}{\theta_H}y_H - t_H\tau_t(\theta_H)y_H - T_H - h(y_H)]$

Proposition 2: The optimum environmental policy for low type over-internalizes the expected environmental externality, in order to distinguish the firm types.

From the First Order Condition of Government problem we obtain:

$t_H = [qy_{LtH}(p_L - \frac{c}{\theta_L} - h_y) + (1-q)y_{HtH}(p_H - \frac{c}{\theta_H} - h_y)]/[q\tau_t(\theta_L)y_{LtH}(1 + \frac{1}{\epsilon_L}) + (1-q)\tau_t(\theta_H)y_{HtH}(1 + \frac{1}{\epsilon_H})]$

From the monopoly problem, we know that in equilibrium we have the following:

$p_H = \frac{c}{\theta_H} + t_H\tau_t(\theta_H) - p_y y_H$

$p_L = \frac{c}{\theta_L} + t_L\tau_t(\theta_L) - p_y y_L$

Substituting the equilibrium price, the environmental policy is the following:

$t_H = [qy_{LtH}(-p_y y_L - h_y) + (1-q)y_{HtH}(-p_y y_H - h_y)]/[qy_{LtH}\tau_t(\theta_L)\frac{1}{\epsilon_L} + (1-q)y_{HtH}\tau_t(\theta_H)\frac{1}{\epsilon_H}]$

We can rewrite it as:

$t_H = \frac{E}{t_t(\theta_H)}$

where $E = [qy_{LtH}(-p_y y_L - h_y) + (1-q)y_{HtH}(-p_y y_H - h_y)]/[qy_{LtH}\frac{\tau_t(\theta_L)}{\tau_t(\theta_H)}\frac{1}{\epsilon_L} + (1-q)y_{HtH}\frac{1}{\epsilon_H}]$

Then,

$t_L = \frac{E}{t_t(\theta_H)} + \frac{T_H - T_L}{\tau_t(\theta_H)y^B}$

Note that when the probability of being in low type is zero (q= 0), then the environmental policy in asymmetric information of high type firm is equal to the environmental policy of complete information.

I am assuming that the externality of high type firm is higher than the externality of low type firm, taking into account how the production changes when environmental policy changes. This means that $y_{LtH}(-p_y y_L - h_y) < y_{HtH}(-p_y y_H - h_y)$. This is true if $|-h_y - p_y y_L| << |-h_y - p_y y_H|$, which is possible because function h is convex and even if $|y_{HtH}| \leq |y_{LtH}|$, I am assuming that the inequality holds.

Also I am assuming that $y_{HtH}\frac{1}{\epsilon_H} \leq y_{LtH}\frac{\tau_t(\theta_L)}{\tau_t(\theta_H)}\frac{1}{\epsilon_L}$, which it is possible because $\frac{\tau_t(\theta_L)}{\tau_t(\theta_H)} > 1$ and $|y_{HtH}| < |y_{LtH}|$, and even if $|\frac{1}{\epsilon_L}| \leq |\frac{1}{\epsilon_H}|$, I am assuming that the inequality holds. Therefore, when a positive probability exists of being in low type, the environmental policy of high type in asymmetric information is lower than in complete information. This is the “separating” effect in high type.

In contrast, when the probability of being in low type is one (q=1), then the environmental policy in asymmetric information of high type firm is equal to the environmental policy of complete information of low type. However, the environmental policy of low type in asymmetric information will be always higher than the environmental policy in complete information, in order to sustain the IC of high, allowing the separating equilibrium. This is the “separating” effect in low type.

Therefore, the “separating” effect in the Autarky case with asymmetric information for low type over-internalizes its externality, in order to hold the IC of high. However, the “separating” effect for high type under-internalizes its externality, in order that the cost of the incentive constraint of high be lower.

I divide the optimum marginal environmental policy in a marginal tax and a lump sum tax. We need to have these two components, in order to ensure the existence of the separating equilibrium.

It will be charge a marginal tax until y^B, where $y^B < y^*_\theta$ and y^*_θ is the optimum production output for local firm with efficient level θ. If the firm produces more than y_B, then government charges the lump sum tax.

$t_H^{AI} = \frac{E}{\tau_t(\theta_H)}$, until y^B

$T_H^{AI} = Ey_H^D$, where $y_H^D = y_H^* - y^B$

$t_L^{AI} = \frac{E}{\tau_t(\theta_H)} + \frac{T_H - T_L}{\tau_t(\theta_H)y^B}$, until y^B

$T_L^{AI} = E\frac{\tau_t(\theta_L)}{\tau_t(\theta_H)}y_L^D + \frac{T_H - T_L}{\tau_t(\theta_H)y^B}\tau_t(\theta_L)y_L^D$, where $y_L^D = y_L^* - y^B$

Calculating T_L^{AI}, we get the following:

$T_L^{AI} = Ey^B\frac{\tau_t(\theta_L)y_L^D}{\tau_t(\theta_H)y^B + \tau_t(\theta_L)y_L^D} + Ey_H^D\frac{\tau_t(\theta_L)y_L^D}{\tau_t(\theta_H)y^B + \tau_t(\theta_L)y_L^D}$

Note that y^B is the same for any efficient level, but $y_L^D < y_H^D$, since $y_L^* < y_H^*$.

In order to prove that this menu of contracts is equilibrium, I need to demonstrate that the four inequalities hold.

The incentives constrain for high type holds by the way that the marginal tax for low type firm was constructed.

The individual rationality of low type firm condition is:

$\pi_L^{AI} \geq 0$

$py_L - \frac{c}{\theta_L}y_L - t_L^{AI}t_t(\theta_L)y^B - T_L^{AI} \geq 0$

Then the necessary and sufficient condition for IR of low type firm to hold is:

$py_L - \frac{c}{\theta_L}y_L \geq Ey_H[\frac{\tau_t(\theta_L)}{\tau_t(\theta_H)} - \frac{\tau_t(\theta_L)y_L^D}{\tau_t(\theta_H)y^B + \tau_t(\theta_L)y_L^D}(\frac{\tau_t(\theta_L)}{\tau_t(\theta_H)} - 1)]$

We impose this condition. Note from complete information case that $py_L - \frac{c}{\theta_L}y_L - y^B\epsilon_L(-h_y - p_y y_L) - y_L^D\epsilon_L(-h_y - p_y y_L) > 0$.

The individual rationality of high type holds, as a consequence of individual rationality of low type and the incentives constraint of high type hold. Because we can rewrite these conditions as follows:

$\pi_H^{AI} \geq \pi_H^D = py_H - \frac{c}{\theta_H}y_H - t_H^{AI}\tau_t(\theta_H)y^B - \frac{T_H^{AI} - T_L^{AI}}{\tau_t(\theta_H)y^B}\tau_t(\theta_H)y^B - T_L^{AI}$

$> py_H - \frac{c}{\theta_H} y_H - t_H^{AI} \tau_t(\theta_L) y^B - \frac{T_H^{AI} - T_L^{AI}}{\tau_t(\theta_H) y^B} \tau_t(\theta_L) y^B - T_L^{AI}$

$> py_L - \frac{c}{\theta_L} y_L - t_H^{AI} \tau_t(\theta_L) y^B - \frac{T_H^{AI} - T_L^{AI}}{\tau_t(\theta_H) y^B} \tau_t(\theta_L) y^B - T_L^{AI} = \pi_L^{AI} \geq 0$

The incentive constraint of low type firm condition is:

$\pi_L^D \leq \pi_L^{AI}$

$py_L - \frac{c}{\theta_L} y_L - t_H^{AI} \tau_t(\theta_L) y_B - T_H^{AI} \leq py_L - \frac{c}{\theta_L} y_L - t_H^{AI} \tau_t(\theta_L) y_B - T_H^{AI} \frac{\tau_t(\theta_L)}{\tau_t(\theta_H)} + T_L^{AI} (\frac{\tau_t(\theta_L)}{\tau_t(\theta_H)} - 1)$

Then the necessary and sufficient condition for IC of low type firm to hold is:

$\tau_t(\theta_H) y_H^D \leq \tau_t(\theta_L) y_L^D$

We impose this condition. ■

2.4 International Oligopoly Competition

In this section, I analyze the optimal environmental policies that the government chooses in order to maximize the welfare, when the local firm competes à la Cournot with a foreign firm. First I present the case with complete information and later the case with asymmetric information.

2.4.1 Complete Information case

In the international oligopoly competition with complete information, the model is a two stage game. In the first stage, local government chooses its environmental policy with complete information, t_θ. In the second stage, after observing the environmental policy, Firm (1) and Firm (2) choose their output for Home market, y and x, respectively. Firm (1) output depends on the firm type, there is high type firm output, y_H, and low type firm output, y_L. Let the total output for Home market be $A_\theta = x + y_\theta$. Notice that the output of firm 2 to Home Market are exports.

In this case, the Welfare function is the following:

$W_{CC} = u(A_\theta) - pA_\theta + py_\theta - (\frac{c}{\theta} + \tau(\theta))y_\theta - T_\theta - h(y_\theta)$

Where A_θ is the second stage equilibrium output.

It will be solved with a Subgame Perfect Equilibrium, so it will be calculated with backward induction.

Second stage: given the environmental policy t_θ, firm (1) and firm (2) maximize their benefits by choosing their output for Home market, y and x respectively.

First stage: Home government maximizes its national Welfare by choosing the environmental policy t_θ.

Proposition 3: The optimum environmental policy under-internalizes the environmental externality, because of the "rent-shifting" effect.

From First Order Condition of Government problem we obtain:

$t = \frac{1}{\tau_t(\theta)} \frac{\epsilon_\theta}{\epsilon_\theta + 1}(-h_y + p - \frac{c}{\theta} - x_\theta p_A(1 + \frac{x_t}{y_t}))$

In this case, the environmental policy is equal to net marginal externality, minus the marginal benefit of foreign firm, which is lower than the environmental policy in the Autarky case with complete information. This means that the government has incentives to under-internalize the pollution externality, in order that the local firm gains more market.

From the Cournot problem, we know that in equilibrium we have the following:

$p = \frac{c}{\theta} + \tau(\theta) - p_A y_\theta(1 + x_y)$

Substituting the equilibrium price, the environmental policy is the following:

$t = \frac{\epsilon_\theta}{\tau_t(\theta)}(-h_y - p_A y_\theta(1 + x_y) - p_A x_\theta(1 + x_y))$ ■

The optimum marginal environmental policy is equal to the net marginal environmental externality minus the marginal benefit of Foreign firm, taking into account the production elasticity and the efficiency level. This environmental policy under-internalizes the pollution externality, in order to make more competitive the Home firm, and then it could gain more market against Foreign firm, this is the “rent-shifting” effect.

I am assuming that the externality of high type firm is higher than the externality of low type firm, taking into account the supply elasticity. This means that $t_L\tau_t(\theta_L) < t_H\tau_t(\theta_H)$, so $\epsilon_L(-h_y - p_A A_L(1+x_y)) < \epsilon_H(-h_y - p_A A_H(1+x_y))$. This is true if $|(-h_y - p_A A_L(1+x_y))| << |(-h_y - p_A A_H(1+x_y))|$, which is possible because function h is convex and even if $|\epsilon_H| \leq |\epsilon_L|$, I am assuming that the inequality holds.

I divide the optimum marginal environmental policy into a marginal tax and a lump sum tax.

A marginal tax will be charged until y^B, where $y^B \leq y^*_\theta$ and y^*_θ is the optimum production output for local firm with efficient level θ. If the firm produces more than y_B, then government charges the lump sum tax.

$t^{CC} = \frac{\epsilon_\theta}{\tau_t(\theta)}(-h_y - p_A A_\theta(1+x_y))$, until y^B

$T^{CC} = y^D_\theta \epsilon_\theta(-h_y - p_A A_\theta(1+x_y))$, where $y^D_\theta = y^*_\theta - y^B$

Note that y^B is the same for any efficiency level, but $y^D_L < y^D_H$, since $y^*_L < y^*_H$

2.4.2 Asymmetric Information case

In the international oligopoly competition with asymmetric information, the model is a three stage game. In the first stage, local government chooses its environmental policy contracts and offers two different, $(t_L, T_L), (t_H, T_H)$. In the second stage, after observing the environment contracts, Firm (1) chooses its contract. In the third stage, after observing the environmental policy contract that is chosen, Firm (1) and Firm (2) choose their output for Home market y and x, respectively. Let the total output for home market be $A = x + y$. Notice that the output of Firm (2) to Home market is exports.

In this case, the Welfare function is the following:

$E(W^{CI}) = q[u(A_L) - pA_L + py_L - (\frac{c}{\theta_L} + \tau_L(\theta_L))y_L - T_L - h(y_L)] + (1-q)[u(A_H) - pA_H + py_H - (\frac{c}{\theta_H} + \tau_H(\theta_H))y_H - T_H - h(y_H)]$

Where A_L and A_H are the third stage equilibrium output.

It will be solved with a Subgame Perfect Equilibrium, so it will be calculated with backward induction.

Third stage: given the chosen environmental contract (t_θ, T_θ), Firm (1) and Firm (2) maximize their benefits by choosing its output for Home market y and x, respectively.

Second stage: given the environmental contracts menu offered by government (t_θ, T_θ), Firm (1) chooses the contract that maximizes its benefits. Notice that the marginal cost of Firm (1) is the marginal cost of producing plus the marginal environmental policy $\frac{c}{\theta} + \tau(\theta)$, where θ is the efficiency level of the Home firm.

First stage: Home government maximizes its national Welfare by choosing the environmental policy contracts menu (t_θ, T_θ), in order to differentiate the firm types.

The government cannot use the first best contract, the one that uses with complete information, because the high type firm has incentives to deviate.[4]

It is necessary that the individual rationality (IR) and the incentive constraint (IC) hold with asymmetric information, in order to achieve equilibrium.

$IR_H : 0 \leq py_H - (\frac{c}{\theta_H} + \tau_H(\theta_H)) * y_H - T_H$

$IR_L : 0 \leq py_L - (\frac{c}{\theta_L} + \tau_L(\theta_L)) * y_L - T_L$

$IC_H : py_H - (\frac{c}{\theta_H} + \tau_L(\theta_H)) * y_H - T_L \leq py_H - (\frac{c}{\theta_H} + \tau_H(\theta_H)) * y_H - T_H$

$IC_L : py_L - (\frac{c}{\theta_L} + \tau_H(\theta_L)) * y_L - T_H \leq py_L - (\frac{c}{\theta_L} + \tau_L(\theta_L)) * y_L - T_L$

Because the high type firm is the one that has incentives to deviate, the IC of high is used to calculate the marginal tax for low type firm.

[4] Go to Appendix to see the proof.

$py_H - \frac{c}{\theta_H} * y_H - t_L\tau_t(\theta_H)y^B - T_L = py_H - \frac{c}{\theta_H}y_H - t_H\tau_t(\theta_H)y^B - T_H$

Therefore, the condition for marginal tax for low type firm, in order to incentive constraint of high type firm holds is the following:

$t_L^{CI} = t_H^{CI} + \frac{T_H^{CI} - T_L^{CI}}{\tau_t(\theta_H)y^B}$

Taking into account the incentives constraint of high type firm, the expected Welfare that government needs to maximize with asymmetric information is the following:

$E(W^{CI}) = q[u(A_L) - px_L - \frac{c}{\theta_L}y_L - t_H\tau_t(\theta_L)y_L - \frac{T_H - T_L}{\tau_t(\theta_H)y^B}\tau_t(\theta_L)y_L - T_L - h(y_L)] + (1-q)[u(A_H) - px_H - \frac{c}{\theta_H}y_H - t_H\tau_t(\theta_H)y_H - T_H - h(y_H)]$

Proposition 4: The optimum environmental policy for low type firm under-internalizes the environmental externality, because of the “rent-shifting” effect, but over-internalizes the environmental externality, because of the “separating” effect.

From the First Order Condition of Government problem, we obtain:

$t_H = [qy_{LtH}(p_L - \frac{c}{\theta_L} - h_y - p_A x_L(1 + x_y) - (T_H - T_L)\frac{\tau_t(\theta_L)}{\tau_t(\theta_H)y^B}) + (1-q)y_{HtH}(p_H - \frac{c}{\theta_H} - h_y - p_A x_H(1 + x_y))]/[q\tau_t(\theta_L)y_{LtH}(1 + \frac{1}{\epsilon_L}) + (1-q)\tau_t(\theta_H)y_{HtH}(1 + \frac{1}{\epsilon_H})]$

From Cournot problem, we know that in equilibrium we have the following:

$p_H = \frac{c}{\theta_H} + t_H\tau_t(\theta_H) - p_A y_H(1 + x_y)$

$p_L = \frac{c}{\theta_L} + t_L\tau_t(\theta_L) - p_A y_L(1 + x_y)$

Substituting the equilibrium price, the environmental policy is the following:

$t_H = [qy_{LtH}(-p_A A_L(1 + x_y) - h_y) + (1-q)y_{HtH}(-p_A A_H(1 + x_y) - h_y)]/[qy_{LtH}\tau_t(\theta_L)\frac{1}{\epsilon_L} + (1-q)y_{HtH}\tau_t(\theta_H)\frac{1}{\epsilon_H}]$

We can rewrite it as:

$t_H = \frac{F}{\tau_t(\theta_H)}$

where $F = [qy_{LtH}(-p_A A_L(1+x_y) - h_y) + (1-q)y_{HtH}(-p_A A_H(1+x_y) - h_y)]/[qy_{LtH}\frac{\tau_t(\theta_L)}{\tau_t(\theta_H)}\frac{1}{\epsilon_L} + (1-q)y_{HtH}\frac{1}{\epsilon_H}]$

Then,

$t_L = \frac{F}{\tau_t(\theta_H)} + \frac{T_H - T_L}{\tau_t(\theta_H)y^B}$

Note that when the probability of being in low type is zero (q= 0), then the environmental policy in asymmetric information of high type firm is equal to the environmental policy of complete information.

I am assuming that the externality of high type firm is higher than the externality of low type firm, taking into account how the production changes when the environmental policy changes. This means that $y_{LtH}(-p_A A_L(1+x_y) - h_y) < y_{HtH}(-p_A A_H(1+x_y) - h_y)$. This is true if $|(-p_A A_L(1+x_y) - h_y)| << |(-p_A A_H(1+x_y) - h_y)|$, which it is possible because function h is convex and even if $|y_{HtH}| \leq |y_{LtH}|$, I am assuming that the inequality holds.

Also I am assuming that $y_{HtH}\frac{1}{\epsilon_H} \leq y_{LtH}\frac{\tau_t(\theta_L)}{\tau_t(\theta_H)}\frac{1}{\epsilon_L}$, which is possible because $\frac{\tau_t(\theta_L)}{\tau_t(\theta_H)} > 1$ and $|y_{HtH}| < |y_{LtH}|$, and even if $|\frac{1}{\epsilon_L}| \leq |\frac{1}{\epsilon_H}|$, I am assuming that the inequality holds. Therefore, when a positive probability exists of being in low type, the environmental policy of high type in asymmetric information is lower than in complete information.

Therefore in Cournot case with asymmetric information, the environmental policy of high type firm under-internalizes its externality for two reasons. The first one is to gain more market against Foreign firm, because of the "rent-shifting" effect. The second one is to hold the incentive constraint of high type firm, decreasing its cost, because of the "separating" effect.

In contrast, when the probability of being in low type is one (q= 1), then the environmental policy in asymmetric information of high type firm is equal to the environmental policy of complete information of low type firm. However, the environmental policy of low type in asymmetric information will be always higher than the environmental policy in complete information, in order to sustain the IC of high, allowing the separating equilibrium.

Therefore in Cournot case with asymmetric information, the environmental policy of low type firm may under or over-internalize the externality, depending on which effect is stronger. The first one, the "rent-shifting" effect, is to gain more market share against foreign firm, so this one under-internalizes the environmental externality. The second one, the "separating" effect, is to hold the incentive constraint of high type firm, so this one over-internalizes the environmental externality.

I divide the optimum marginal environmental policy into a marginal tax and a lump sum tax. We need to have these two components, in order to ensure the existence of the separating equilibrium.

A marginal tax will be charged until y^B, where $y^B < y^*_\theta$ and y^*_θ is the optimum production output for Home firm with efficient level θ. If the firm produces more than y_B, then government charges the lump sum tax.

$t_H^{CI} = \frac{F}{\tau_t(\theta_H)}$, until y^B

$T_H^{CI} = Fy_H^D$, where $y_H^D = y_H^* - y^B$

$t_L^{CI} = \frac{F}{\tau_t(\theta_H)} + \frac{T_H^{CI} - T_L^{CI}}{\tau_t(\theta_H)y^B}$, until y^B

$T_L^{CI} = F\frac{\tau_t(\theta_L)}{\tau_t(\theta_H)}y_L^D + \frac{T_H^{CI} - T_L^{CI}}{\tau_t(\theta_H)y^B}\tau_t(\theta_L)y_L^D$, where $y_L^D = y_L^* - y^B$

Calculating T_L^{CI}, we get the following:

$T_L^{CI} = Fy^B\frac{\tau_t(\theta_L)y_L^D}{\tau_t(\theta_H)y^B + \tau_t(\theta_L)y_L^D} + Fy_H^D\frac{\tau_t(\theta_L)y_L^D}{\tau_t(\theta_H)y^B + \tau_t(\theta_L)y_L^D}$

Note that y^B is the same for any efficiency level, but $y_L^D < y_H^D$, since $y_L^* < y_H^*$.

In order to prove that this menu of contracts is equilibrium, I need to demonstrate that the four inequalities hold.

The incentives constrain for high type firm holds by the way that the marginal tax for low type firm was constructed.

The IR of low type condition is: $\pi_L^{CI} \geq 0$

$py_L - \frac{c}{\theta_L}y_L - t_L^{CI}\tau_t(\theta_L)y^B - T_L^{CI} \geq 0$

Then the necessary and sufficient condition for IR of low type to hold is:

$$py_L - \frac{c}{\theta_L}y_L \geq Fy_H[\frac{\tau_t(\theta_L)}{\tau_t(\theta_H)} - \frac{\tau_t(\theta_L)y_L^D}{\tau_t(\theta_H)y^B+\tau_t(\theta_L)y_L^D}(\frac{\tau_t(\theta_L)}{\tau_t(\theta_H)} - 1)]$$

We impose this condition. Note from complete information case that $py_L - \frac{c}{\theta_L}y_L - y^B\epsilon_L(-h_y - p_A A_L(1+x_y)) - y_L^D\epsilon_L(-h_y - p_A A_L(1+x_y)) > 0$.

The individual rationality of high holds, as consequence of individual rationality of low type and the incentives constraint of high type firm hold. Because we can rewrite these conditions as follows:

$$\pi_H^{CI} \geq \pi_H^D = py_H - \frac{c}{\theta_H}y_H - t_H^{CI}\tau_t(\theta_H)y^B - \frac{T_H^{CI}-T_L^{CI}}{\tau_t(\theta_H)y^B}\tau_t(\theta_H)y^B - T_L^{CI}$$

$$> py_H - \frac{c}{\theta_H}y_H - t_H^{CI}\tau_t(\theta_L)y^B - \frac{T_H^{CI}-T_L^{CI}}{\tau_t(\theta_H)y^B}\tau_t(\theta_L)y^B - T_L^{CI}$$

$$> py_L - \frac{c}{\theta_L}y_L - t_H^{CI}\tau_t(\theta_L)y^B - \frac{T_H^{CI}-T_L^{CI}}{\tau_t(\theta_H)y^B}\tau_t(\theta_L)y^B - T_L^{CI} = \pi_L^{CI} \geq 0$$

The incentive constraint of low type condition firm is:

$$\pi_L^D \leq \pi_L^{CI}$$

$$py_L - \frac{c}{\theta_L}y_L - t_H^{CI}\tau_t(\theta_L)y_B - T_H^{CI} \leq py_L - \frac{c}{\theta_L}y_L - t_H^{CI}\tau_t(\theta_L)y_B - T_H^{CI}\frac{\tau_t(\theta_L)}{\tau_t(\theta_H)} + T_L^{CI}(\frac{\tau_t(\theta_L)}{\tau_t(\theta_H)} - 1)$$

Then the necessary and sufficient condition for IC of low type firm to hold is:

$$\tau_t(\theta_H)y_H^D \leq \tau_t(\theta_L)y_L^D$$

We impose this condition. ■

2.5 Conclusion

This paper shows that in international oligopoly competition with complete information, the optimum environmental policy under-internalizes the externality, for every level of firm efficiency. This result is consistent with the conclusions of Conrad [5].

However, with asymmetric information, I find that the optimum environmental policy is affected by two effects: the “rent-shifting effect” and the “separating effect”.

For the case of a low efficiency type local firm, the “separating effect” makes government to over-internalize the pollution externality. Therefore, if the “separating effect” is stronger than the “rent-shifting effect”, the environmental policy for the low type may over-internalize the pollution externality. The conclusion is that government does not always have an incentive to under-internalize its pollution externality.

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2.6 Appendix

Single Crossing Property

$\pi = py - (\frac{c}{\theta} + \tau(\theta)) * y - T$

How cost changes when the environmental policy changes:

$\frac{\partial - \pi}{\partial t} = \tau_t(\theta) * y$

And because $\tau_t(\theta) > 0$, then the derivation is positive. Therefore, with an increment of environmental policy, the cost increases.

How the marginal cost changes when the type changes:

$\frac{\partial^2 - \pi}{\partial \theta \partial t} = \tau_{t\theta}(\theta) * y$

And because $\tau_{t\theta}(\theta) < 0$, then the derivation is negative. Therefore, the high type has less marginal cost, as compared with low type. Consequently, high type has incentives to differentiate him from low type ■

Proof that it exist incentives to deviate in the Autarky case

If High type says the true, then its benefits are the following:

$\pi_H^{FB} = py_H - \frac{c}{\theta_H} y_H - \frac{\epsilon_H}{\tau_t(\theta_H)}(-h_y - p_y y_H) * \tau_t(\theta_H) y^B - \epsilon_H y_H^D(-h_y - p_y y_H)$

If High type deviates, then its benefits are the following:

$\pi_H^{D} = py_H - \frac{c}{\theta_H} y_H - \frac{\epsilon_L}{\tau_t(\theta_L)}(-h_y - p_y y_L) * \tau_t(\theta_H) y^B - \epsilon_L y_L^D(-h_y - p_y y_L)$

Because $\tau_t(\theta_H) < \tau_t(\theta_L)$, then $\frac{\tau_t(\theta_H)}{\tau_t(\theta_L)} < 1$ and also $\epsilon_L(-h_y - p_y y_L) < \epsilon_H(-h_y - p_y y_H)$ and $y_L^D < y_H^D$. Therefore, $\pi_H^{FB} < \pi_H^D$ ■

Proof that it exist incentives to deviate in the Cournot model

If High type says the true, then its benefits are the following:

$$\pi_H^{FB} = py_H - \frac{c}{\theta_H} y_H - \frac{\epsilon_H}{\tau_t(\theta_H)}(-h_y - p_A A_H(1 + x_y)) * \tau_t(\theta_H) y^B - \epsilon_H y_H^D(-h_y - p_A A_H(1 + x_y))$$

If High type deviates, then its benefits are the following:

$$\pi_H^D = py_H - \frac{c}{\theta_H} y_H - \frac{\epsilon_L}{\tau_t(\theta_L)}(-h_y - p_A A_L(1 + x_y)) * \tau_t(\theta_H) y^B - \epsilon_L y_L^D(-h_y - p_A A_L(1 + x_y))$$

Because $\tau_t(\theta_H) < \tau_t(\theta_L)$, then $\frac{\tau_t(\theta_H)}{\tau_t(\theta_L)} < 1$ and also $\epsilon_L(-h_y - p_A A_L(1 + x_y)) < \epsilon_H(-h_y - p_A A_H(1 + x_y))$ and $y_L^D < y_H^D$. Therefore, $\pi_H^{FB} < \pi_H^D$ ■

Chapter 3

Environmental Policy in a model of International Trade with competition à la Stackelberg. Comparison with the Cournot competition and the Autarky case

3.1 Introduction

There has been much concern about the strategic use of environmental policy to achieve competitive advantage in international trade. Conrad [4] demonstrates that the optimal environmental policy under-internalizes pollution externality under conditions of international oligopoly competition.

Subsequent studies by Kennedy [8] and Copeland and Taylor [6] examine the strategic use of environmental policy under conditions of free trade and transboundary pollution; studies by [10], Maggi [9] and Qiu [11] examine the effects of incomplete information; and Ulph [12] examines strategic behavior within firms to determine their investment levels in Research and Development.

Some empirical literature like Copeland and Taylor [5] [6], and Grossman and Krueger [7], seeks to measure pollution externality using quantitative data in order to determine whether or not trade agreements increase pollution.

In additional, other literature is concerned with how different market structures modify the strategic use of environmental policy. Barrett [1] examines different market structures in a unilateral commerce model. When the market has a Cournot structure, he finds that there is a strategic incentive for government to choose a weak environmental policy. He also finds that when the market has a Stackelberg structure, the leader firm country does not have an incentive to behave strategically, as the local firm does not need any help to win the market. Choosing a weak environmental policy is thus a costly subsidy.

Burguet and Sempere [3] analyze the effects on environmental policy of gradually changing commercial tariffs in a Cournot market structure. They find two opposite effects, one that gives incentives to reduce the environmental policy in order to make local industry competitive against imports in the market, and the other that gives incentives to increase environmental policy because of the intensity of production. Because the model is symmetrical and both of these opposite effects occur, it is not possible to know which effect will win, so environmental policy could increase or decrease.

In this chapter, I investigate the effects of a gradual reduction of commerce tariffs in a Stackelberg structure. Adding asymmetric competition to the Burguet and Sempere model [3], I seek to determine whether or not one effect dominates in the leader firm country while the other effect dominates in the follower firm country.

I find that the environmental policy of the leader firm country is always higher than the environmental policy of the follower firm country. Moreover, I also find that the environmental policy of the leader firm country increases as commerce tariffs decrease, from low levels of both damage and abatement efficiency. This is the opposite of what happens to the environmental policy of the follower firm country, which decreases its environmental policy as commerce tariffs decrease, at low levels of both damage and abatement efficiency.

This result suggests that the leader firm country has the dominant effect of reducing its pollution externality. By contrast, the follower firm country has the dominant effect of increasing firm efficiency to gain more market share.

In this paper, I compare the results of the Stackelberg model with those of the Cournot model. I find that the environmental policy of the country with a firm that competes à la Cournot is weaker than the environmental policy of the leader firm country in the Stackelberg model, but it is stronger than the environmental policy of the follower firm country.

The previous result is explained because a country with a firm that competes à la Cournot does not have the same incentive to reduce its pollution externality as a leader firm country, since production of the Cournot firm is lower than that of the leader firm in the Stackelberg model. Additionally, a country with a firm in the Cournot model does not have the same incentive to gain competitiveness as a follower firm country. When firms have the same costs, Cournot firms have half of the market share.

In the case of low levels of both abatement efficiency and damage, a country with a firm that competes à la Cournot decreases its environmental policy as commerce tariffs decrease, in order to gain market. This result is consistent with literature that analyzes the behavior of countries with Cournot firms. When they find strategic behavior, countries have incentives to diminish their environmental policy in order to make their firm more efficient and to gain more market share.

However, in the case of medium or high levels of both abatement efficiency and damage, a country with a firm that competes à la Cournot increases its environmental policy as commerce tariffs decrease. This decreases its pollution externality as it has a high level of production.

In addition, I compare oligopolistic competition in the Autarky case. I find that the environmental policy of the leader firm country in the Stackelberg model, and that of a country with a firm in the Cournot model, are both higher than the environmental policy of the Autarky country, since the production of these firms in imperfect competition are higher than production in the Autarky case.

This paper can help to make sense of why some governments have incentives to under-internalize their environmental externality despite the fact that they enter into environmental agreements like the Kyoto protocol. Moreover, this model of Stackelberg market competition could explain why countries with leader technology firms, as in Europe or Canada, have high environmental policy, and why countries with follower technology firms, like in India or Latin America, have low environmental policy.

This chapter is structured in the following manner. In Section 2, I present the model. In Section 3, the results of the Stackelberg model are analyzed and compared with the results of other market structures. In Section 4, I draw some conclusions.

3.2 The Stackelberg Competition model and Adaptations of the model to other Market Structures

This game is based on the Burguet and Sempere model [3] with Stackelberg market structure. I assume that there are two countries: Home (1) and Foreign (2). There is only one tradable good, which is produced by two firms. Firm (1) is in Home country and Firm (2) is in Foreign. Each firm sells in both countries, but they only produce in their own countries. Neither exchange rates nor transportation costs are in play. Firms compete in quantities in both markets. Firm (1) is the Leader and Firm (2) is the Follower in both markets.

Demand is the same in each country. The price for the good is given by the inverse demand function and, for simplicity, it is linear, $P(Q) = A - Q$. The price in Home is $P(Q_1)$ and $P(Q_2)$ in Foreign.

Firms (1) and (2) have a constant marginal cost. The marginal cost is affected by the environmental regulation. For simplicity, I will assume that marginal cost coincides with the value of the environmental policy.

Governments choose their environmental policy, c_i, in order to maximize the social welfare of its country. Imports to each country are subject to a tariff, t_i, which for further simplicity, we assume is exogenous and the same in both countries, t.

The model is a three stage game. In the first stage, Home and Foreign governments simultaneously choose their environmental policy, c_1^s and c_2^s respectively. In the second stage, after observing the environment policies, Leader Firm chooses its output for Home and for Foreign market, q_{11} and q_{12} respectively. In the third stage, after observing the environmental policies and the production of Leader Firm, Follower Firm (2) chooses its output for Home and for Foreign market, q_{21} and q_{22} respectively. Let the total output in Home market be $Q_1 = q_{11} + q_{21}$ and in Foreign market be $Q_2 = q_{12} + q_{22}$.

The production of the good creates emissions, e_i^s, which are local. Emissions, e_i^s, are equal to output produced in the country minus the abatement. That is, $e_i^s = ((q_{i1} + q_{i2}) - bc_i^s)$, where b is the level of efficiency of the abatement policy and also a positive constant. Based in Ulph [12], the environmental externality can be modeled as the total pollution that creates the emissions $\frac{d}{2}((q_{i1} + q_{i2}) - bc_i^s)^2$, where d is the level of damage that provokes the emissions and is a positive constant. The Social Cost, h_i, is defined as the sum of the private cost from the firm plus the environmental externality.

Therefore, the Social Cost is:

$$h_i(c_i^s, q_{i1} + q_{i2}) = (q_{i1} + q_{i2})\, c_i^s + \frac{d}{2}\left((q_{i1} + q_{i2}) - bc_i^s\right)^2$$

where $i \in \{1, 2\}$. The Social Cost function is convex with respect to environmental policy.[1]

Finally, Welfare is the total social surplus minus social cost, plus exports, minus imports, plus any

[1] Go to Appendix to see the proof.

tariff revenue. Therefore, the Welfare function is:

$$W_i^S = \int_0^{q_{1io}+q_{2io}} P_i dQ_i - h_i(c_i^s, q_{i1o}+q_{i2o}) - P_i q_{jio} + P_j q_{ijo} - t(q_{ijo} - q_{jio})$$

where $i \epsilon \{1,2\}$, q_{iio} and q_{ijo} are the second stage equilibrium output if $i = 1$ and are the third stage equilibrium output if $i = 2$.

The five terms on the right-hand side are, respectively, gross (domestic) surplus, social cost of domestic output, the value of imports, the value of exports, and the net tariff revenue.

The model will be solved with a Subgame Perfect Equilibrium, therefore it will be calculated with backward induction. This problem has a unique equilibrium as the Welfare function is strictly concave.[2]

Third stage: given the environment policies, c_1^s and c_2^s, and the output of Firm 1, Firm 2 maximizes its benefits by choosing its output for Home market q_{21} and for Foreign Market q_{22}. Notice that the output of firm 2 to Home Market is exports. Therefore, marginal cost of Firm 2 for Foreign market is just c_2^s, instead of marginal cost for Home market, which is $c_2^s + t$.

Second stage: given the environment policies, c_1^s and c_2^s, Firm 1 maximizes its benefits by choosing its output for Home market q_{11} and for Foreign Market q_{12}. Notice that the output of Firm 1 to Foreign Market is exports. Therefore, marginal cost of Firm 1 for Home market is just c_1^s, instead of marginal cost for Foreign market that is $c_1^s + t$.

First stage: governments maximize their national Welfare by choosing their environmental policy, c_i^s.[3]

3.2.1 The Cournot competition model

The results obtained in the Stackelberg Model are compared with the results obtained with the Cournot Model.

[2]Go to Appendix to see the proof.

[3]Go to Appendix to see the specific solution of the Stackelberg model. It is not presented in the text because of its complexity.

3.2. THE STACKELBERG COMPETITION MODEL AND ADAPTATIONS OF THE MODEL TO OTHER MARKET STRUCTURES

It is necessary to analyze the results with the Cournot competition, in order to see if these results are equivalent to the ones that have already found in literature.

Additionally, it is important to compare the results obtained in the Stackelberg case with the Cournot competition, to analyze if the Stackelberg model, an asymmetric competition, can tell us which effect dominates the leader country and which one dominates the follower country, explaining something more of the behavior of the countries, when they are in a different market structure.

The Cournot model is a two stage game. In the first stage, Home and Foreign governments simultaneously choose their environmental policy, c_1^c and c_2^c respectively. In the second stage, after observing the environment policies, Home and Foreign firm choose their output for Home and for Foreign market, q_{11} and q_{12}, and q_{21} and q_{22} respectively.

The production of the good creates emissions, e_i^c, which are local. Emissions, e_i^c, are equal to output produced in the country minus the abatement. That is, $e_i^c = ((q_{i1} + q_{i2}) - bc_i^s)$, where b is the level of efficiency of the abatement policy and also a positive constant.

The Social cost is the same as the Stackelberg case.

The Welfare function is:

$$W_i^c = \int_0^{q_{1io}+q_{2io}} P_i dQ_i - h_i(c_i^c, q_{i1o} + q_{i2o}) - P_i q_{jio} + P_j q_{ijo} - t(q_{ijo} - q_{jio})$$

where $i \in \{1, 2\}$, q_{iio} and q_{ijo} are the second stage equilibrium output.

The five terms on the right-hand side are, respectively, gross (domestic) surplus, social cost of domestic output, the value of imports, the value of exports, and the net tariff revenue.

The model will be solved with a Subgame Perfect Equilibrium. Therefore, it will be calculated with backward induction. This problem has a unique equilibrium, because the Welfare function is strictly concave.[4]

Second stage: given the environment policies, c_1^c and c_2^c, Home and Foreign Firm maximize their

[4]Go to Appendix to see the proof.

benefits by choosing their output for Home market q_{11} and q_{21} respectively and for Foreign Market q_{12} and q_{22} respectively. Notice that the output of Home Firm to Foreign market are exports, so its marginal cost for Foreign market is $c_1^c + t$. Analogous, the output of Foreign Firm to Home market are exports, so its marginal cost for Home market is $c_2^c + t$.

First stage: Governments maximize their national Welfare by choosing their environmental policy, c_i^c.[5]

3.2.2 The Autarky case

The Stackelberg case and the Cournot competition results are compared with the Autarky case. The objective of this is to compare the strategic behavior of environmental policy in oligopoly competition with the behavior of no competition, and analyze if these results are equivalent with the ones that have already found in literature.

The Autarky case is a two stage game. In the first stage, Home government chooses its environmental policy, c_1^a. In the second stage, after observing the environment policy, Home monopoly firm chooses its output for Home market, q_{11}.

The production of the good creates emissions, e_i^a, which are local. Emissions, e_i^a, are equal to output produced in the country minus the abatement. That is, $e_i^a = ((q_{i1} + q_{i2}) - bc_i^s)$, where b is the level of efficiency of the abatement policy and also a positive constant.

The Social Cost is the same as the Stackelberg case.

The Welfare function is:

$W_1^a = \int_0^{q_{11o}} P_1 dQ_1 - h_1(c_1^a, q_{11o})$

where q_{11o} is the second stage equilibrium output.

The terms on the right-hand side are, respectively, gross (domestic) surplus and social cost of domestic output.

[5]Go to Appendix to see the specific solution of the Cournot model. It is not presented in the text because of its complexity.

The model will be solved with a Subgame Perfect Equilibrium. Therefore, it will be calculated with backward induction. This problem has a unique equilibrium, because the Welfare function is strictly concave.[6]

Second stage: given the environment policy, c_1^a, Home monopoly firm maximizes its benefits by choosing its output for Home market q_{11}.

First stage: Government maximize its national Welfare by choosing its environmental policy, c_1^a.[7]

3.3 Results of the Environmental policy in the Stackelberg model. Comparison with the Cournot model and the Autarky case

In this section, I present the main results of the Stackelberg model. I also compare the behavior of these ones with the results of using other market structures, specifically with the Autarky case and the competition à la Cournot.

3.3.1 How environmental policy of the countries that compete à la Stackelberg changes, as commerce tariffs decrease. Comparison with the Cournot model.

The environmental policy of the leader firm country has the dominant effect of reducing the pollution externality, as commerce tariffs decrease. Instead, the environmental policy of the follower firm country has the dominant effect of increasing its firm competitiveness, as commerce tariffs decrease.

Proposition 1: $\frac{\partial c_1^s}{\partial t} < \frac{\partial c^c}{\partial t} < \frac{\partial c_2^s}{\partial t}$ *when* $b \geq 0.5$ *and* $d \geq 0.6$. [8]

The proof of all propositions in this chapter is in the Appendix.

[6] Go to Appendix to see the proof.

[7] Go to Appendix to see the specific solution of Autarky case. It is not presented in the text because of its complexity.

[8] Go to Appendix to see the proof that: $\frac{\partial c_1^s}{\partial t} > \frac{\partial c^c}{\partial t} > \frac{\partial c_2^s}{\partial t}$ when $b < 0.5$ and $d < 0.6$..

The leader firm country in the Stackelberg model has the strongest effect of increasing the environmental policy, as commerce tariffs decrease. Contrary to the follower firm country in the Stackelberg model, that has the strongest effect of decreasing the environmental policy, as commerce tariffs decrease.

Instead, those countries that compete à la Cournot have an incentive which is lower than the leader firm country to increase the environmental policy, but higher than the country with follower firm to increase the environmental policy, when the commerce tariffs decrease.

This behavior is explained because the leader firm country in the Stackelberg model produces a very high amount of output, its priority being to reduce its pollution externality.

Additionally, the country with a firm in the Cournot model has less incentives of increasing its environmental policy than the leader firm country, as commerce tariffs decrease, because the firm that competes à la Cournot produces less amount of output than the leader firm in the Stackelberg model. Therefore, its emissions are lower.

Instead, the follower firm country in the Stackelberg model has more incentives of decreasing its environmental policy, as commerce tariff decrease, which is a consequence that it needs to increase its firm efficiency to gain more market share, because its production level is low.

Finally, the country with the firm in the Cournot model has less incentives of decreasing its environmental policy than the follower firm country in the Stackelberg model, as commerce tariffs decrease, because it does not need to increase as much its firm competitiveness, given its firm has half of the market.

Proposition 2: $\frac{\partial c_1^s}{\partial t} < 0$ *when* $b \geq 0.94$ *and* $d \geq 0.94$.

The leader firm country that competes à la Stackelberg increases its environmental policy when the commerce tariffs decrease, from low levels of both damage and abatement efficiency.

This behavior is because the government of the leader firm country prefers decreasing its pollution externality than to increase its firm efficiency. Therefore, from low levels of both damage and abatement efficiency, it increases its environmental policy, when the commerce tariffs decrease.

3.3. RESULTS OF THE ENVIRONMENTAL POLICY IN THE STACKELBERG MODEL. COMPARISON WITH THE COURNOT MODEL AND THE AUTARKY CASE

Proposition 3: $\frac{\partial c_1^s}{\partial t} > 0$ *when* $0.6 \leq b \leq 0.93$ *and* $0.6 \leq d \leq 0.93$.

In the case when both abatement efficiency and damage levels are very low, the government of the leader firm country decreases its environmental policy, as commerce tariffs decrease. This is explained because even its production level is high, its environmental externality is not high and it needs a lot of policy to reduce this few pollution, because the efficiency to abate is very low, therefore it prefers to gain more market share.

Although the government of the leader firm country decreases its environmental policy, as commerce tariffs decrease at very low levels of both abatement efficiency and damage, its environmental policy is the one that has the smallest reduction, as compared with the decrease of the environmental policy of the follower firm country or the firm in the Cournot model (see proposition 1). This is because the incentive of the leader firm country to reduce its environmental policy is higher than the other countries, because it has the highest production level.

Proposition 4: $\frac{\partial c_2^s}{\partial t} > 0$ *when* $0.6 \leq b \leq 1.3$ *and* $0.6 \leq d \leq 1.2$.

In the case when both abatement efficiency and damage levels are low, the government of the follower firm country prefers to gain more market share than decreasing its pollution externality, because its production level is low. The levels of both damage and abatement efficiency where the country of the follower firm decreases its environmental policy are higher than the ones where the country of the leader firm increases its environmental policy, as commerce tariffs decrease.

Proposition 5: $\frac{\partial c_2^s}{\partial t} < 0$ *when* $b \geq 1.4$ *and* $d \geq 1.2$.

The government of the follower firm country increases its environmental policy, as commerce tariffs decrease from medium levels of both abatement efficiency and damage, because it needs to reduce its environmental externality.

Although government increases its environmental policy as commerce tariffs decrease, from medium levels of both abatement efficiency and damage, the increment of the environmental policy of this country is still the lowest, compared with the leader firm country and with the country with the Cournot firm (see proposition 1). This is because the incentive of the follower firm country to obtain more market share is higher than the other countries, because its level of production is the lowest one.

Proposition 6: $\frac{\partial c^c}{\partial t} > 0$ *when* $0.6 \leq b \leq 1$ *and* $0.6 \leq d \leq 1$.

The government of the country with the firm that competes à la Cournot prefers to increase its firm competitiveness, but only when pollution externality is not high. This is because it wants to increase its market share, because it has a medium-high production level. Therefore, only with low levels of both damage and abatement efficiency, it decreases its environmental policy as commerce tariffs decrease.

Note that the maximum level of both damage and abatement efficiency where the country with the Cournot firm decreases its environmental policy, as commerce tariffs decrease, is higher than the one where the country with leader firm decreases its environmental policy, but lower than the one where the follower firm country decreases its environmental policy, as commerce tariffs decrease.

The government of the country with the firm that competes à la Cournot increases its environmental policy as commerce tariffs decrease from low-medium levels of both damage and abatement efficiency.

Proposition 7: $\frac{\partial c^c}{\partial t} < 0$ *when* $b \geq 1.1$ *and* $d \geq 1.1$.

Countries with a firm that competes à la Cournot increase their environmental policy, when the commerce tariffs decrease, at levels of damage and abatement efficiency which are higher as compared with a country with a leader firm, but at levels which are lower as compared with a country with a follower firm.

The reason of this is because its levels of production is medium-high. Although, it does not have as high a production as the leader firm; it has a production which is higher than the follower firm in the Stackelberg model. Therefore, it does not have as high the pollution externality as the leader firm; but it has a higher pollution than the follower firm country.

Consequently, the minimum level of both damage and abatement efficiency where the country with the Cournot firm increases its environmental policy, as commerce tariffs decrease, is higher than the one of the leader firm country increases its environmental policy, but lower than the minimum where the follower firm country increases its environmental policy, as commerce tariffs decrease.

This result is consistent with Burguet and Sempere [3] , who find that the environmental policy of the country with the Cournot firm can increase or decrease, as commerce tariffs decrease, depending on which effect dominates, the one to have a more competitive firm or the one to diminish the externality.

3.3.2 Environmental policy and emissions of the countries that compete à la Stackelberg. Comparison with the Cournot model and the Autarky case

The leader firm country in the Stackelberg model has the highest environmental policy, which is the opposite of both the Autarky country and the follower firm country that has the lowest one.

Proposition 8: $c_1^s > c^c > c_2^s$ *and* $c^c > c^a$.

In order to prove proposition 8, I prove the following three lemmas.

Lemma 8.1: $c_1^s > c^c$ *when* $b \geq 0.5$, $d \geq 0.6$. *and* $A \geq 1$. [9]

The leader firm country has a higher environmental policy than the country with the firm that competes à la Cournot, from low levels of both damage and abatement efficiency, and a minimum demand size.

The reason is that the government of the leader firm country has more incentives to decrease its pollution externality, instead of increasing its competitiveness, because it has a high level of production.

Instead, the firm that competes à la Cournot does not have as much production as the leader firm that competes à la Stackelberg; therefore, it does not have such a high level of emissions, having fewer incentives to reduce its pollution externality.

[9] Go to Appendix to see the proof that: $c_1^s < c^c$ *when* $b < 0.5$, $d < 0.6$ and $A \geq 2$.

Additionally, the country with a firm in the Cournot model has more incentives to have a more efficient firm, to obtain more market, because its production is lower than the leader firm country.

For the result to be true, it is necessary to have a minimum level of demand size, because with a very low demand size, the leader firm country in the Stackelberg model has as priority to have the most of the market share, because the pollution externality is not high, so it wants to increase its production as much as possible.

Lemma 8.2: $c^c > c_2^s$ *when* $b \geq 2$, $d \geq 2.2$ *and* $A \geq 1$. [10]

The country with a firm that competes à la Cournot has a higher environmental policy than the follower firm country, from medium levels of both damage and abatement efficiency, and a minimum demand size.

The government of the follower firm country has more incentives to increase its firm competitiveness, instead of decreasing its pollution externality, in order to gain more market share and to increase its production.

Instead, the country with the firm in the Cournot model does not have an environmental policy which is as low as that in the follower firm country. This is because the firm that competes à la Cournot has more market share than the follower firm in the Stackelberg model, so it does not need to increase its firm efficiency as much as the follower firm country.

For the result to be true, it is necessary to have medium levels of both damage and abatement efficiency and a minimum of demand size, in order for the environmental policy of the country that competes à la Cournot to be higher than the environmental policy of the follower firm country in the Stackelberg model. This is because with low levels of both damage and abatement efficiency, and a very low demand size, the competition between Cournot firms is very intense, because the pollution externality is not high, so they want to increase their production.

Although the follower firm in the Stackelberg model has a low environmental policy, when the demand size is very low, it is not the lowest environmental policy, because the leader firm has an

[10] When $b < 2$ and $d < 2.2$, it is not possible to determine if $c^c > c_2^s$ or $c^c < c_2^s$, since the functions are discontinuous in that domain. Go to Appendix to see the specific discontinuous areas.

environmental policy very low to gain as more market as possible, because the pollution externality is very low.

Lemma 8.3: $c^c > c^a$ *when* $b \geq 1.1$, $d \geq 1.1$ *and* $A \geq 3$. [11]

The country with the firm that competes à la Cournot has a higher environmental policy than the Autarky country, from low levels of both damage and abatement efficiency, and a minimum demand size.

The government of the Autarky country chooses a low environmental policy, to increase its production and to increase its consumption surplus, because it has a low production level, as consequence that the domestic monopoly does not have any international competition.

For the result to be true, it is necessary to have a minimum of demand size, in order to have the environmental policy of the country that competes à la Cournot higher than the environmental policy of the Autarky country. This is because with low demand size, the competition between the Cournot firms is very intense, because they want to increase its production, having a very low environmental policy. This motivation does not exist in the Autarky country.

The leader firm country that competes à la Stackelberg could have the highest emission level, which is the opposite of both the Autarky country and the follower firm country, which has the lowest one.

Proposition 9: $e_1^s > e^c > e_2^s$ *and* $e^c > e^a$.

In order to prove proposition 9, I prove the following three lemmas.

Lemma 9.1: $e_1^s > e^c$ *when* $b \geq 4.3$, $d \geq 1$ *and* $A \geq 27$. [12]

For the result to be true, it is necessary to have medium-high level of abatement efficiency and at least medium demand size, in order to the leader firm country that competes à la Stackelberg have more emissions than the country with the firm in the Cournot model.

[11] When $b < 1.1$ and $d < 1.1$, it is not possible to determine if $c^c > c^a$ or $c^c < c^a$, since the functions are discontinuous in that domain. Go to Appendix to see the specific discontinuous areas.

[12] When $b < 4.3$ and $d < 1$, it is not possible to determine if $e_1^s > e^c$ or $e_1^s < e^c$, since the functions are discontinuous in that domain. Go to Appendix to see the specific discontinuous areas.

Medium-high levels of abatement efficiency and at least a medium demand size for the leader firm country that competes à la Stackelberg are necessary to have more emissions than the country with the firm in the Cournot model. This is because with high levels of abatement efficiency, it is necessary less environmental policy to reduce the pollution. Consequently, the production of the leader firm is higher than the one with low level of abatement efficiency.

Therefore, the production of the leader firm is much higher than the one of the firm that competes à la Cournot. However, the environmental policy of the leader firm country is higher than the one of the country with the Cournot firm (see Proposition 5). This difference is not high enough to compensate the difference in production.

Lemma 9.2: $e^c > e_2^s$ *when* $b \geq 0.8$, $d \geq 0.7$ *and* $A \geq 0$. [13]

The country with the Cournot firm has higher emissions than the follower firm country that competes à la Stackelberg. This is because even the environmental policy of the follower firm country in the Stackelberg model is lower than the one with the Cournot firm (see Proposition 5); the follower firm has a production which is lower than the Cournot firm, from the low levels of both damage and abatement efficiency for any demand size.

Lemma 9.3: $e^c > e^a$ *when* $b \geq 2.5$, $d \geq 0.4$ *and* $A \geq 10$. [14]

The environmental policy of the country with a Cournot firm is higher than the one in the Autarky case (see Proposition 5), although the emissions of the country with the Cournot firm are higher than the ones in the Autarky country, from the medium abatement efficiency level and a minimum demand size. For the result to be true, it is necessary to have at least medium demand size, because the production in the Autarky country is higher than the production of the Cournot firm at low level of demand size. This is because the Autarky country does not have any competition, so its firm has all the market.

Therefore, with more demand size, the production of the Cournot firm increases more than the one of the monopoly in the Autarky country, because the Cournot firm produces for its market and for

[13] When $b < 0.8$ and $d < 0.7$, it is not possible to determine if $e^c > e_2^s$ or $e^c < e_2^s$, since the functions are discontinuous in that domain. Go to Appendix to see the specific discontinuous areas.

[14] When $b < 2.5$ and $d < 0.4$, it is not possible to determine if $e^c > e^a$ or $e^c < e^a$, since the functions are discontinuous in that domain. Go to Appendix to see the specific discontinuous areas.

foreign market; even the environmental policy of the country with the Cournot firm is higher (see Proposition 5). As a result, emissions of the country with a Cournot firm are higher than the ones in the Autarky country.

3.4 Conclusions

This chapter shows that, as commerce tariffs decrease, a country with a firm in the Cournot model can decrease or increase its environmental policy, depending on the effect that dominates. This result is consistent with the conclusions of Burguet and Sempere [3].

However, with an asymmetric oligopoly model, as under Stackelberg competition, I show that the leader firm country has the dominant effect of reducing its pollution externality, whereas the follower firm country has the dominant effect of increasing its firm competitiveness to gain more market share.

I also find that, as commerce tariffs decrease, the environmental policy of a country with a firm that competes à la Cournot increases less than in the leader firm country that competes à la Stackelberg, but increases more than in the follower firm country in the Stackelberg model.

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3.5 Appendix

Proof that the social cost function is convex

Social cost function is the following:

$h_i(c_i, q_{i1} + q_{i2}) = (q_{i1} + q_{i2})c_i + \frac{d}{2}((q_{i1} + q_{i2}) - bc_i)^2)$

First Order Condition:

$\frac{\partial h_i}{\partial c_i} = (q_{i1} + q_{i2}) + c_i\frac{\partial(q_{i1}+q_{i2})}{\partial c_i} + \frac{d}{2}(2(q_{i1} + q_{i2})\frac{\partial(q_{i1}+q_{i2})}{\partial c_i} - 2b(q_{i1} + q_{i2}) - 2bc_i\frac{\partial(q_{i1}+q_{i2})}{\partial c_i} + 2b^2c_i)$

Second Order Condition:

$\frac{\partial^2 h_i}{\partial {c_i}^2} = 2\frac{\partial(q_{i1}+q_{i2})}{\partial c_i} + d\frac{\partial(q_{i1}+q_{i2})}{\partial c_i}^2 - 2db\frac{\partial(q_{i1}+q_{i2})}{\partial c_i} + db^2$

Therefore,

$\frac{\partial^2 h_1}{\partial {c_1}^2} = -2(2) + 4d + 2db(2) + db^2$

$\frac{\partial^2 h_2}{\partial {c_1}^2} = -2(\frac{3}{2}) + d(\frac{9}{4}) + 2db(\frac{3}{2}) + db^2$

Because from second and third stage equilibrium we obtain that $\frac{\partial(q_{11}+q_{12})}{\partial c_1} = -2$ and $\frac{\partial(q_{21}+q_{22})}{\partial c_2} = -\frac{3}{2}$

In order to have $h_i(c_i, q_{i1} + q_{i2})$ convex, it is sufficient that $b > \frac{1}{2}$ and $d > \frac{3}{4}$ ■

Proof that Welfare function is concave

Welfare function is the following:

W_i= $\int_0^{q_{1io}+q_{2io}} P_i \, dQ_i - h_i(c_i, q_{i1o} + q_{i2o}) - P_iq_{jio} + P_jq_{ijo} - t(q_{ijo} - q_{jio})$

Because the inverse demand function is linear, we obtain the following:

$W_i = A(q_{1io} + q_{2io}) - \frac{(q_{i1o}+q_{i2o})^2}{2} - h_i - P_iq_{jio} + P_jq_{ijo} - t(q_{ijo} - q_{jio})$

First Order Condition:

$\frac{\partial W_i}{\partial c_i} = A\frac{\partial(q_{1io}+q_{2io})}{\partial c_i} - (q_{1io}+q_{2io})\frac{\partial(q_{1io}+q_{2io})}{\partial c_i} - \frac{\partial h_i}{\partial c_i} - P_i\frac{\partial q_{jio}}{\partial c_i} - q_{jio}\frac{\partial P_i}{\partial Q_i}\frac{\partial Q_i}{\partial c_i} + P_j\frac{\partial q_{ijo}}{\partial c_i} + q_{ijo}\frac{\partial P_j}{\partial Q_j}\frac{\partial Q_j}{\partial c_i} - t(\frac{\partial q_{ijo}}{\partial c_i} - \frac{\partial q_{jio}}{\partial c_i})$

Second Order Condition:

$\frac{\partial^2 W_i}{\partial {c_i}^2} = -\frac{\partial(q_{1io}+q_{2io})}{\partial c_i}2 - \frac{\partial^2 h_i}{\partial {c_i}^2} - \frac{\partial q_{jio}}{\partial c_i}\frac{\partial P_i}{\partial Q_i}\frac{\partial Q_i}{\partial c_i} + \frac{\partial q_{ijo}}{\partial c_i}\frac{\partial P_j}{\partial Q_j}\frac{\partial Q_j}{\partial c_i}$

Therefore,

$\frac{\partial^2 W_1}{\partial {c_1}^2} = -(-\frac{1}{2})^2 - \frac{\partial^2 h_1}{\partial {c_1}^2} - \frac{1}{2}(\frac{1}{2}) - \frac{1}{2}$

$\frac{\partial^2 W_2}{\partial {c_2}^2} = -(-\frac{1}{4})^2 - \frac{\partial^2 h_2}{\partial {c_2}^2} - \frac{1}{2}(\frac{1}{4}) - \frac{3}{4}(\frac{1}{4})$

Because from second and third stage equilibrium we obtain that $\frac{\partial q_{1i}}{\partial c_1} = -1$, $\frac{\partial q_{2i}}{\partial c_1} = \frac{1}{2}$, $\frac{\partial q_{1i}}{\partial c_2} = \frac{1}{2}$ and $\frac{\partial q_{2i}}{\partial c_2} = -\frac{3}{4}$

In order to have W_i concave, it is sufficient that h_i is convex ■

Results of Stackelberg Model

The equilibrium results are the following:

$c_{1o} = \frac{(8A(64-(164+125b+22b^2)d+8(2+b)^2(3+2b)d^2)-(445-2(459+368b+72b^2)d+32(2+b)^2(3+2b)d^2)t)}{(8(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))}$

$c_{2o} = \frac{(4A(64-(164+122b+15b^2)d+4(24+34b+15b^2+2b^3)d^2)-(243-2(253+204b+32b^2)d+8(24+34b+15b^2+2b^3)d^2)t)}{(4(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))}$

$q_{11o} = \frac{(8Abd(-23+24d+22b^2d+4b^3d+b(-11+40d))+(229-2(267+302b+93b^2)d+4(72+186b+167b^2+62b^3+8b^4)d^2)t)}{(4(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))}$

$q_{12o} = \frac{(8Abd(-23+24d+22b^2d+4b^3d+b(-11+40d))-(155-2(225+358b+126b^2)d+4(72+234b+247b^2+106b^3+16b^4)d^2)t)}{(4(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))}$

$q_{21o} = \frac{(8Abd(-13+24d+15b^2d+2b^3d+b(-9+34d))-(121-(342+538b+207b^2)d+(240+796b+826b^2+340b^3+48b^4)d^2)t)}{(4(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))}$

$q_{22o} = \frac{(8Abd(-13+24d+15b^2d+2b^3d+b(-9+34d))+(199-2(239+281b+79b^2)d+(240+604b+554b^2+220b^3+32b^4)d^2)t)}{(4(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))}$

$W_{1o} = -(-64A^2b^2d(-4096+8(2329+2522b+673b^2)d-(27840+59600b+47361b^2+16556b^3+2148b^4)d^2+4(6+7b+2b^2)^2(96+80b+17b^2)d^3)+16Abd(2784b^6d^3+384b^7d^3+8b^5d^2(-347+432d)-4b^4d^2(2603+6218d)+b^2d(1981+143658d-181280d^2)+2b^3d(3627+9584d-53728d^2)-8(-3275+12562d-15336d^2+5760d^3)-4b(1842+19453d-58266d^2+36384d^3))t+(-104712+$

$(388656+229448b+126521b^2)d-4(103992+36864b-31853b^2-8030b^3+669b^4)d^2-4(-14736+204040b+548455b^2+538300b^3+252144b^4+55880b^5+4512b^6)d^3+16(6+7b+2b^2)^2(112+1032b+1367b^2+504b^3+48b^4)d^4)t^2)/(128(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)^2)$

$W_{2o} = -(-16A^2b^2d(-4096 + 16(1389 + 1282b + 301b^2)d - (36352 + 69312b + 49284b^2 + 15380b^3 + 1761b^4)d^2 + 4(6+7b+2b^2)^2(128+80b+13b^2)d^3) + 8Abd(336b^6d^3 + 16b^5d^2(-37 + 183d) + b^4d^2(-5369+8628d) + 4b^2d(1916+3067d-3228d^2) + b^3d(3641-10956d+6816d^2) - 8(-1751 + 5728d - 5776d^2 + 1728d^3) - 12b(446 + 1737d - 4832d^2 + 2256d^3))t + (-31060 + (120804+92708b+57189b^2)d-2(79488+114844b+114390b^2+59077b^3+10839b^4)d^2+(79552+147328b+173140b^2+141732b^3+63745b^4+12256b^5+448b^6)d^3+4(6+7b+2b^2)^2(-80+104b+43b^2+96b^3+32b^4)d^4)t^2)/(32(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)^2).$

Results of the Cournot Model

The equilibrium results are the following:

$c_{1o} = \frac{(2A(-5+(4+3b)d)+(8-(4+3b)d)t)}{(-10+(8+18b+9b^2)d)}$

$c_{2o} = \frac{(2A(-5+(4+3b)d)+(8-(4+3b)d)t)}{(-10+(8+18b+9b^2)d)}$

$q_{11o} = \frac{(Ab(4+3b)d+(-6+(4+7b+3b^2)d)t)}{/(-10+(8+18b+9b^2)d)}$

$q_{21o} = \frac{(Ab(4+3b)d-(-4+(4+11b+6b^2)d)t)}{(-10+(8+18b+9b^2)d)}$

$q_{12o} = \frac{(Ab(4+3b)d-(-4+(4+11b+6b^2)d)t)}{(-10+(8+18b+9b^2)d)}$

$q_{22o} = \frac{(Ab(4+3b)d+(-6+(4+7b+3b^2)d)t)}{(-10+(8+18b+9b^2)d)}$

$W_{1o} = \frac{-(-4A^2b^2d(-25+2(4+3b)^2d)+2A(-2+b)bd(-26+(4+3b)^2d)t+(-28+4(5-b+7b^2)d+b(2+b)(4+3b)^2d^2)t^2)}{(2(-10+(8+18b+9b^2)d)^2)}$

$W_{2o} = \frac{-(-4A^2b^2d(-25+2(4+3b)^2d)+2A(-2+b)bd(-26+(4+3b)^2d)t+(-28+4(5-b+7b^2)d+b(2+b)(4+3b)^2d^2)t^2)}{(2(-10+(8+18b+9b^2)d)^2)}.$

Results of Autarky case

The equilibrium results are the following:

$c_{1o} = \frac{(A(-3+d+2bd))}{(-3+d+4bd+4b^2d)}$

$Q_{1o} = \frac{(Ab(1+2b)d)}{(-3+(1+2b)^2d)}$

$W_{1o} = \frac{(3A^2b^2d)}{(2(-3+(1+2b)^2d))}$.

Proposition 1

Claim: $\frac{\partial c_2^s}{\partial t} > \frac{\partial c^c}{\partial t}$ and $\frac{\partial c^c}{\partial t} > \frac{\partial c_1^s}{\partial t}$.

In oder to prove proposition 1, I prove the following 2 lemmas.

Lemma 1.1: $\frac{\partial c^c}{\partial t} < \frac{\partial c_2^s}{\partial t}$ *when* $b \geq 0.5$ *and* $d \geq 0.6$.

$\frac{\partial c_2^s}{\partial t} - \frac{\partial c^c}{\partial t} = \frac{(382-(732+646b+491b^2)d+(272+644b+970b^2+524b^3+64b^4)d^2+8b^2(24+46b+29b^2+6b^3)d^3)}{(4(-10+(8+18b+9b^2)d)(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))}$

I need to prove that the $B = (382 - (732 + 646b + 491b^2)d + (272 + 644b + 970b^2 + 524b^3 + 64b^4)d^2 + 8b^2(24 + 46b + 29b^2 + 6b^3)d^3) > 0$ and $D = (4(-10 + (8 + 18b + 9b^2)d)(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)) > 0$.

Step 1. To prove that $B > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 0.5 \\ d \geq 0.6}} B(b, d)$$

The minimum value that B can attain is 25.27, when $b = 0.5$ and $d = 0.6$

Step 2. To prove that $D > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 0.5 \\ d \geq 0.6}} D(b, d)$$

The minimum value that D can attain is 23.87, when $b = 0.5$ and $d = 0.6$ ■

Lemma 1.2: $\frac{\partial c_1^s}{\partial t} < \frac{\partial c^c}{\partial t}$ *when* $b \geq 0.5$ *and* $d \geq 0.6$.

$\frac{\partial c^c}{\partial t} - \frac{\partial c_1^s}{\partial t} = \frac{(-354+(196-246b+773b^2)d+(208+84b-902b^2-936b^3-272b^4)d^2+32b(24+70b+75b^2+35b^3+6b^4)d^3)}{(8(-10+(8+18b+9b^2)d)(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))}$

I need to prove that the $E = (-354 + (196 - 246b + 773b^2)d + (208 + 84b - 902b^2 - 936b^3 -$

$272b^4)d^2 + 32b(24 + 70b + 75b^2 + 35b^3 + 6b^4)d^3) > 0$ and $F = (8(-10 + (8 + 18b + 9b^2)d)(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)) > 0$.

Step 1. To prove that $E > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 0.5 \\ d \geq 0.6}} E(b, d)$$

The minimum value that E can attain is 51.45, when $b = 0.5$ and $d = 0.6$

Step 2. To prove that $F > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 0.5 \\ d \geq 0.6}} F(b, d)$$

The minimum value that F can attain is 47.74, when $b = 0.5$ and $d = 0.6$ ■

Proof of $\frac{\partial c_1^s}{\partial t} > \frac{\partial c^c}{\partial t} > \frac{\partial c_2^s}{\partial t}$ *when* $b < 0.5$ *and* $d < 0.6$.

Claim: $\frac{\partial c_1^s}{\partial t} > \frac{\partial c^c}{\partial t} > \frac{\partial c_2^s}{\partial t}$ when $b < 0.5$ and $d < 0.6$.

In oder to prove the claim, I prove the following 2 lemmas.

Lemma 1: $\frac{\partial c_2^s}{\partial t} < \frac{\partial c^c}{\partial t}$ *when* $0.05 \leq b \leq 0.4$ *and* $0.05 \leq d \leq 0.4$.

$$\frac{\partial c_2^s}{\partial t} - \frac{\partial c^c}{\partial t} = \frac{(382 - (732 + 646b + 491b^2)d + (272 + 644b + 970b^2 + 524b^3 + 64b^4)d^2 + 8b^2(24 + 46b + 29b^2 + 6b^3)d^3)}{(4(-10 + (8 + 18b + 9b^2)d)(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2))}$$

I need to prove that the $B_1 = (382 - (732 + 646b + 491b^2)d + (272 + 644b + 970b^2 + 524b^3 + 64b^4)d^2 + 8b^2(24 + 46b + 29b^2 + 6b^3)d^3) > 0$ and $D_1 = (4(-10 + (8 + 18b + 9b^2)d)(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)) < 0$.

Step 1. To prove that $B_1 > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{0.05 \leq b \leq 0.4 \\ 0.05 \leq d \leq 0.4}} B_1(b, d)$$

The minimum value that B_1 can attain is 35.10, when $b = 0.4$ and $d = 0.4$

Step 2. To prove that $D_1 < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{0.05 \leq b \leq 0.4 \\ 0.05 \leq d \leq 0.4}} D_1(b, d)$$

The maximum value that D_1 can attain is -1.15, when $b = 0.4$ and $d = 0.4$ ■

Lemma 2: $\frac{\partial c^c}{\partial t} < \frac{\partial c_1^s}{\partial t}$ *when* $0.41 \leq b \leq 0.49$ *and* $0.4 \leq d \leq 0.5$.

$$\frac{\partial c^c}{\partial t} - \frac{\partial c_1^s}{\partial t} = \frac{(-354+(196-246b+773b^2)d+(208+84b-902b^2-936b^3-272b^4)d^2+32b(24+70b+75b^2+35b^3+6b^4)d^3)}{(8(-10+(8+18b+9b^2)d)(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))}$$

I need to prove that the $E_1 = (-354 + (196 - 246b + 773b^2)d + (208 + 84b - 902b^2 - 936b^3 - 272b^4)d^2 + 32b(24 + 70b + 75b^2 + 35b^3 + 6b^4)d^3) < 0$ and $F_1 = (8(-10 + (8 + 18b + 9b^2)d)(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)) > 0$.

Step 1. To prove that $E_1 < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{0.41 \leq b \leq 0.49 \\ 0.4 \leq d \leq 0.5}} E_1(b, d)$$

The maximum value that E_1 can attain is -88.46, when $b = 0.49$ and $d = 0.5$

Step 1. To prove that $F_1 > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{0.41 \leq b \leq 0.49 \\ 0.4 \leq d \leq 0.5}} F_1(b, d)$$

The minimum value that F_1 can attain is 3.37, when $b = 0.41$ and $d = 0.4$ ■

Proposition 2

Claim: $\frac{\partial c_1^s}{\partial t} < 0$ when $b \geq 0.94$ and $d \geq 0.94$

$$\frac{\partial c_1^s}{\partial t} = \frac{-445+2(459+368b+72b^2)d-32(2+b)^2(3+2b)d^2}{8(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)}$$

I need to prove that the $G = -445 + 2(459 + 368b + 72b^2)d - 32(2 + b)^2(3 + 2b)d^2 < 0$ and $H = 8(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2) > 0$.

Step 1. To prove that $G < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b \geq 0.94 \\ d \geq 0.94}} G(b, d)$$

The maximum value that G can attain is -4.82, when $b = 0.94$ and $d = 0.94$

Step 2. To prove that $H > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 0.94 \\ d \geq 0.94}} H(b, d)$$

The minimum value that H can attain is 2,247.41, when $b = 0.94$ and $d = 0.94$ ■

Proposition 3

Claim: $\frac{\partial c_1^s}{\partial t} > 0$ when $0.6 \leq b \leq 0.93$ and $0.5 \leq d \leq 0.93$.

$\frac{\partial c_1^s}{\partial t} = \frac{-445+2(459+368b+72b^2)d-32(2+b)^2(3+2b)d^2}{8(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)}$

I need to prove that the $G_1 = -445 + 2(459 + 368b + 72b^2)d - 32(2 + b)^2(3 + 2b)d^2 > 0$ and $H_1 = 8(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2) > 0$.

Step 1. To prove that $G_1 > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{0.6 \leq b \leq 0.93 \\ 0.5 \leq d \leq 0.93}} G_1(b, d)$$

The minimum value that G_1 can attain is 6.39, when $b = 0.93$ and $d = 0.93$

Step 2. To prove that $H_1 > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{0.6 \leq b \leq 0.93 \\ 0.5 \leq d \leq 0.93}} H_1(b,d)$$

The minimum value that H_1 can attain is 2.13, when $b = 0.6$ and $d = 0.5$ ■

Proposition 4

Claim: $\frac{\partial c_2^s}{\partial t} > 0$ when $0.6 \leq b \leq 1.3$ and $0.5 \leq d \leq 1.2$

$\frac{\partial c_2^s}{\partial t} = \frac{-243+2(253+204b+32b^2)d-8(24+34b+15b^2+2b^3)d^2}{4(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)}$

I need to prove that the $I_1 = -243 + 2(253 + 204b + 32b^2)d - 8(24 + 34b + 15b^2 + 2b^3)d^2 > 0$ and $K_1 = 4(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2) > 0$.

Step 1. To prove that $I_1 > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{0.6 \leq b \leq 1.3 \\ 0.5 \leq d \leq 1.2}} I_1(b,d)$$

The minimum value that I_1 can attain is 2.16, when $b = 1.3$ and $d = 1.2$

Step 2. To prove that $K_1 > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{0.6 \leq b \leq 1.3 \\ 0.5 \leq d \leq 1.2}} K_1(b,d)$$

The minimum value that K_1 can attain is 1.07, when $b = 0.6$ and $d = 0.5$ ■

Proposition 5

Claim: $\frac{\partial c_2^s}{\partial t} < 0$ when $b \geq 1.4$ and $d \geq 1.2$

$\frac{\partial c_2^s}{\partial t} = \frac{-243+2(253+204b+32b^2)d-8(24+34b+15b^2+2b^3)d^2}{4(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)}$

I need to prove that the $I = -243 + 2(253 + 204b + 32b^2)d - 8(24 + 34b + 15b^2 + 2b^3)d^2 < 0$

and $K = 4(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2) > 0$.

Step 1. To prove that $I < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b \geq 1.4 \\ d \geq 1.2}} I(b, d)$$

The maximum value that I can attain is -26.57, when $b = 1.4$ and $d = 1.2$

Step 2. To prove that $K > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 1.4 \\ d \geq 1.2}} K(b, d)$$

The minimum value that K can attain is $5,354.68$, when $b = 1.4$ and $d = 1.2$ ■

Proposition 6

Claim: $\frac{\partial c^c}{\partial t} > 0$ when $0.6 \leq b \leq 1$ and $0.5 \leq d \leq 1$

$\frac{\partial c^c}{\partial t} = \frac{8-(4+3b)d}{-10+(8+18b+9b^2)d}$

I need to prove that the $L_1 = 8 - (4 + 3b)d > 0$ and $M_1 = -10 + (8 + 18b + 9b^2)d > 0$.

Step 1. To prove that $L_1 > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{0.6 \leq b \leq 1 \\ 0.5 \leq d \leq 1}} L_1(b, d)$$

The minimum value that L_1 can attain is 1, when $b = 1$ and $d = 1$

Step 2. To prove that $M_1 > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{0.6 \leq b \leq 1 \\ 0.5 \leq d \leq 1}} M_1(b, d)$$

The minimum value that M_1 can attain is 1.02, when $b = 0.6$ and $d = 0.5$ ■

Proposition 7

Claim: $\frac{\partial c^c}{\partial t} < 0$ when $b \geq 1.1$ and $d \geq 1.1$

$\frac{\partial c^c}{\partial t} = \frac{8-(4+3b)d}{-10+(8+18b+9b^2)d}$

I need to prove that the $L = 8 - (4 + 3b)d < 0$ and $M = -10 + (8 + 18b + 9b^2)d > 0$.

Step 1. To prove that $L < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b \geq 1.1 \\ d \geq 1.1}} L(b, d)$$

The maximum value that L can attain is -0.03, when $b = 1.1$ and $d = 1.1$

Step 2. To prove that $M > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 1.1 \\ d \geq 1.1}} M(b, d)$$

The minimum value that M can attain is 32.56, when $b = 1.1$ and $d = 1.1$ ■

Proposition 8

Claim: $c_1^s > c^c > c_2^s$ and $c^c > c^a$

To prove it, I only need to show that $c_1^s - c^c > 0$, $c^c - c_2^s > 0$ and $c^c - c^a > 0$.

Lemma 8.1: $c_1^s > c^c$ *when* $b \geq 0.5$, $d \geq 0.6$ *and* $A \geq 1$.

$c_1^s - c^c = (8Abd(48b^4d^2 + 2b^3d(-19 + 140d) + 3b^2d(-41 + 200d) + 2(-91 - 4d + 96d^2) + 2b(33 - 59d + 280d^2)) - (-354 + (196 - 246b + 773b^2)d + (208 + 84b - 902b^2 - 936b^3 - 272b^4)d^2 + 32b(24 + 70b + 75b^2 + 35b^3 + 6b^4)d^3)t)/(8(-10 + (8 + 18b + 9b^2)d)(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2))$

I need to prove that the $N = (8Abd(48b^4d^2 + 2b^3d(-19 + 140d) + 3b^2d(-41 + 200d) + 2(-91 - 4d + 96d^2) + 2b(33 - 59d + 280d^2)) - (-354 + (196 - 246b + 773b^2)d + (208 + 84b - 902b^2 - 936b^3 - 272b^4)d^2 + 32b(24 + 70b + 75b^2 + 35b^3 + 6b^4)d^3)t) > 0$ and $O = (8(-10 + (8 + 18b + 9b^2)d)(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)) > 0$.

Step 1. To prove that $N(A, b, d, t) > 0$, where $A > 0$ and $0 \leq t \leq 1$. I solve the following optimization problem with constraints:

$$\underset{b,d}{\text{Min}}\ N(A, b, d, t)$$

subject to

$$b \geq 0.5, \tag{3.1}$$

$$d \geq 0.6, \tag{3.2}$$

$$A = 1, \tag{3.3}$$

$$t = 1 \tag{3.4}$$

The minimum value that N can attain is 13.59, when $b = 0.5$, $d = 0.6$, $A = 1$ and $t = 1$.

Step 2. $\frac{\partial N(A,b,d,t)}{\partial A} = 8bd(48b^4d^2 + 2b^3d(-19 + 140d) + 3b^2d(-41 + 200d) + 2(-91 - 4d + 96d^2) + 2b(33 - 59d + 280d^2))$

To prove that $N_A(b, d) = 8bd(48b^4d^2 + 2b^3d(-19 + 140d) + 3b^2d(-41 + 200d) + 2(-91 - 4d + 96d^2) + 2b(33 - 59d + 280d^2)) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 0.5 \\ d \geq 0.6}} N_A(b, d)$$

The minimum value that N_A can attain is 65.04, when $b = 0.5$ and $d = 0.6$

Step 3. $\frac{\partial N(A,b,d,t)}{\partial t} = 354 - (196 - 246b + 773b^2)d - (208 + 84b - 902b^2 - 936b^3 - 272b^4)d^2 -$

$32b(24 + 70b + 75b^2 + 35b^3 + 6b^4)d^3$

To prove that $N_t(b, d) = 354 - (196 - 246b + 773b^2)d - (208 + 84b - 902b^2 - 936b^3 - 272b^4)d^2 - 32b(24 + 70b + 75b^2 + 35b^3 + 6b^4)d^3 < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b \geq 0.5 \\ d \geq 0.6}} N_t(b, d)$$

The maximum value that N_t can attain is -51.45, when $b = 0.5$ and $d = 0.6$

Step 4. To prove that $O(b, d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 0.5 \\ d \geq 0.6}} O(b, d)$$

The minimum value that O can attain is 47.74, when $b = 0.5$ and $d = 0.6$ ■

Lemma 8.2: *$c^c > c_2^s$ when $b \geq 2$, $d \geq 2.2$ and $A \geq 1$.*

$c^c - c_2^s = (4Abd(212 - 256d + 24b^4d^2 + 2b^2d(-55 + 92d) + b^3d(-25 + 116d) + 4b(1 - 68d + 24d^2)) - (382 - (732 + 646b + 491b^2)d + (272 + 644b + 970b^2 + 524b^3 + 64b^4)d^2 + 8b^2(24 + 46b + 29b^2 + 6b^3)d^3)t)/(4(-10 + (8 + 18b + 9b^2)d)(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2))$

I need to prove that the $R = (4Abd(212 - 256d + 24b^4d^2 + 2b^2d(-55 + 92d) + b^3d(-25 + 116d) + 4b(1 - 68d + 24d^2)) - (382 - (732 + 646b + 491b^2)d + (272 + 644b + 970b^2 + 524b^3 + 64b^4)d^2 + 8b^2(24 + 46b + 29b^2 + 6b^3)d^3)t) > 0$ and $S = (4(-10 + (8 + 18b + 9b^2)d)(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)) > 0$.

Step 1. To prove that $R(A, b, d, t) > 0$, where $A > 0$ and $0 \leq t \leq 1$. I solve the following optimization problem with constraints:

$$\underset{b,d}{\text{Min}}\ R(A,b,d,t)$$

subject to

$$b \geq 2, \tag{3.5}$$

$$d \geq 2.2, \tag{3.6}$$

$$A = 1, \tag{3.7}$$

$$t = 1 \tag{3.8}$$

The minimum value that R can attain is 337.84, when $b = 2$, $d = 2.2$, $A = 1$ and $t = 1$.

Step 2. $\frac{\partial R(A,b,d,t)}{\partial A} = 4bd(212 - 256d + 24b^4d^2 + 2b^2d(-55 + 92d) + b^3d(-25 + 116d) + 4b(1 - 68d + 24d^2))$

To prove that $R_A(b,d) = 4bd(212 - 256d + 24b^4d^2 + 2b^2d(-55 + 92d) + b^3d(-25 + 116d) + 4b(1 - 68d + 24d^2)) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 2 \\ d \geq 2.2}} R_A(b,d)$$

The minimum value that R_A can attain is $138,927$, when $b = 2$ and $d = 2.2$

Step 3. $\frac{\partial R(A,b,d,t)}{\partial t} = -382 + (732 + 646b + 491b^2)d - (272 + 644b + 970b^2 + 524b^3 + 64b^4)d^2 - 8b^2(24 + 46b + 29b^2 + 6b^3)d^3$

To prove that $R_t(b,d) = -382 + (732 + 646b + 491b^2)d - (272 + 644b + 970b^2 + 524b^3 + 64b^4)d^2 - 8b^2(24 + 46b + 29b^2 + 6b^3)d^3 < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b \geq 2 \\ d \geq 2.2}} R_t(b,d)$$

The maximum value that R_t can attain is $-138,590$, when $b = 2$ and $d = 2.2$

Step 4. To prove that $S(b,d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 2 \\ d \geq 2.2}} S(b,d)$$

The minimum value that S can attain is $8,092,090$, when $b = 2$ and $d = 2.2$ ■

Lemma 8.3: *$c^c > c^a$ when $b \geq 1.1$, $d \geq 1.1$ and $A \geq 3$.*

$c^c - c^a = \frac{Abd(6b^2d+4(4+d)+b(-13+11d))-(24-(20+41b+32b^2)d+(4+3b)(d+2bd)^2)t}{(-3+(1+2b)^2d)(-10+(8+18b+9b^2)d)}$

I need to prove that the $T = Abd(6b^2d + 4(4+d) + b(-13+11d)) - (24 - (20+41b+32b^2)d + (4+3b)(d+2bd)^2)t > 0$ and $U = (-3 + (1+2b)^2d)(-10 + (8+18b+9b^2)d) > 0$.

Step 1. To prove that $T(A,b,d,t) > 0$, where $A > 0$ and $0 \leq t \leq 1$. I solve the following optimization problem with constraints:

$$\operatorname*{Min}_{b,d} T(A,b,d,t)$$

subject to

$$b \geq 1.1, \tag{3.9}$$

$$d \geq 1.1, \tag{3.10}$$

$$A = 3, \tag{3.11}$$

$$t = 1 \tag{3.12}$$

The minimum value that T can attain is 99.20, when $b = 1.1$, $d = 1.1$, $A = 3$ and $t = 1$.

Step 2. $\frac{\partial T(A,b,d,t)}{\partial A} = bd(6b^2d + 4(4+d) + b(-13+11d))$

To prove that $T_A(b,d) = bd(6b^2d+4(4+d)+b(-13+11d)) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b\geq 1.1\\ d\geq 1.1}} T_A(b,d)$$

The minimum value that T_A can attain is 33.15, when $b = 1.1$ and $d = 1.1$

Step 3. $\frac{\partial T(A,b,d,t)}{\partial t} = -24 + (20+41b+32b^2)d - (4+3b)(d+2bd)^2$

To prove that $T_t(b,d) = -24+(20+41b+32b^2)d-(4+3b)(d+2bd)^2 < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b\geq 1.1\\ d\geq 1.1}} T_t(b,d)$$

The maximum value that T_t can attain is -0.25, when $b = 1.1$ and $d = 1.1$

Step 4. To prove that $U(b,d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b\geq 1.1\\ d\geq 1.1}} U(b,d)$$

The minimum value that U can attain is 269.07, when $b = 1.1$ and $d = 1.1$ ■

Proof of $c_1^s < c^c$ *when* $b < 0.5$ *and* $0.4 \leq d \leq 0.55$, $A \geq 2$

Claim: $c_1^s < c^c$ when $0.41 \leq b \leq 0.45$ and $d < 0.6$, $A \geq 2$

$c_1^s - c^c = (8Abd(48b^4d^2+2b^3d(-19+140d)+3b^2d(-41+200d)+2(-91-4d+96d^2)+2b(33-59d+280d^2)) - (-354+(196-246b+773b^2)d+(208+84b-902b^2-936b^3-272b^4)d^2+32b(24+70b+75b^2+35b^3+6b^4)d^3)t)/(8(-10+(8+18b+9b^2)d)(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))$

I need to prove that the $N_1 = (8Abd(48b^4d^2+2b^3d(-19+140d)+3b^2d(-41+200d)+2(-91-4d+96d^2)+2b(33-59d+280d^2)) - (-354+(196-246b+773b^2)d+(208+84b-902b^2-

$936b^3 - 272b^4)d^2 + 32b(24 + 70b + 75b^2 + 35b^3 + 6b^4)d^3)t) < 0$ and $O_1 = (8(-10 + (8 + 18b + 9b^2)d)(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)) > 0$.

Step 1. To prove that $N_1(A, b, d, t) < 0$, where $A > 0$ and $0 \leq t \leq 1$. I solve the following optimization problem with constraints:

$$\underset{b,d}{\text{Max}}\ N_1(A, b, d, t)$$

subject to

$$0.41 \leq b \leq 0.45, \tag{3.13}$$

$$0.4 \leq d \leq 0.55, \tag{3.14}$$

$$A = 2, \tag{3.15}$$

$$t = 1 \tag{3.16}$$

The maximum value that N_1 can attain is -34.16, when $b = 0.45$, $d = 0.55$, $A = 2$ and $t = 1$

Step 2. $\frac{\partial N_1(A,b,d,t)}{\partial A} = 8bd(48b^4d^2 + 2b^3d(-19 + 140d) + 3b^2d(-41 + 200d) + 2(-91 - 4d + 96d^2) + 2b(33 - 59d + 280d^2))$

To prove that $N_{1A}(b, d) = 8bd(48b^4d^2 + 2b^3d(-19 + 140d) + 3b^2d(-41 + 200d) + 2(-91 - 4d + 96d^2) + 2b(33 - 59d + 280d^2)) < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{0.41 \leq b \leq 0.45 \\ 0.4 \leq d \leq 0.55}} N_{1A}(b, d)$$

The maximum value that N_{1A} can attain is -43.82, when $b = 0.45$ and $d = 0.55$

Step 3. $\frac{\partial N_1(A,b,d,t)}{\partial t} = 354 - (196 - 246b + 773b^2)d - (208 + 84b - 902b^2 - 936b^3 - 272b^4)d^2 - 32b(24 + 70b + 75b^2 + 35b^3 + 6b^4)d^3$

To prove that $N_{1t}(b,d) = 354-(196-246b+773b^2)d-(208+84b-902b^2-936b^3-272b^4)d^2-32b(24+70b+75b^2+35b^3+6b^4)d^3 > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{0.41\leq b\leq 0.45\\ 0.4\leq d\leq 0.55}} N_{1t}(b,d)$$

The minimum value that N_{1t} can attain is 53.48, when $b = 0.45$ and $d = 0.55$

Step 4. To prove that $O_1(b,d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{0.41\leq b\leq 0.45\\ 0.4\leq d\leq 0.55}} O_1(b,d)$$

The minimum value that O_1 can attain is 1.63, when $b = 0.45$ and $d = 0.55$ ■

Discontinuous areas of $c^c - c_2^s$

$c^c - c_2^s = (4Abd(212-256d+24b^4d^2+2b^2d(-55+92d)+b^3d(-25+116d)+4b(1-68d+24d^2))-(382-(732+646b+491b^2)d+(272+644b+970b^2+524b^3+64b^4)d^2+8b^2(24+46b+29b^2+6b^3)d^3)t)/(4(-10+(8+18b+9b^2)d)(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))$

When $(4(-10+(8+18b+9b^2)d)(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)) = 0$ are the areas where $c^c - c_2^s$ is discontinuous.

The following equations are these areas:

$d = \frac{10}{(8+18b+9b^2)}$

$d = \frac{(164+220b+73b^2-Sqrt[2320+480b+1688b^2+3448b^3+1233b^4])}{(2(96+280b+276b^2+112b^3+16b^4))}$

$d = \frac{(164+220b+73b^2+Sqrt[2320+480b+1688b^2+3448b^3+1233b^4])}{(2(96+280b+276b^2+112b^3+16b^4))}$ ■

Discontinuous areas of $c^c - c^a$

$c^c - c^a = \frac{Abd(6b^2d+4(4+d)+b(-13+11d))-(24-(20+41b+32b^2)d+(4+3b)(d+2bd)^2)t}{(-3+(1+2b)^2d)(-10+(8+18b+9b^2)d)}$

When $(-3+(1+2b)^2d)(-10+(8+18b+9b^2)d) = 0$ are the areas where $c^c - c^a$ is discontinuous.

The following equations are these areas:

$d = \frac{3}{(1+2b)^2}$

$d = \frac{10}{8+18b+9b^2}$ ■

Proposition 9

Claim: $e_1^s > e^c > e_2^s$ and $e^c > e^a$

In order to prove proposition 9, I prove the following three Lemmas.

Lemma 9.1: $e_1^s > e^c$ *when* $b \geq 4.3$, $d \geq 1$ *and* $A \geq 27$.

$e_1^s - e^c = (8Abd(408 + b(238 - 856d) - 384d + 5b^3d + 38b^4d - 2b^2(33 + 239d)) + (-604b^4d^2 - 272b^5d^2 + b^3d(773 + 1186d) + 2b^2d(67 + 1750d) + 24(-19 + 10d + 8d^2) + b(-354 - 852d + 2048d^2))t)/(8(-10+(8+18b+9b^2)d)(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))$

I need to prove that the $X_1 = (8Abd(408 + b(238 - 856d) - 384d + 5b^3d + 38b^4d - 2b^2(33 + 239d)) + (-604b^4d^2 - 272b^5d^2 + b^3d(773 + 1186d) + 2b^2d(67 + 1750d) + 24(-19 + 10d + 8d^2) + b(-354 - 852d + 2048d^2))t) > 0$ and $Y_1 = (8(-10 + (8 + 18b + 9b^2)d)(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)) > 0$.

Step 1. To prove that $X_1(A, b, d, t) > 0$, where $A > 0$ and $0 \leq t \leq 1$. I solve the following optimization problem with constraints:

$$\underset{b,d}{\text{Min}}\ X_1(A, b, d, t)$$

subject to

$$b \geq 4.3, \tag{3.17}$$

$$d \geq 1, \tag{3.18}$$

$$A = 27, \tag{3.19}$$

$$t = 1 \tag{3.20}$$

The minimum value that X_1 can attain is $267,576$, when $b = 4.3$, $d = 1$, $A = 27$ and $t = 1$.

Step 2. $\frac{\partial X_1(A,b,d,t)}{\partial A} = 8bd(408 + b(238 - 856d) - 384d + 5b^3d + 38b^4d - 2b^2(33 + 239d))$

To prove that $X_{1A}(b,d) = 8bd(408+b(238-856d)-384d+5b^3d+38b^4d-2b^2(33+239d)) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 4.3 \\ d \geq 1}} X_{1A}(b,d)$$

The minimum value that X_{1A} can attain is $23,977.4$, when $b = 4.3$ and $d = 1$

Step 3. $\frac{\partial X_1(A,b,d,t)}{\partial t} = -604b^4d^2 - 272b^5d^2 + b^3d(773 + 1186d) + 2b^2d(67 + 1750d) + 24(-19 + 10d + 8d^2) + b(-354 - 852d + 2048d^2)$

To prove that $X_{1t}(b,d) = -604b^4d^2 - 272b^5d^2 + b^3d(773+1186d) + 2b^2d(67+1750d) + 24(-19+10d + 8d^2) + b(-354 - 852d + 2048d^2) < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b \geq 4.3 \\ d \geq 1}} X_{1t}(b,d)$$

The maximum value that X_{1t} can attain is $-379,815$, when $b = 4.3$ and $d = 1$

Step 4. To prove that $Y_1(b,d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 4.3 \\ d \geq 1}} Y_1(b,d)$$

The minimum value that Y_1 can attain is $35,560,300$, when $b = 4.3$ and $d = 1$ ■

Lemma 9.2: *$e^c > e_2^s$ when $b \geq 0.8$, $d \geq 0.7$ and $A \geq 0$.*

$e^c - e_2^s = (4Abd(-8 + 64d + 76b^3d + 25b^4d + 4b(-47 + 24d) + b^2(-4 + 84d)) + (671b^4d^2 + 64b^5d^2 + 2b^2d(-521 + 1234d) + b^3d(-491 + 2016d) + 4(67 - 168d + 80d^2) + 2b(191 - 820d + 688d^2))t)/(4(-10+(8+18b+9b^2)d)(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))$

I need to prove that the $X_2 = (4Abd(-8 + 64d + 76b^3d + 25b^4d + 4b(-47 + 24d) + b^2(-4 + 84d)) + (671b^4d^2 + 64b^5d^2 + 2b^2d(-521 + 1234d) + b^3d(-491 + 2016d) + 4(67 - 168d + 80d^2) + 2b(191 - 820d + 688d^2))t) > 0$ and $Y_2 = (4(-10 + (8 + 18b + 9b^2)d)(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)) > 0$.

Step 1.To prove that $X_2(A, b, d, t) > 0$, where $A > 0$ and $0 \leq t \leq 1$. I solve the following optimization problem with constraints:

$$\underset{b,d}{\text{Min}}\ X_2(A, b, d, t)$$

subject to

$$b \geq 0.8, \tag{3.21}$$

$$d \geq 0.7, \tag{3.22}$$

$$A = 0.01, \tag{3.23}$$

$$t = 0 \tag{3.24}$$

The minimum value that X_2 can attain is 0.2159, when $b = 0.8$, $d = 0.7$, $A = 0.01$ and $t = 0$.

Step 2. $\frac{\partial X_2(A,b,d,t)}{\partial A} = 4bd(-8 + 64d + 76b^3d + 25b^4d + 4b(-47 + 24d) + b^2(-4 + 84d))$

To prove that $X_{2A}(b, d) = 4bd(-8 + 64d + 76b^3d + 25b^4d + 4b(-47 + 24d) + b^2(-4 + 84d)) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 0.8 \\ d \geq 0.7}} X_{2A}(b, d)$$

The minimum value that X_{2A} can attain is 21.59, when $b = 0.8$ and $d = 0.7$

Step 3. $\frac{\partial X_2(A,b,d,t)}{\partial t} = 671b^4d^2 + 64b^5d^2 + 2b^2d(-521 + 1234d) + b^3d(-491 + 2016d) + 4(67 - 168d + 80d^2) + 2b(191 - 820d + 688d^2)$

To prove that $X_{2t}(b,d) = 671b^4d^2 + 64b^5d^2 + 2b^2d(-521+1234d) + b^3d(-491+2016d) + 4(67-168d+80d^2) + 2b(191-820d+688d^2) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b\geq 0.8\\ d\geq 0.7}} X_{2t}(b,d)$$

The minimum value that X_{2t} can attain is 662.89, when $b = 0.8$ and $d = 0.7$

Step 4. To prove that $Y_2(b,d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b\geq 0.8\\ d\geq 0.7}} Y_2(b,d)$$

The minimum value that Y_2 can attain is $2,640.09$, when $b = 0.8$ and $d = 0.7$ ■

Lemma 9.3: $e^c > e^a$ *when* $b \geq 2.5$, $d \geq 0.4$ *and* $A \geq 10$

$$e^c - e^a = \frac{Ab(-14-14b+13b^2)d - 2(1+4b)(-3+(1+2b)^2d)t}{(-3+(1+2b)^2d)(-10+(8+18b+9b^2)d)}$$

I need to prove that the $X_3 = Ab(-14-14b+13b^2)d - 2(1+4b)(-3+(1+2b)^2d)t > 0$ and $Y_3 = (-3+(1+2b)^2d)(-10+(8+18b+9b^2)d) > 0$.

Step 1. To prove that $X_3(A,b,d,t) > 0$, where $A > 0$ and $0 \leq t \leq 1$. I solve the following optimization problem with constraints:

$$\underset{b,d}{\text{Min}}\ X_3(A, b, d, t)$$

subject to

$$b \geq 2.5, \tag{3.25}$$

$$d \geq 0.4, \tag{3.26}$$

$$A = 10, \tag{3.27}$$

$$t = 1 \tag{3.28}$$

The minimum value that X_3 can attain is 71.7, when $b = 2.5$, $d = 0.4$, $A = 10$ and $t = 1$.

Step 2. $\frac{\partial X_3(A,b,d,t)}{\partial A} = b(-14 - 14b + 13b^2)d$

To prove that $X_{3A}(b, d) = b(-14 - 14b + 13b^2)d > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 2.5 \\ d \geq 0.4}} X_{3A}(b, d)$$

The minimum value that X_{3A} can attain is 32.25, when $b = 2.5$ and $d = 0.4$

Step 3. $\frac{\partial X_3(A,b,d,t)}{\partial t} = -2(1 + 4b)(-3 + (1 + 2b)^2 d)$

To prove that $X_{3t}(b, d) = -2(1 + 4b)(-3 + (1 + 2b)^2 d) < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b \geq 2.5 \\ d \geq 0.4}} X_{3t}(b, d)$$

The maximum value that X_{3t} can attain is -250.8, when $b = 2.5$ and $d = 0.4$

Step 4. To prove that $Y_3(b, d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 2.5 \\ d \geq 0.4}} Y_3(b, d)$$

The minimum value that Y_3 can attain is 384.18, when $b = 2.5$ and $d = 0.4$ ■

Discontinuous areas of $e^s_1 - e^c$

$e^s_1 - e^c = (8Abd(408 + b(238 - 856d) - 384d + 5b^3d + 38b^4d - 2b^2(33 + 239d)) + (-604b^4d^2 - 272b^5d^2 + b^3d(773 + 1186d) + 2b^2d(67 + 1750d) + 24(-19 + 10d + 8d^2) + b(-354 - 852d + 2048d^2))t)/(8(-10+(8+18b+9b^2)d)(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))$

When $(8(-10+(8+18b+9b^2)d)(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)) = 0$ are the areas where $e^s_1 - e^c$ is discontinuous.

The following equations are these areas:

$d = \frac{10}{(8+18b+9b^2)}$

$d = \frac{(164+220b+73b^2-Sqrt[2320+480b+1688b^2+3448b^3+1233b^4])}{(2(96+280b+276b^2+112b^3+16b^4))}$

$d = \frac{(164+220b+73b^2+Sqrt[2320+480b+1688b^2+3448b^3+1233b^4])}{(2(96+280b+276b^2+112b^3+16b^4))}$ ■

Discontinuous areas of $e^c - e^s_2$

$e^c - e^s_2 = (4Abd(-8 + 64d + 76b^3d + 25b^4d + 4b(-47 + 24d) + b^2(-4 + 84d)) + (671b^4d^2 + 64b^5d^2 + 2b^2d(-521 + 1234d) + b^3d(-491 + 2016d) + 4(67 - 168d + 80d^2) + 2b(191 - 820d + 688d^2))t)/(4(-10+(8+18b+9b^2)d)(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))$

When $(4(-10+(8+18b+9b^2)d)(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)) = 0$ are the areas where $e^c - e^s_2$ is discontinuous.

The following equations are these areas:

$d = \frac{10}{(8+18b+9b^2)}$

$d = \frac{(164+220b+73b^2-Sqrt[2320+480b+1688b^2+3448b^3+1233b^4])}{(2(96+280b+276b^2+112b^3+16b^4))}$

$d = \frac{(164+220b+73b^2+Sqrt[2320+480b+1688b^2+3448b^3+1233b^4])}{(2(96+280b+276b^2+112b^3+16b^4))}$ ■

Discontinuous areas of $e^c - e^a$

$e^c - e^a = \frac{Ab(-14-14b+13b^2)d-2(1+4b)(-3+(1+2b)^2d)t}{(-3+(1+2b)^2d)(-10+(8+18b+9b^2)d)}$

When $(-3+(1+2b)^2d)(-10+(8+18b+9b^2)d) = 0$ are the areas where $e^c - e^a$ is discontinuous.

The following equations are these areas:

$d = \frac{3}{(1+2b)^2}$

$d = \frac{10}{(8+18b+9b^2)}$ ■

Chapter 4

Welfare in a model of International Trade with competition à la Stackelberg. Comparison with the Cournot competition and the Autarky case

4.1 Introduction

Debates over how the strategic use of environmental policies are affected by international trade are also concerned with the repercussions for social welfare in the countries. Brander and Spencer [2] [3] develop a Cournot model in which they establish the existence of "rent-shifting" incentives for a strategic trade policy, and conclude that this strategic behavior could reduce global welfare.

Eaton and Grossman [7] find that in a Cournot duopoly it is optimal to subsidize both production and export tax in order to increase social welfare. Conrad [5] finds that when there is a free trade agreement in a duopoly model, it is optimal to under-internalize the pollution externality, because environmental policy can be by government as an instrument to subsidize local firms. Walz [11] finds that although some countries use "ecological dumping" to subsidize its local firms, countries that export will still prefer free trade agreements as these trade agreements increase social welfare.

Furthermore, Markusen, Morey and Olewiler [9] find that an optimal environmental policy needs to not only take the level of pollution into account, but also the relative cost that the environmental policy represents to the local firm as compared to other countries with low environmental policies. Therefore, if the local environmental policy is higher than in the foreign country, the firm will shut down its local plant and move to the foreign country, with the consequence that local welfare decreases.

In contrast, Antweiler, Copeland and Taylor [1] discover three effects that change levels of pollution."The scale effect", which is the effect of production intensity; "the composition effect", which is the effect of the composition of the goods that a country produces; and "the technique effect", which is the effect of the intensity of the abatement. Antweiler, Copeland and Taylor [1] [6] find, through these effects, free trade decreases world pollution.

Moreover, Grossman and Krueger [8] search for a correlation between growth and pollution, where trade is an important variable for growth. They find that environmental quality deteriorates in the beginning of economic growth, but improves in a subsequent phase together with, by consequence, an increase in social welfare.

Additionally, Burguet and Sempere [4] also analyze the effect of gradually reducing commerce tariffs on country welfare. They find that welfare will typically increase as a result of trade liberalization. But they also find exceptions depending on the environmental instrument that government chooses. If government chooses a standard regulation, and as commerce tariffs decrease, this may cause a parallel decrease in welfare.

I investigate the effects of a gradual reduction of commerce tariffs in a Stackelberg model. In my second chapter, where I incorporate asymmetric competition, I analyzed whether or not one effect dominates in the leader firm country's environmental policy, and if the another effect dominates in the follower firm country's environmental policy.

In that chapter, I find that the environmental policy of the leader firm country is always higher than the environmental policy of the follower firm country.

I also compare the results of the Stackelberg model with the Cournot model. I find that the environmental policy of a country with a firm competing à la Cournot is lower than the environmental policy of the leader firm country in a Stackelberg model. However, a country with a firm competing à la Cournot is higher than the policy of the follower firm country.

In this chapter, I analyze the welfare of the countries. I want to know if trade liberalization increases or decreases social welfare. Moreover, I compare the effect of reducing commerce tariffs on the welfare of countries with different market structures in order to determine whether or not some countries have greater incentives to engage in free trade.

As commerce tariffs decrease in the case of low or medium abatement efficiency levels, I find that welfare of the leader firm country increases less than the welfare of the follower firm country in the Stackelberg model. This result suggests that the follower firm country has greater incentives for a trade agreement in a Stackelberg model, as it gains more market share while decreasing its environmental policy.

In this paper, I also compare the results of the Stackelberg model with the Cournot model. In the case of medium or high abatement efficiency level, I find that, as commerce tariffs decrease, the welfare of the follower firm country that competes à la Stackelberg increases less than the welfare of the country with a firm that competes à la Cournot.

The previous result suggests that the countries that gain most from trade agreements are those with a Cournot market structure. In the Cournot model, as commerce tariffs decrease, countries gain market with an environmental policy that is not as low as in the follower firm country, as the former do not need to increase production as much as the follower firm country. Additionally, its production level is lower than a leader firm country. Therefore, its environmental externality is lower than in a leader firm country that competes à la Stackelberg.

Finally, in this model, all countries with imperfect competition have higher welfare than the Autarky country. Because their level of production is higher, they have a greater surplus than in the Autarky case.

This model explains how environmental policy affects the social welfare of countries, and which countries have greater incentives to enter into environmental agreements, such as the Kyoto protocol, depending on the international competition structure. This paper is relevant for developing more efficient environmental agreements.

The chapter is organized in the following manner. In the following Section 2, the model is introduced. In Section 3, the results of the Stackelberg model are analyzed and compared with other results of other market structures. Finally, conclusions are drawn in Section 4.

4.2 The Stackelberg competition model and adaptations of the model to other market structures

This game is based on the Burguet and Sempere model [4] with Stackelberg market structure. I assume that there are two countries: Home (1) and Foreign (2). There is only one tradable good, which is produced by two firms. Firm (1) is in Home country and Firm (2) is in Foreign. Each firm sells in both countries, but they only produce in their own countries. Neither exchange rates nor transportation costs are in play. Firms compete in quantities in both markets. Firm (1) is the Leader and Firm (2) is the Follower in both markets.

Demand is the same in each country. The price for the good is given by the inverse demand function and, for simplicity, it is linear, $P(Q) = A - Q$. The price in Home is $P(Q_1)$ and $P(Q_2)$ in Foreign.

Firms (1) and (2) have a constant marginal cost. The marginal cost is affected by the environmental regulation. For simplicity, I will assume that marginal cost coincides with the value of the environmental policy.

Governments choose their environmental policy, c_i, in order to maximize the social welfare of its country. Imports to each country are subject to a tariff t_i, which for further simplicity, we assume is exogenous and the same in both countries.

The model is a three stage game. In the first stage, Home and Foreign governments simultaneously choose their environmental policy, c_1^s and c_2^s respectively. In the second stage, after observing the environment policies, Leader Firm chooses its output for Home and for Foreign market, q_{11} and q_{12} respectively. In the third stage, after observing the environmental policies and the production of Leader Firm, Follower Firm (2) chooses its output for Home and for Foreign market, q_{21} and q_{22} respectively. Let the total output in Home market be $Q_1 = q_{11} + q_{21}$ and in Foreign market be $Q_2 = q_{12} + q_{22}$.

The production of the good creates emissions, e_i^s, which are local. Emissions, e_i^s, are equal to output produced in the country minus the abatement. That is, $e_i^s = ((q_{i1} + q_{i2}) - bc_i^s)$, where b is the level of efficiency of the abatement policy and also a positive constant. Based in Ulph [10], the environmental externality can be modeled as the total pollution that creates the emissions $\frac{d}{2}((q_{i1} + q_{i2}) - bc_i^s)^2$, where d is the level of damage that provokes the emissions and is a positive constant. The Social Cost, h_i, is defined as the sum of the private cost from the firm plus the environmental externality.

Therefore, the Social Cost is:

$$h_i(c_i^s, q_{i1} + q_{i2}) = (q_{i1} + q_{i2})\, c_i^s + \frac{d}{2}\left((q_{i1} + q_{i2}) - bc_i^s\right)^2$$

where $i \in \{1, 2\}$. The Social Cost function is convex with respect to environmental policy.[1]

Finally, Welfare is the total social surplus minus social cost, plus exports, minus imports, plus any

[1] Go to Appendix to see the proof.

tariff revenue. Therefore, the Welfare function is:

$$W_i^S = \int_0^{q_{1io}+q_{2io}} P_i dQ_i - h_i(c_i^s, q_{i1o} + q_{i2o}) - P_i q_{jio} + P_j q_{ijo} - t(q_{ijo} - q_{jio})$$

where $i \in \{1, 2\}$, q_{iio} and q_{ijo} are the second stage equilibrium output if $i = 1$ and are the third stage equilibrium output if $i = 2$.

The five terms on the right-hand side are, respectively, gross (domestic) surplus, social cost of domestic output, the value of imports, the value of exports, and the net tariff revenue.

The model will be solved with a Subgame Perfect Equilibrium, therefore it will be calculated with backward induction. This problem has a unique equilibrium as the Welfare function is strictly concave.[2]

Third stage: given the environment policies, c_1^s and c_2^s, and the output of Firm 1, Firm 2 maximizes its benefits by choosing its output for Home market q_{21} and for Foreign Market q_{22}. Notice that the output of firm 2 to Home Market is exports. Therefore, marginal cost of Firm 2 for Foreign market is just c_2^s, instead of marginal cost for Home market, which is $c_2^s + t$.

Second stage: given the environment policies, c_1^s and c_2^s, Firm 1 maximizes its benefits by choosing its output for Home market q_{11} and for Foreign Market q_{12}. Notice that the output of Firm 1 to Foreign Market is exports. Therefore, marginal cost of Firm 1 for Home market is just c_1^s, instead of marginal cost for Foreign market that is $c_1^s + t$.

First stage: governments maximize their national Welfare by choosing their environmental policy, c_i^s.[3]

4.2.1 The Cournot competition model

The results obtained in the Stackelberg Model are compared with the results obtained with the Cournot Model.

[2]Go to Appendix to see the proof.
[3]Go to Appendix to see the specific solution of the Stackelberg model. It is not presented in the text because of its complexity.

It is necessary to analyze the results with the Cournot competition, in order to see if these results are equivalent to the ones that have already found in literature.

Additionally, it is important to compare the results obtained in the Stackelberg case with the Cournot competition, to analyze if the Stackelberg model, an asymmetric competition, can tell us which effect dominates the leader country and which one dominates the follower country, explaining something more of the behavior of the countries, when they are in a different market structure.

The Cournot model is a two stage game. In the first stage, Home and Foreign governments simultaneously choose their environmental policy, c_1^c and c_2^c respectively. In the second stage, after observing the environment policies, Home and Foreign firm choose their output for Home and for Foreign market, q_{11} and q_{12}, and q_{21} and q_{22} respectively.

The production of the good creates emissions, e_i^c, which are local. Emissions, e_i^c, are equal to output produced in the country minus the abatement. That is, $e_i^c = ((q_{i1} + q_{i2}) - bc_i^s)$, where b is the level of efficiency of the abatement policy and also a positive constant.

The Social cost is the same as the Stackelberg case.

The Welfare function is:

$$W_i^c = \int_0^{q_{1io}+q_{2io}} P_i dQ_i - h_i(c_i^c, q_{i1o} + q_{i2o}) - P_i q_{jio} + P_j q_{ijo} - t(q_{ijo} - q_{jio})$$

where $i \,\epsilon\, \{1, 2\}$, q_{iio} and q_{ijo} are the second stage equilibrium output.

The five terms on the right-hand side are, respectively, gross (domestic) surplus, social cost of domestic output, the value of imports, the value of exports, and the net tariff revenue.

The model will be solved with a Subgame Perfect Equilibrium. Therefore, it will be calculated with backward induction. This problem has a unique equilibrium, because the Welfare function is strictly concave.[4]

Second stage: given the environment policies, c_1^c and c_2^c, Home and Foreign Firm maximize their

[4]Go to Appendix to see the proof.

benefits by choosing their output for Home market q_{11} and q_{21} respectively and for Foreign Market q_{12} and q_{22} respectively. Notice that the output of Home Firm to Foreign market are exports, so its marginal cost for Foreign market is $c_1^c + t$. Analogous, the output of Foreign Firm to Home market are exports, so its marginal cost for Home market is $c_2^c + t$.

First stage: Governments maximize their national Welfare by choosing their environmental policy, c_i^c.[5]

4.2.2 The Autarky case

The Stackelberg case and the Cournot competition results are compared with the Autarky case. The objective of this is to compare the strategic behavior of environmental policy in oligopoly competition with the behavior of no competition, and analyze if these results are equivalent with the ones that have already found in literature.

The Autarky case is a two stage game. In the first stage, Home government chooses its environmental policy, c_1^a. In the second stage, after observing the environment policy, Home monopoly firm chooses its output for Home market, q_{11}.

The production of the good creates emissions, e_i^a, which are local. Emissions, e_i^a, are equal to output produced in the country minus the abatement. That is, $e_i^a = ((q_{i1} + q_{i2}) - bc_i^s)$, where b is the level of efficiency of the abatement policy and also a positive constant.

The Social Cost is the same as the Stackelberg case.

The Welfare function is:

$W_1^a = \int_0^{q_{11o}} P_1 dQ_1 - h_1(c_1^a, q_{11o})$

where q_{11o} is the second stage equilibrium output.

The terms on the right-hand side are, respectively, gross (domestic) surplus and social cost of domestic output.

[5]Go to Appendix to see the specific solution of the Cournot model. It is not presented in the text because of its complexity.

The model will be solved with a Subgame Perfect Equilibrium. Therefore, it will be calculated with backward induction. This problem has a unique equilibrium, because the Welfare function is strictly concave.[6]

Second stage: given the environment policy, c_1^a, Home monopoly firm maximizes its benefits by choosing its output for Home market q_{11}.

First stage: Government maximize its national Welfare by choosing its environmental policy, c_1^a.[7]

4.3 Results of the Welfare in the Stackelberg model. Comparison with the Cournot model and the Autarky case

In this section, I present the Welfare results analysis of this model. I also compare the behavior of the welfare of the Stackelberg countries with other market structures, specifically with the Autarky case and the competition à la Cournot.

4.3.1 How welfare of the countries that compete à la Stackelberg changes, as commerce tariffs decrease. Comparison with the Cournot model

The welfare of the leader firm country and the one of the country with the Cournot firm increase more than the welfare of the follower firm country, as commerce tariffs decrease.

Proposition 1: $\frac{\partial W_2^s}{\partial t} > \frac{\partial W_1^s}{\partial t}$ *and* $\frac{\partial W_2^s}{\partial t} > \frac{\partial W^c}{\partial t}$

In order to prove proposition 1, I prove the following two lemmas.

Lemma 1.1: $\frac{\partial W_2^s}{\partial t} > \frac{\partial W_1^s}{\partial t}$ *when* $b \geq 4.8$, $d \geq 0.1$ *and* $A \geq 0$. [8]

The proof of all propositions in this chapter is in the Appendix.

[6] Go to Appendix to see the proof.

[7] Go to Appendix to see the specific solution of Autarky case. It is not presented in the text because of its complexity.

[8] When $b < 4.8$ and $d < 0.1$, it is not possible to determine if $\frac{\partial W_2^s}{\partial t} > \frac{\partial W_1^s}{\partial t}$ or $\frac{\partial W_2^s}{\partial t} < \frac{\partial W_1^s}{\partial t}$, since the functions are discontinuous in that domain. Go to Appendix to see the specific discontinuous areas.

Welfare of the leader firm country that competes à la Stackelberg increases more than the one of the follower firm country that competes à la Stackelberg, as commerce tariffs decrease, at a high level of abatement efficiency.

This behavior is explained because at low levels of abatement efficiency, the leader firm country that competes à la Stackelberg has a high environmental policy, to reduce its pollution externality (see Chapter 3, Proposition 1). This is because it has a high level of production, besides the fact that high amounts of policy are needed to reduce the externality, because the pollution abatement is inefficient. Therefore, as commerce tariffs decrease, its environmental policy increases more, because its priority is to reduce its pollution externality, increasing very little its welfare.

Instead, at low level of abatement efficiency, the country where the follower firm is located is more benefited by having a low environmental policy, in order to have a more competitive firm rather than reducing its environmental externality, because its production level is low. Therefore, as commerce tariffs decrease, its environmental policy decreases, in order to gain more market share, increasing its welfare a lot.

Finally, at high level of abatement efficiency, the environmental policy of the leader firm country is medium, because it is easier to reduce pollution. Consequently, this country gains more market share, compared with itself at low levels of abatement efficiency, and also has control over its pollution externality.

Therefore, as commerce tariffs decrease, its environmental policy increases, which it is not high because of the high level of abatement efficiency, reducing its environmental externality, but still having a very high production level. Consequently, its welfare increases more than the welfare of the follower firm country.

Lemma 1.2: $\frac{\partial W_2^s}{\partial t} > \frac{\partial W^c}{\partial t}$ *when* $b \geq 2.9$, $d \geq 0.9$ *and* $A \geq 0.1$.[9]

The welfare of the country with the firm that competes à la Cournot increases more than that of the follower firm country in the Stackelberg model, as commerce tariffs decrease, from medium level of abatement efficiency.

[9]When $b < 2.9$ and $d < 0.9$, it is not possible to determine if $\frac{\partial W_2^s}{\partial t} > \frac{\partial W^c}{\partial t}$ or $\frac{\partial W_2^s}{\partial t} < \frac{\partial W^c}{\partial t}$, since the functions are discontinuous in that domain. Go to Appendix to see the specific discontinuous areas.

This behavior is explained because at low levels of abatement efficiency, the country with the firm in the Cournot model has a high-medium environmental policy to reduce its pollution externality, which is higher than in the follower firm country (see Chapter 3, Proposition 1). This is because the Cournot firm has a high-medium production level and it needs high amounts of policy to reduce the emissions, because the pollution abatement is inefficient.

Instead, at low level of abatement efficiency, the follower firm country in the Stackelberg model has incentives to increase its market share, because its production level is low and, consequently, its environmental policy is also low. Therefore, as commerce tariffs decrease, it decreases its environmental policy, in order for its firm to gain competitiveness to increase its production, increasing its welfare.

In the case of medium level of abatement efficiency, the environmental policy of the country with a firm that competes à la Cournot is lower, because it is easier to reduce pollution; consequently, this country gains more market, compared with itself at low level of abatement efficiency, and also has control over its pollution externality.

Therefore, as commerce tariffs decrease, it increases its environmental policy, which is not high because of the high level of abatement efficiency, reducing its environmental externality, but still having a high production level. Consequently, its welfare increases more than the welfare of the follower firm country.

A country with a Cournot firm needs a lower level of abatement efficiency than the leader firm country in the Stackelberg model, in order to increase its welfare more than the follower firm country, as commerce tariffs decrease. This is a consequence of the fact that a country with a Cournot firm does not have as high an environmental policy as a country with a leader firm, because the production of the Cournot firm is lower than the leader firm in the Stackelberg model (see Chapter 3, Proposition 1).

Proposition 2: $\frac{\partial W_1^s}{\partial t} < 0$ *when* $b \geq 3.7$, $d \geq 1$ *and* $A \geq 0$.[10]

The Welfare of the leader firm country in the Stackelberg model increases, when the commerce

[10]When $b < 3.7$ and $d < 1$, it is not possible to determine if $\frac{\partial W_1^s}{\partial t} < 0$ or $\frac{\partial W_1^s}{\partial t} > 0$, since the functions are discontinuous in that domain. Go to Appendix to see the specific discontinuous areas.

tariffs decrease, from high-medium level of abatement efficiency.

This behavior is explained because, at low levels of abatement efficiency, the leader firm country has a very high environmental policy, because its priority is to reduce its environmental externality, caused by its high level of production. Therefore, a reduction in commerce tariffs makes that the environmental policy increase more, making the cost of production higher; consequently, welfare does not increase.

Instead, at high-medium level of abatement efficiency, the leader firm country has a lower environmental policy, compared with the one at low level of abatement efficiency, because it is easier to reduce pollution. Therefore, it has a high production level, at medium cost, with a pollution externality which is not high. Consequently, because it already has a very high production level, as commerce tariffs decrease, its environmental policy increases, in order to have a controlled pollution externality, thus increasing its welfare.

Proposition 3: $\frac{\partial W_2^s}{\partial t} < 0$ *when* $b \geq 1.1$, $d \geq 1$ *and* $A \geq 0$. [11]

The welfare of the follower firm country in the Stackelberg model increases, when the commerce tariffs decrease from low level of both damage and abatement efficiency.

This behavior is explained because at low levels of both abatement efficiency and damage, the follower firm country has a low environmental policy, in order to gain market share, because its production level is low; therefore, its environmental externality is low. Consequently, as commerce tariffs decrease, it reduces its environmental policy, in order to its firm be more efficient and to gain more market share, thus increasing its welfare.

In the case of medium or high levels of both damage and abatement efficiency, as commerce tariffs decrease, its environmental policy increases, but it still has the lowest environmental policy, compared with the other countries, in order to gain more market share, but also having a low pollution externality, thus increasing its welfare.

Proposition 4: $\frac{\partial W^c}{\partial t} < 0$ *when* $b \geq 2.1$, $d \geq 1$ *and* $A \geq 0$. [12]

[11] When $b < 1.1$ and $d < 1$, it is not possible to determine if $\frac{\partial W_2^s}{\partial t} < 0$ or $\frac{\partial W_2^s}{\partial t} > 0$, since the functions are discontinuous in that domain. Go to Appendix to see the specific discontinuous areas.

[12] When $b < 2.1$ and $d < 1$, it is not possible to determine if $\frac{\partial W^c}{\partial t} < 0$ or $\frac{\partial W^c}{\partial t} > 0$, since the functions are discontinuous in

The welfare of the country with the firm in the Cournot model increases as commerce tariffs decrease, from medium level of both damage and abatement efficiency.

This behavior is explained because the country with a Country firm has a high environmental policy, when abatement efficiency level is low, because it needs to reduce its environmental externality, caused by its high-medium level of production. Therefore, a reduction in commerce tariffs makes the environmental policy increase more, making the cost of production higher; consequently, welfare does not increase.

Instead, at medium level of abatement efficiency, the country with the Cournot firm has a lower environmental policy, compared with the one at low level of abatement efficiency, because it is easier to reduce the pollution. Therefore, it has a higher production level, at low cost, with a pollution externality that is not high. Consequently, because it already has a high production level, as commerce tariffs decrease, its environmental policy increases, in order to reduce its level of pollution externality, thus increasing its welfare.

A country with a Cournot firm needs a lower level of abatement efficiency, as compared with the leader firm country, in order to increase its welfare, as commerce tariffs decrease. This is because the country with a Cournot firm does not have an environmental policy as high as that of the leader firm country, because the production level of the Cournot firm is lower than that of the leader firm in the Stackelberg model.

4.3.2 Welfare analysis of the countries that compete à la Stackelberg. Comparison with the Cournot model and the Autarky case

Welfare in the leader firm country in the Stackelberg model is higher than the one of the follower firm country, from a high-medium abatement efficiency level.

Proposition 5: $W_1^s > W_2^s$ *when* $b \geq 4.8$, $d \geq 1$ *and* $A \geq 3$.[13]

that domain. Go to Appendix to see the specific discontinuous areas.

[13] When $b < 4.8$ and $d < 1$, it is not possible to determine if $W_1^s > W_2^s$ or $W_1^s < W_2^s$, since the functions are discontinuous in that domain. Go to Appendix to see the specific discontinuous areas.

This behavior is explained because at low level of abatement efficiency, the country with leader firm has a very high environmental policy, in order to reduce its environmental externality. Although it has a high production and controlled environmental externality, it has them at a high cost.

Instead at low levels of abatement efficiency, the follower firm country has a low environmental policy, in order to increase its firm competitiveness and to be able to have a higher market share. Therefore, this country is increasing its production at low cost, and having a low pollution externality, because its production is not high.

However, when the abatement efficiency level is high-medium, the environmental policy of the leader firm country is lower, as compared to the one at low abatement efficiency, because less policy is needed to reduce pollution. Therefore, the leader firm gains more market share and also has a controlled environmental externality, at a low cost, thus having a higher welfare than the follower firm country.

Additionally, at this high-medium level of abatement efficiency, the welfare of the leader firm country increases more than that of the one of the follower firm country, as commerce tariffs decrease.

Proposition 6: $W_1^s > W^c$ *when* $b \geq 7.8$, $d \geq 0.8$ *and* $A \geq 2$.[14]

Welfare in the leader firm country in the Stackelberg model is higher than that in the country with the firm that competes à la Cournot, when abatement efficiency level is high.

This behavior is explained because, at low level of abatement efficiency, the leader firm country has a high environmental policy, in order to reduce its pollution externality, because it needs high amounts of policy to reduce pollution.

Besides, the leader firm has a higher production level than the Cournot firm, so the leader firm country has a controlled environmental externality, at a cost which is higher than in the case of the country with a Cournot firm.

However, when the abatement efficiency level is high, the environmental policy of the leader firm

[14]When $b < 7.8$ and $d < 0.8$, it is not possible to determine if $W_1^s > W^c$ or $W_1^s < W^c$, since the functions are discontinuous in that domain. Go to Appendix to see the specific discontinuous areas.

country is lower, as compared with the one at low abatement efficiency level, because less policy is necessary to reduce pollution. Therefore, the leader firm gains more market share and also has a controlled pollution externality, at a medium cost, thus having a higher welfare than the country with the Cournot firm.

Higher levels of abatement efficiency are necessary, in order to welfare in the leader firm country be higher than in a country with a Cournot firm, as compared with the level of abatement efficiency that is necessary for it to be higher than welfare in the follower firm country, in order to have more production at a lower cost than in the country with the Cournot firm.

Proposition 7: $W^c > W^a$ *when* $b \geq 2.1$, $d \geq 0.6$ *and* $A \geq 3$.[15]

Welfare in the Autarky country is the lowest, as compared with a different market structure, from medium abatement efficiency level.

This behavior is explained because the Cournot firm produces more than the Autarky case. Despite the fact that a country with a Cournot firm has a higher environmental policy than the Autarky country (see Chapter 3, Proposition 8), its emissions are higher (see Chapter 3, Proposition 9). Therefore, even the country with the Cournot firm has an environmental externality which is higher than the Autarky country, it has a higher welfare, because Cournot firm has a higher level of production.

At least medium levels of abatement efficiency and a minimum demand size are necessary, because the production in the Autarky country is higher than the production in the Cournot firm, at low level of abatement efficiency and with a low demand size. This is because the Autarky country does not have any competition, so its firm has all the market.

Therefore, with a higher demand size, the production of the Cournot firm increases more than the case of the monopoly in the Autarky country, because the Cournot firm produces both for its market and for the foreign market.

[15]When $b < 2.1$ and $d < 0.6$, it is not possible to determine if $W^c > W^a$ or $W^c < W^a$, since the functions are discontinuous in that domain. Go to Appendix to see the specific discontinuous areas.

4.4 Conclusions

This paper shows that in the Stackelberg model, as commerce tariffs decrease, the welfare of the leader firm country increases less than the welfare of the follower firm country, at a low or medium abatement efficiency level.

In the case of medium or high abatement efficiency levels, as commerce tariffs decrease, I find that the welfare of the follower firm country that competes à la Stackelberg increases less than the welfare of the country with the firm that competes à la Cournot.

Therefore, countries whose firms face competition à la Cournot increase their welfare more by subscribing to trade agreements rather than countries whose firms face Stackelberg competition.

Finally, in this model, all countries with imperfect competition have higher welfare than the Autarky country.

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4.5 Appendix

Proof that the social cost function is convex

Social cost function is the following:

$h_i(c_i, q_{i1}+q_{i2}) = (q_{i1}+q_{i2})c_i + \frac{d}{2}((q_{i1}+q_{i2}) - bc_i)^2)$

First Order Condition:

$\frac{\partial h_i}{\partial c_i} = (q_{i1}+q_{i2}) + c_i\frac{\partial(q_{i1}+q_{i2})}{\partial c_i} + \frac{d}{2}(2(q_{i1}+q_{i2})\frac{\partial(q_{i1}+q_{i2})}{\partial c_i} - 2b(q_{i1}+q_{i2}) - 2bc_i\frac{\partial(q_{i1}+q_{i2})}{\partial c_i} + 2b^2c_i)$

Second Order Condition:

$\frac{\partial^2 h_i}{\partial {c_i}^2} = 2\frac{\partial(q_{i1}+q_{i2})}{\partial c_i} + d\frac{\partial(q_{i1}+q_{i2})}{\partial c_i}^2 - 2db\frac{\partial(q_{i1}+q_{i2})}{\partial c_i} + db^2$

Therefore,

$\frac{\partial^2 h_1}{\partial {c_1}^2} = -2(2) + 4d + 2db(2) + db^2$

$\frac{\partial^2 h_2}{\partial {c_1}^2} = -2(\frac{3}{2}) + d(\frac{9}{4}) + 2db(\frac{3}{2}) + db^2$

Because from second and third stage equilibrium we obtain that $\frac{\partial(q_{11}+q_{12})}{\partial c_1} = -2$ and $\frac{\partial(q_{21}+q_{22})}{\partial c_2} = -\frac{3}{2}$

In order to have $h_i(c_i, q_{i1}+q_{i2})$ convex, it is sufficient that $b > \frac{1}{2}$ and $d > \frac{3}{4}$ ■

Proof that Welfare function is concave

Welfare function is the following:

$W_i = \int_0^{q_{1io}+q_{2io}} P_i \, dQ_i - h_i(c_i, q_{i1o}+q_{i2o}) - P_i q_{jio} + P_j q_{ijo} - t(q_{ijo} - q_{jio})$

Because the inverse demand function is linear, we obtain the following:

$W_i = A(q_{1io}+q_{2io}) - \frac{(q_{i1o}+q_{i2o})^2}{2} - h_i - P_i q_{jio} + P_j q_{ijo} - t(q_{ijo} - q_{jio})$

First Order Condition:

$\frac{\partial W_i}{\partial c_i} = A\frac{\partial(q_{1io}+q_{2io})}{\partial c_i} - (q_{1io}+q_{2io})\frac{\partial(q_{1io}+q_{2io})}{\partial c_i} - \frac{\partial h_i}{\partial c_i} - P_i\frac{\partial q_{jio}}{\partial c_i} - q_{jio}\frac{\partial P_i}{\partial Q_i}\frac{\partial Q_i}{\partial c_i} + P_j\frac{\partial q_{ijo}}{\partial c_i} + q_{ijo}\frac{\partial P_j}{\partial Q_j}\frac{\partial Q_j}{\partial c_i} - t(\frac{\partial q_{ijo}}{\partial c_i} - \frac{\partial q_{jio}}{\partial c_i})$

Second Order Condition:

$\frac{\partial^2 W_i}{\partial {c_i}^2} = -\frac{\partial(q_{1io}+q_{2io})}{\partial c_i}2 - \frac{\partial^2 h_i}{\partial {c_i}^2} - \frac{\partial q_{jio}}{\partial c_i}\frac{\partial P_i}{\partial Q_i}\frac{\partial Q_i}{\partial c_i} + \frac{\partial q_{ijo}}{\partial c_i}\frac{\partial P_j}{\partial Q_j}\frac{\partial Q_j}{\partial c_i}$

Therefore,

$\frac{\partial^2 W_1}{\partial {c_1}^2} = -(-\frac{1}{2})^2 - \frac{\partial^2 h_1}{\partial {c_1}^2} - \frac{1}{2}(\frac{1}{2}) - \frac{1}{2}$

$\frac{\partial^2 W_2}{\partial {c_2}^2} = -(-\frac{1}{4})^2 - \frac{\partial^2 h_2}{\partial {c_2}^2} - \frac{1}{2}(\frac{1}{4}) - \frac{3}{4}(\frac{1}{4})$

Because from second and third stage equilibrium we obtain that $\frac{\partial q_{1i}}{\partial c_1} = -1$, $\frac{\partial q_{2i}}{\partial c_1} = \frac{1}{2}$, $\frac{\partial q_{1i}}{\partial c_2} = \frac{1}{2}$ and $\frac{\partial q_{2i}}{\partial c_2} = -\frac{3}{4}$

In order to have W_i concave, it is sufficient that h_i is convex ■

Results of Stackelberg Model

The equilibrium results are the following:

$c_{1o} = \frac{(8A(64-(164+125b+22b^2)d+8(2+b)^2(3+2b)d^2)-(445-2(459+368b+72b^2)d+32(2+b)^2(3+2b)d^2)t)}{(8(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))}$

$c_{2o} = \frac{(4A(64-(164+122b+15b^2)d+4(24+34b+15b^2+2b^3)d^2)-(243-2(253+204b+32b^2)d+8(24+34b+15b^2+2b^3)d^2)t)}{(4(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))}$

$q_{11o} = \frac{(8Abd(-23+24d+22b^2d+4b^3d+b(-11+40d))+(229-2(267+302b+93b^2)d+4(72+186b+167b^2+62b^3+8b^4)d^2)t)}{(4(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))}$

$q_{12o} = \frac{(8Abd(-23+24d+22b^2d+4b^3d+b(-11+40d))-(155-2(225+358b+126b^2)d+4(72+234b+247b^2+106b^3+16b^4)d^2)t)}{(4(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))}$

$q_{21o} = \frac{(8Abd(-13+24d+15b^2d+2b^3d+b(-9+34d))-(121-(342+538b+207b^2)d+(240+796b+826b^2+340b^3+48b^4)d^2)t)}{(4(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))}$

$q_{22o} = \frac{(8Abd(-13+24d+15b^2d+2b^3d+b(-9+34d))+(199-2(239+281b+79b^2)d+(240+604b+554b^2+220b^3+32b^4)d^2)t)}{(4(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2))}$

$W_{1o} = -(-64A^2b^2d(-4096+8(2329+2522b+673b^2)d-(27840+59600b+47361b^2+16556b^3+2148b^4)d^2+4(6+7b+2b^2)^2(96+80b+17b^2)d^3)+16Abd(2784b^6d^3+384b^7d^3+8b^5d^2(-347+432d)-4b^4d^2(2603+6218d)+b^2d(1981+143658d-181280d^2)+2b^3d(3627+9584d-53728d^2)-8(-3275+12562d-15336d^2+5760d^3)-4b(1842+19453d-58266d^2+36384d^3))t+(-104712+$

$(388656+229448b+126521b^2)d-4(103992+36864b-31853b^2-8030b^3+669b^4)d^2-4(-14736+204040b+548455b^2+538300b^3+252144b^4+55880b^5+4512b^6)d^3+16(6+7b+2b^2)^2(112+1032b+1367b^2+504b^3+48b^4)d^4)t^2)/(128(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)^2)$

$W_{2o} = -(-16A^2b^2d(-4096+16(1389+1282b+301b^2)d-(36352+69312b+49284b^2+15380b^3+1761b^4)d^2+4(6+7b+2b^2)^2(128+80b+13b^2)d^3)+8Abd(336b^6d^3+16b^5d^2(-37+183d)+b^4d^2(-5369+8628d)+4b^2d(1916+3067d-3228d^2)+b^3d(3641-10956d+6816d^2)-8(-1751+5728d-5776d^2+1728d^3)-12b(446+1737d-4832d^2+2256d^3))t+(-31060+(120804+92708b+57189b^2)d-2(79488+114844b+114390b^2+59077b^3+10839b^4)d^2+(79552+147328b+173140b^2+141732b^3+63745b^4+12256b^5+448b^6)d^3+4(6+7b+2b^2)^2(-80+104b+43b^2+96b^3+32b^4)d^4)t^2)/(32(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)^2)$

Results of Cournot Model

The equilibrium results are the following:

$c_{1o} = \frac{(2A(-5+(4+3b)d)+(8-(4+3b)d)t)}{(-10+(8+18b+9b^2)d)}$

$c_{2o} = \frac{(2A(-5+(4+3b)d)+(8-(4+3b)d)t)}{(-10+(8+18b+9b^2)d)}$

$q_{11o} = \frac{(Ab(4+3b)d+(-6+(4+7b+3b^2)d)t)}{/(-10+(8+18b+9b^2)d)}$

$q_{21o} = \frac{(Ab(4+3b)d-(-4+(4+11b+6b^2)d)t)}{(-10+(8+18b+9b^2)d)}$

$q_{12o} = \frac{(Ab(4+3b)d-(-4+(4+11b+6b^2)d)t)}{(-10+(8+18b+9b^2)d)}$

$q_{22o} = \frac{(Ab(4+3b)d+(-6+(4+7b+3b^2)d)t)}{(-10+(8+18b+9b^2)d)}$

$W_{1o} = \frac{-(-4A^2b^2d(-25+2(4+3b)^2d)+2A(-2+b)bd(-26+(4+3b)^2d)t+(-28+4(5-b+7b^2)d+b(2+b)(4+3b)^2d^2)t^2)}{(2(-10+(8+18b+9b^2)d)^2)}$

$W_{2o} = \frac{-(-4A^2b^2d(-25+2(4+3b)^2d)+2A(-2+b)bd(-26+(4+3b)^2d)t+(-28+4(5-b+7b^2)d+b(2+b)(4+3b)^2d^2)t^2)}{(2(-10+(8+18b+9b^2)d)^2)}$

Results of Autarky case

The equilibrium results are the following:

$c_{1o} = \frac{(A(-3+d+2bd))}{(-3+d+4bd+4b^2d)}$

$Q_{1o} = \frac{(Ab(1+2b)d)}{(-3+(1+2b)^2d)}$

$W_{1o} = \frac{(3A^2b^2d)}{(2(-3+(1+2b)^2d))}$

Proposition 1

Lemma 1.1: $\frac{\partial W_2^s}{\partial t} > \frac{\partial W_1^s}{\partial t}$ *when* $b \geq 4.8$, $d \geq 0.1$ *and* $A \geq 0$.

$\frac{\partial W_2^s}{\partial t} - \frac{\partial W_1^s}{\partial t} = (8Abd(2b^4(163-21064d)d^2+2112b^6d^3+384b^7d^3-8b^5d^2(199+300d)+b^2d(-13347+119122d-155456d^2)-4b^3d(7-10270d+30272d^2)-8(227+1106d-3784d^2+2304d^3)-4b(-834+9031d-29274d^2+22848d^3))+(19528-(94560+141384b+102235b^2)d+4(54984+192824b+260633b^2+126184b^3+21009b^4)d^2-4(64816+351368b+721595b^2+680032b^3+315889b^4+68136b^5+4960b^6)d^3+64(6+7b+2b^2)^2(48+232b+331b^2+102b^3+4b^4)d^4)t)/(64(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)^2)$

I need to prove that the $B = (8Abd(2b^4(163-21064d)d^2+2112b^6d^3+384b^7d^3-8b^5d^2(199+300d)+b^2d(-13347+119122d-155456d^2)-4b^3d(7-10270d+30272d^2)-8(227+1106d-3784d^2+2304d^3)-4b(-834+9031d-29274d^2+22848d^3))+(19528-(94560+141384b+102235b^2)d+4(54984+192824b+260633b^2+126184b^3+21009b^4)d^2-4(64816+351368b+721595b^2+680032b^3+315889b^4+68136b^5+4960b^6)d^3+64(6+7b+2b^2)^2(48+232b+331b^2+102b^3+4b^4)d^4)t) > 0$ and $D = (64(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)^2) > 0$.

Step 1. To prove that $B(A,b,d,t) > 0$, where $A > 0$ and $0 \leq t \leq 1$. I solve the following optimization problem with constraints:

$$\underset{b,d}{\text{Min}}\ B(A,b,d,t)$$

subject to

$$b \geq 4.8, \tag{4.1}$$

$$d \geq 0.1, \tag{4.2}$$

$$A = 0.01, \tag{4.3}$$

$$t = 0 \tag{4.4}$$

The minimum value that B can attain is 281.97, when $b = 4.8$, $d = 0.1$, $A = 0.01$ and $t = 0$.

Step 2. $\frac{\partial B(A,b,d,t)}{\partial A} = 8bd(2b^4(163 - 21064d)d^2 + 2112b^6d^3 + 384b^7d^3 - 8b^5d^2(199 + 300d) + b^2d(-13347 + 119122d - 155456d^2) - 4b^3d(7 - 10270d + 30272d^2) - 8(227 + 1106d - 3784d^2 + 2304d^3) - 4b(-834 + 9031d - 29274d^2 + 22848d^3))$

To prove that $B_A(b,d) = 8bd(2b^4(163 - 21064d)d^2 + 2112b^6d^3 + 384b^7d^3 - 8b^5d^2(199 + 300d) + b^2d(-13347 + 119122d - 155456d^2) - 4b^3d(7 - 10270d + 30272d^2) - 8(227 + 1106d - 3784d^2 + 2304d^3) - 4b(-834 + 9031d - 29274d^2 + 22848d^3)) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 4.8 \\ d \geq 0.1}} B_A(b,d)$$

The minimum value that B_A can attain is $28,197$, when $b = 4.8$ and $d = 0.1$

Step 3. $\frac{\partial B(A,b,d,t)}{\partial t} = 19528 - (94560 + 141384b + 102235b^2)d + 4(54984 + 192824b + 260633b^2 + 126184b^3 + 21009b^4)d^2 - 4(64816 + 351368b + 721595b^2 + 680032b^3 + 315889b^4 + 68136b^5 + 4960b^6)d^3 + 64(6 + 7b + 2b^2)^2(48 + 232b + 331b^2 + 102b^3 + 4b^4)d^4$

To prove that $B_t(b,d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b\geq 4.8\\ d\geq 0.1}} B_t(b,d)$$

The minimum value that B_t can attain is $50,809.9$, when $b = 4.8$ and $d = 0.1$

Step 4. To prove that $D(b,d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b\geq 4.8\\ d\geq 0.1}} D(b,d)$$

The minimum value that D can attain is $235,001$, when $b = 4.8$ and $d = 0.1$ ■

Lemma 1.2: $\frac{\partial W_2^s}{\partial t} > \frac{\partial W^c}{\partial t}$ *when* $b \geq 2.9$, $d \geq 0.9$ *and* $A \geq 0.1$.

$\frac{\partial W_2^s}{\partial t} - \frac{\partial W^c}{\partial t} = (4Abd(107952b^10d^5 + 9216b^11d^5 + 16b^9d^4(-3923 + 29797d) + 3b^8d^4(-176829 + 254980d) + b^7d^3(107035 - 1601648d - 1199856d^2) - 24b^6d^3(-32614 + 48493d + 338172d^2) + 4b^5d^2(11125 + 562960d + 1238848d^2 - 4440608d^3) - 4b^4d^2(177111 - 733004d - 3820132d^2 + 5550160d^3) - 16b^2d(-56982 + 331663d + 43004d^2 - 867888d^3 + 524096d^4) - 4b^3d(52057 + 779784d - 284180d^2 - 4966208d^3 + 4325312d^4) - 32(17151 - 60408d + 62872d^2 - 4480d^3 - 24320d^4 + 9216d^5) - 16b(-6826 - 129535d + 290096d^2 + 18912d^3 - 318080d^4 + 146688d^5)) + (1270992 - 4(1583716 + 2024716b + 1322145b^2)d + 16(808888 + 2267640b + 2940557b^2 + 1817305b^3 + 504781b^4)d^2 - (14477312 + 68066944b + 140905376b^2 + 157971344b^3 + 101554748b^4 + 35850416b^5 + 5573261b^6)d^3 + 2(4924928 + 34487040b + 103314144b^2 + 171498080b^3 + 172607448b^4 + 108556916b^5 + 41903568b^6 + 9080385b^7 + 843215b^8)d^4 - (3985408 + 37900288b + 152415744b^2 + 344500992b^3 + 486192720b^4 + 448089248b^5 + 272679688b^6 + 107594728b^7 + 26014865b^8 + 3380960b^9 + 165824b^10)d^5 - 4(24 + 46b + 29b^2 + 6b^3)^2(-320 - 2592b - 7492b^2 - 10068b^3 - 6077b^4 - 1056b^5 + 32b^6)d^6)t)/(16(-10 + (8 + 18b + 9b^2)d)^2(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)^2)$

I need to prove that the $E = (4Abd(107952b^10d^5 + 9216b^11d^5 + 16b^9d^4(-3923 + 29797d) +$

$3b^8d^4(-176829+254980d)+b^7d^3(107035-1601648d-1199856d^2)-24b^6d^3(-32614+48493d+338172d^2)+4b^5d^2(11125+562960d+1238848d^2-4440608d^3)-4b^4d^2(177111-733004d-3820132d^2+5550160d^3)-16b^2d(-56982+331663d+43004d^2-867888d^3+524096d^4)-4b^3d(52057+779784d-284180d^2-4966208d^3+4325312d^4)-32(17151-60408d+62872d^2-4480d^3-24320d^4+9216d^5)-16b(-6826-129535d+290096d^2+18912d^3-318080d^4+146688d^5))+(1270992-4(1583716+2024716b+1322145b^2)d+16(808888+2267640b+2940557b^2+1817305b^3+504781b^4)d^2-(14477312+68066944b+140905376b^2+157971344b^3+101554748b^4+35850416b^5+5573261b^6)d^3+2(4924928+34487040b+103314144b^2+171498080b^3+172607448b^4+108556916b^5+41903568b^6+9080385b^7+843215b^8)d^4-(3985408+37900288b+152415744b^2+344500992b^3+486192720b^4+448089248b^5+272679688b^6+107594728b^7+26014865b^8+3380960b^9+165824b^10)d^5-4(24+46b+29b^2+6b^3)^2(-320-2592b-7492b^2-10068b^3-6077b^4-1056b^5+32b^6)d^6)t)>0$ and $F=(16(-10+(8+18b+9b^2)d)^2(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)^2)>0$.

Step 1. To prove that $E(A,b,d,t)>0$, where $A>0$ and $0\leq t\leq 1$. I solve the following optimization problem with constraints:

$$\underset{b,d}{\text{Min}}\ E(A,b,d,t)$$

subject to

$$b\geq 2.9, \tag{4.5}$$

$$d\geq 0.9, \tag{4.6}$$

$$A=0.1, \tag{4.7}$$

$$t=0 \tag{4.8}$$

The minimum value that E can attain is $272,741,000$, when $b=2.9$, $d=0.9$, $A=0.1$ and $t=0$.

Step 2. $\frac{\partial E(A,b,d,t)}{\partial A}=4bd(107952b^10d^5+9216b^11d^5+16b^9d^4(-3923+29797d)+3b^8d^4(-176829+$

$254980d) + b^7d^3(107035 - 1601648d - 1199856d^2) - 24b^6d^3(-32614 + 48493d + 338172d^2) + 4b^5d^2(11125 + 562960d + 1238848d^2 - 4440608d^3) - 4b^4d^2(177111 - 733004d - 3820132d^2 + 5550160d^3) - 16b^2d(-56982 + 331663d + 43004d^2 - 867888d^3 + 524096d^4) - 4b^3d(52057 + 779784d - 284180d^2 - 4966208d^3 + 4325312d^4) - 32(17151 - 60408d + 62872d^2 - 4480d^3 - 24320d^4 + 9216d^5) - 16b(-6826 - 129535d + 290096d^2 + 18912d^3 - 318080d^4 + 146688d^5))$

To prove that $E_A(b,d) = 4bd(107952b^10d^5 + 9216b^11d^5 + 16b^9d^4(-3923 + 29797d) + 3b^8d^4(-176829 + 254980d) + b^7d^3(107035 - 1601648d - 1199856d^2) - 24b^6d^3(-32614 + 48493d + 338172d^2) + 4b^5d^2(11125 + 562960d + 1238848d^2 - 4440608d^3) - 4b^4d^2(177111 - 733004d - 3820132d^2 + 5550160d^3) - 16b^2d(-56982 + 331663d + 43004d^2 - 867888d^3 + 524096d^4) - 4b^3d(52057 + 779784d - 284180d^2 - 4966208d^3 + 4325312d^4) - 32(17151 - 60408d + 62872d^2 - 4480d^3 - 24320d^4 + 9216d^5) - 16b(-6826 - 129535d + 290096d^2 + 18912d^3 - 318080d^4 + 146688d^5)) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 2.9 \\ d \geq 0.9}} E_A(b,d)$$

The minimum value that E_A can attain is $2,727,410,000$, when $b = 2.9$ and $d = 0.9$

Step 3. $\frac{\partial E(A,b,d,t)}{\partial t} = 1270992 - 4(1583716 + 2024716b + 1322145b^2)d + 16(808888 + 2267640b + 2940557b^2 + 1817305b^3 + 504781b^4)d^2 - (14477312 + 68066944b + 140905376b^2 + 157971344b^3 + 101554748b^4 + 35850416b^5 + 5573261b^6)d^3 + 2(4924928 + 34487040b + 103314144b^2 + 171498080b^3 + 172607448b^4 + 108556916b^5 + 41903568b^6 + 9080385b^7 + 843215b^8)d^4 - (3985408 + 37900288b + 152415744b^2 + 344500992b^3 + 486192720b^4 + 448089248b^5 + 272679688b^6 + 107594728b^7 + 26014865b^8 + 3380960b^9 + 165824b^10)d^5 - 4(24 + 46b + 29b^2 + 6b^3)^2(-320 - 2592b - 7492b^2 - 10068b^3 - 6077b^4 - 1056b^5 + 32b^6)d^6$

To prove that $E_t(b,d) = 1270992 - 4(1583716 + 2024716b + 1322145b^2)d + 16(808888 + 2267640b + 2940557b^2 + 1817305b^3 + 504781b^4)d^2 - (14477312 + 68066944b + 140905376b^2 + 157971344b^3 + 101554748b^4 + 35850416b^5 + 5573261b^6)d^3 + 2(4924928 + 34487040b + 103314144b^2 + 171498080b^3 + 172607448b^4 + 108556916b^5 + 41903568b^6 + 9080385b^7 + 843215b^8)d^4 - (3985408 + 37900288b + 152415744b^2 + 344500992b^3 + 486192720b^4 + 448089248b^5 + 272679688b^6 + 107594728b^7 +$

$260148656^8 + 3380960b^9 + 165824b^10)d^5 - 4(24+46b+29b^2+6b^3)^2(-320-2592b-7492b^2-10068b^3-6077b^4-1056b^5+32b^6)d^6 > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 2.9 \\ d \geq 0.9}} E_t(b,d)$$

The minimum value that E_t can attain is $301,341,000,000$, when $b = 2.9$ and $d = 0.9$

Step 4. To prove that $F(b,d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 2.9 \\ d \geq 0.9}} F(b,d)$$

The minimum value that F can attain is $4,149,000,000,000$, when $b = 2.9$ and $d = 0.9$ ■

Discontinuous areas of $\frac{\partial W_2^s}{\partial t} - \frac{\partial W_1^s}{\partial t}$

$\frac{\partial W_2^s}{\partial t} - \frac{\partial W_1^s}{\partial t} = (8Abd(2b^4(163-21064d)d^2+2112b^6d^3+384b^7d^3-8b^5d^2(199+300d)+b^2d(-13347+119122d-155456d^2)-4b^3d(7-10270d+30272d^2)-8(227+1106d-3784d^2+2304d^3)-4b(-834+9031d-29274d^2+22848d^3))+(19528-(94560+141384b+102235b^2)d+4(54984+192824b+260633b^2+126184b^3+21009b^4)d^2-4(64816+351368b+721595b^2+680032b^3+315889b^4+68136b^5+4960b^6)d^3+64(6+7b+2b^2)^2(48+232b+331b^2+102b^3+4b^4)d^4)t)/(64(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)^2)$

When $(64(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)^2) = 0$ are the areas where $\frac{\partial W_2^s}{\partial t} - \frac{\partial W_1^s}{\partial t}$ is discontinuous

The following equations are these areas:

$d = \frac{(164+220b+73b^2-Sqrt[2320+480b+1688b^2+3448b^3+1233b^4])}{(2(96+280b+276b^2+112b^3+16b^4))}$

$d = \frac{(164+220b+73b^2+Sqrt[2320+480b+1688b^2+3448b^3+1233b^4])}{(2(96+280b+276b^2+112b^3+16b^4))}$

Discontinuous areas of $\frac{\partial W_2^s}{\partial t} - \frac{\partial W^c}{\partial t}$

$\frac{\partial W_2^s}{\partial t} - \frac{\partial W^c}{\partial t} = (4Abd(107952b^10d^5 + 9216b^11d^5 + 16b^9d^4(-3923 + 29797d) + 3b^8d^4(-176829 + 254980d) + b^7d^3(107035 - 1601648d - 1199856d^2) - 24b^6d^3(-32614 + 48493d + 338172d^2) + 4b^5d^2(11125 + 562960d + 1238848d^2 - 4440608d^3) - 4b^4d^2(177111 - 733004d - 3820132d^2 + 5550160d^3) - 16b^2d(-56982 + 331663d + 43004d^2 - 867888d^3 + 524096d^4) - 4b^3d(52057 + 779784d - 284180d^2 - 4966208d^3 + 4325312d^4) - 32(17151 - 60408d + 62872d^2 - 4480d^3 - 24320d^4 + 9216d^5) - 16b(-6826 - 129535d + 290096d^2 + 18912d^3 - 318080d^4 + 146688d^5)) + (1270992 - 4(1583716 + 2024716b + 1322145b^2)d + 16(808888 + 2267640b + 2940557b^2 + 1817305b^3 + 504781b^4)d^2 - (14477312 + 68066944b + 140905376b^2 + 157971344b^3 + 101554748b^4 + 35850416b^5 + 5573261b^6)d^3 + 2(4924928 + 34487040b + 103314144b^2 + 171498080b^3 + 172607448b^4 + 108556916b^5 + 41903568b^6 + 9080385b^7 + 843215b^8)d^4 - (3985408 + 37900288b + 152415744b^2 + 344500992b^3 + 486192720b^4 + 448089248b^5 + 272679688b^6 + 107594728b^7 + 26014865b^8 + 3380960b^9 + 165824b^10)d^5 - 4(24 + 46b + 29b^2 + 6b^3)^2(-320 - 2592b - 7492b^2 - 10068b^3 - 6077b^4 - 1056b^5 + 32b^6)d^6)t)/(16(-10 + (8 + 18b + 9b^2)d)^2(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)^2)$

When $(16(-10+(8+18b+9b^2)d)^2(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)^2) = 0$ are the areas where $\frac{\partial W_2^s}{\partial t} - \frac{\partial W^c}{\partial t}$ is discontinuous

The following equations are these areas:

$d = \frac{10}{(8+18b+9b^2)}$

$d = \frac{(164+220b+73b^2-Sqrt[2320+480b+1688b^2+3448b^3+1233b^4])}{(2(96+280b+276b^2+112b^3+16b^4))}$

$d = \frac{(164+220b+73b^2+Sqrt[2320+480b+1688b^2+3448b^3+1233b^4])}{(2(96+280b+276b^2+112b^3+16b^4))}$

Proposition 2

Claim: $\frac{\partial W_1^s}{\partial t} < 0$ when $b \geq 3.7$, $d \geq 1$ and $A \geq 0$.

$\frac{\partial W_1^s}{\partial t} = -(16Abd(2784b^6d^3 + 384b^7d^3 + 8b^5d^2(-347 + 432d) - 4b^4d^2(2603 + 6218d) + b^2d(1981 + 143658d - 181280d^2) + 2b^3d(3627 + 9584d - 53728d^2) - 8(-3275 + 12562d - 15336d^2 +$

$5760d^3) - 4b(1842 + 19453d - 58266d^2 + 36384d^3)) + 2(-104712 + (388656 + 229448b + 126521b^2)d - 4(103992 + 36864b - 31853b^2 - 8030b^3 + 669b^4)d^2 - 4(-14736 + 204040b + 548455b^2 + 538300b^3 + 252144b^4 + 55880b^5 + 4512b^6)d^3 + 16(6 + 7b + 2b^2)^2(112 + 1032b + 1367b^2 + 504b^3 + 48b^4)d^4)t)/(128(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)^2)$

I need to prove that the $G = -(16Abd(2784b^6d^3 + 384b^7d^3 + 8b^5d^2(-347 + 432d) - 4b^4d^2(2603 + 6218d) + b^2d(1981 + 143658d - 181280d^2) + 2b^3d(3627 + 9584d - 53728d^2) - 8(-3275 + 12562d - 15336d^2 + 5760d^3) - 4b(1842 + 19453d - 58266d^2 + 36384d^3)) + 2(-104712 + (388656 + 229448b + 126521b^2)d - 4(103992 + 36864b - 31853b^2 - 8030b^3 + 669b^4)d^2 - 4(-14736 + 204040b + 548455b^2 + 538300b^3 + 252144b^4 + 55880b^5 + 4512b^6)d^3 + 16(6 + 7b + 2b^2)^2(112 + 1032b + 1367b^2 + 504b^3 + 48b^4)d^4)t) < 0$ and $H = (128(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)^2) > 0$.

Step 1. To prove that $G(A, b, d, t) < 0$, where $A > 0$ and $0 \leq t \leq 1$. I solve the following optimization problem with constraints:

$$\underset{b,d}{\text{Max}}\ G(A, b, d, t)$$

subject to

$$b \geq 3.7, \tag{4.9}$$

$$d \geq 1, \tag{4.10}$$

$$A = 0.01, \tag{4.11}$$

$$t = 0 \tag{4.12}$$

The maximum value that G can attain is $-38,820.3$, when $b = 3.7$, $d = 1$, $A = 0.01$ and $t = 0$.

Step 2. $\frac{\partial G(A,b,d,t)}{\partial A} = -16bd(2784b^6d^3 + 384b^7d^3 + 8b^5d^2(-347 + 432d) - 4b^4d^2(2603 + 6218d) + b^2d(1981 + 143658d - 181280d^2) + 2b^3d(3627 + 9584d - 53728d^2) - 8(-3275 + 12562d - 15336d^2 + 5760d^3) - 4b(1842 + 19453d - 58266d^2 + 36384d^3))$

To prove that $G_A(b,d) = -16bd(2784b^6d^3 + 384b^7d^3 + 8b^5d^2(-347 + 432d) - 4b^4d^2(2603 + 6218d) + b^2d(1981 + 143658d - 181280d^2) + 2b^3d(3627 + 9584d - 53728d^2) - 8(-3275 + 12562d - 15336d^2 + 5760d^3) - 4b(1842 + 19453d - 58266d^2 + 36384d^3)) < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b\geq 3.7\\ d\geq 1}} G_A(b,d)$$

The maximum value that G_A can attain is $-3,882,030$, when $b = 3.7$ and $d = 1$

Step 3. $\frac{\partial G(A,b,d,t)}{\partial t} = -2(-104712 + (388656 + 229448b + 126521b^2)d - 4(103992 + 36864b - 31853b^2 - 8030b^3 + 669b^4)d^2 - 4(-14736 + 204040b + 548455b^2 + 538300b^3 + 252144b^4 + 55880b^5 + 4512b^6)d^3 + 16(6 + 7b + 2b^2)^2(112 + 1032b + 1367b^2 + 504b^3 + 48b^4)d^4)$

To prove that $G_t(b,d) = -2(-104712 + (388656 + 229448b + 126521b^2)d - 4(103992 + 36864b - 31853b^2 - 8030b^3 + 669b^4)d^2 - 4(-14736 + 204040b + 548455b^2 + 538300b^3 + 252144b^4 + 55880b^5 + 4512b^6)d^3 + 16(6 + 7b + 2b^2)^2(112 + 1032b + 1367b^2 + 504b^3 + 48b^4)d^4) < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b\geq 3.7\\ d\geq 1}} G_t(b,d)$$

The maximum value that G_t can attain is $-5,373,610,000$, when $b = 3.7$ and $d = 1$

Step 4. To prove that $H(b,d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b\geq 3.7\\ d\geq 1}} H(b,d)$$

The minimum value that H can attain is $17,428,200,000$, when $b = 3.7$ and $d = 1$ ∎

Discontinuous areas of $\frac{\partial W_1^s}{\partial t}$

$\frac{\partial W_1^s}{\partial t} = -(16Abd(2784b^6d^3 + 384b^7d^3 + 8b^5d^2(-347 + 432d) - 4b^4d^2(2603 + 6218d) + b^2d(1981 +$

$143658d - 181280d^2) + 2b^3d(3627 + 9584d - 53728d^2) - 8(-3275 + 12562d - 15336d^2 + 5760d^3) - 4b(1842 + 19453d - 58266d^2 + 36384d^3)) + 2(-104712 + (388656 + 229448b + 126521b^2)d - 4(103992 + 36864b - 31853b^2 - 8030b^3 + 669b^4)d^2 - 4(-14736 + 204040b + 548455b^2 + 538300b^3 + 252144b^4 + 55880b^5 + 4512b^6)d^3 + 16(6 + 7b + 2b^2)^2(112 + 1032b + 1367b^2 + 504b^3 + 48b^4)d^4)t)/(128(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)^2)$

When $(128(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)^2) = 0$ are the areas where $\frac{\partial W_1^s}{\partial t}$ is discontinuous

The following equations are these areas:

$d = \frac{(164+220b+73b^2-Sqrt[2320+480b+1688b^2+3448b^3+1233b^4])}{(2(96+280b+276b^2+112b^3+16b^4))}$

$d = \frac{(164+220b+73b^2+Sqrt[2320+480b+1688b^2+3448b^3+1233b^4])}{(2(96+280b+276b^2+112b^3+16b^4))}$

Proposition 3

Claim: $\frac{\partial W_2^s}{\partial t} < 0$ when $b \geq 1.1$, $d \geq 1$ and $A \geq 0$.

$\frac{\partial W_2^s}{\partial t} = -(8Abd(336b^6d^3 + 16b^5d^2(-37 + 183d) + b^4d^2(-5369 + 8628d) + 4b^2d(1916 + 3067d - 3228d^2) + b^3d(3641 - 10956d + 6816d^2) - 8(-1751 + 5728d - 5776d^2 + 1728d^3) - 12b(446 + 1737d - 4832d^2 + 2256d^3)) + 2(-31060 + (120804 + 92708b + 57189b^2)d - 2(79488 + 114844b + 114390b^2 + 59077b^3 + 10839b^4)d^2 + (79552 + 147328b + 173140b^2 + 141732b^3 + 63745b^4 + 12256b^5 + 448b^6)d^3 + 4(6 + 7b + 2b^2)^2(-80 + 104b + 43b^2 + 96b^3 + 32b^4)d^4)t)/(32(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)^2)$

I need to prove that the $I = -(8Abd(336b^6d^3 + 16b^5d^2(-37 + 183d) + b^4d^2(-5369 + 8628d) + 4b^2d(1916 + 3067d - 3228d^2) + b^3d(3641 - 10956d + 6816d^2) - 8(-1751 + 5728d - 5776d^2 + 1728d^3) - 12b(446 + 1737d - 4832d^2 + 2256d^3)) + 2(-31060 + (120804 + 92708b + 57189b^2)d - 2(79488 + 114844b + 114390b^2 + 59077b^3 + 10839b^4)d^2 + (79552 + 147328b + 173140b^2 + 141732b^3 + 63745b^4 + 12256b^5 + 448b^6)d^3 + 4(6 + 7b + 2b^2)^2(-80 + 104b + 43b^2 + 96b^3 + 32b^4)d^4)t) < 0$ and $J = (32(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)^2) > 0$.

Step 1. To prove that $I(A, b, d, t) < 0$, where $A > 0$ and $0 \leq t \leq 1$. I solve the following optimization problem with constraints:

$$\begin{aligned}
&\operatorname*{Max}_{b,d}\ I(A, b, d, t) \\
&\text{subject to} \\
&b \geq 1.1, && (4.13) \\
&d \geq 1, && (4.14) \\
&A = 0.01, && (4.15) \\
&t = 0 && (4.16)
\end{aligned}$$

The maximum value that I can attain is $-1,998.88$, when $b = 1.1$, $d = 1$, $A = 0.01$ and $t = 0$.

Step 2. $\frac{\partial I(A,b,d,t)}{\partial A} = -8bd(336b^6d^3 + 16b^5d^2(-37 + 183d) + b^4d^2(-5369 + 8628d) + 4b^2d(1916 + 3067d - 3228d^2) + b^3d(3641 - 10956d + 6816d^2) - 8(-1751 + 5728d - 5776d^2 + 1728d^3) - 12b(446 + 1737d - 4832d^2 + 2256d^3))$

To prove that $I_A(b, d) = -8bd(336b^6d^3 + 16b^5d^2(-37+183d) + b^4d^2(-5369+8628d) + 4b^2d(1916 + 3067d - 3228d^2) + b^3d(3641 - 10956d + 6816d^2) - 8(-1751 + 5728d - 5776d^2 + 1728d^3) - 12b(446 + 1737d - 4832d^2 + 2256d^3)) < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b \geq 1.1 \\ d \geq 1}} I_A(b, d)$$

The maximum value that I_A can attain is $-199,888$, when $b = 1.1$ and $d = 1$

Step 3. $\frac{\partial I(A,b,d,t)}{\partial t} = -2(-31060 + (120804 + 92708b + 57189b^2)d - 2(79488 + 114844b + 114390b^2 + 59077b^3 + 10839b^4)d^2 + (79552 + 147328b + 173140b^2 + 141732b^3 + 63745b^4 + 12256b^5 + 448b^6)d^3 + 4(6 + 7b + 2b^2)^2(-80 + 104b + 43b^2 + 96b^3 + 32b^4)d^4)$

To prove that $I_t(b,d) = -2(-31060 + (120804 + 92708b + 57189b^2)d - 2(79488 + 114844b + 114390b^2 + 59077b^3 + 10839b^4)d^2 + (79552 + 147328b + 173140b^2 + 141732b^3 + 63745b^4 + 12256b^5 + 448b^6)d^3 + 4(6 + 7b + 2b^2)^2(-80 + 104b + 43b^2 + 96b^3 + 32b^4)d^4) < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b \geq 1.1 \\ d \geq 1}} I_t(b,d)$$

The maximum value that I_t can attain is $-816,858$, when $b = 1.1$ and $d = 1$

Step 4. To prove that $J(b,d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 1.1 \\ d \geq 1}} J(b,d)$$

The minimum value that J can attain is $7,376,720$, when $b = 1.1$ and $d = 1$ ■

Discontinuous areas of $\frac{\partial W_2^s}{\partial t}$

$\frac{\partial W_2^s}{\partial t} = -(8Abd(336b^6d^3 + 16b^5d^2(-37 + 183d) + b^4d^2(-5369 + 8628d) + 4b^2d(1916 + 3067d - 3228d^2) + b^3d(3641 - 10956d + 6816d^2) - 8(-1751 + 5728d - 5776d^2 + 1728d^3) - 12b(446 + 1737d - 4832d^2 + 2256d^3)) + 2(-31060 + (120804 + 92708b + 57189b^2)d - 2(79488 + 114844b + 114390b^2 + 59077b^3 + 10839b^4)d^2 + (79552 + 147328b + 173140b^2 + 141732b^3 + 63745b^4 + 12256b^5 + 448b^6)d^3 + 4(6 + 7b + 2b^2)^2(-80 + 104b + 43b^2 + 96b^3 + 32b^4)d^4)t)/(32(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)^2$

When $(32(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)^2) = 0$ are the areas where $\frac{\partial W_2^s}{\partial t}$ is discontinuous

The following equations are these areas:

$d = \frac{(164+220b+73b^2-Sqrt[2320+480b+1688b^2+3448b^3+1233b^4])}{(2(96+280b+276b^2+112b^3+16b^4))}$

$d = \frac{(164+220b+73b^2+Sqrt[2320+480b+1688b^2+3448b^3+1233b^4])}{(2(96+280b+276b^2+112b^3+16b^4))}$

Proposition 4

Claim: $\frac{\partial W^c}{\partial t} < 0$ when $b \geq 2.1$, $d \geq 1$ and $A \geq 0$.

$\frac{\partial W^c}{\partial t} = \frac{-2A(-2+b)bd(-26+(4+3b)^2d)-2(-28+4(5-b+7b^2)d+b(2+b)(4+3b)^2d^2)t}{2(-10+(8+18b+9b^2)d)^2}$

I need to prove that the $K = -2A(-2+b)bd(-26+(4+3b)^2d) - 2(-28+4(5-b+7b^2)d + b(2+b)(4+3b)^2d^2)t < 0$ and $L = 2(-10+(8+18b+9b^2)d)^2 > 0$.

Step 1. To prove that $K(A,b,d,t) < 0$, where $A > 0$ and $0 \leq t \leq 1$. I solve the following optimization problem with constraints:

$$\underset{b,d}{\text{Max}}\ K(A,b,d,t)$$

subject to

$$b \geq 2.1, \tag{4.17}$$

$$d \geq 1, \tag{4.18}$$

$$A = 0.01, \tag{4.19}$$

$$t = 0 \tag{4.20}$$

The maximum value that K can attain is -0.34, when $b = 2.1$, $d = 1$, $A = 0.01$ and $t = 0$.

Step 2. $\frac{\partial K(A,b,d,t)}{\partial A} = -2(-2+b)bd(-26+(4+3b)^2d)$

To prove that $K_A(b,d) = -2(-2+b)bd(-26+(4+3b)^2d) < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b\geq 2.1\\ d\geq 1}} K_A(b,d)$$

The maximum value that K_A can attain is -33.64, when $b = 2.1$ and $d = 1$

Step 3. $\frac{\partial K(A,b,d,t)}{\partial t} = -2(-28+4(5-b+7b^2)d+b(2+b)(4+3b)^2d^2)$

To prove that $K_t(b,d) = -2(-28+4(5-b+7b^2)d+b(2+b)(4+3b)^2d^2) < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b\geq 2.1\\ d\geq 1}} K_t(b,d)$$

The maximum value that K_t can attain is $-2,041.03$, when $b = 2.1$ and $d = 1$

Step 4. To prove that $L(b,d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b\geq 2.1\\ d\geq 1}} L(b,d)$$

The minimum value that L can attain is $11,397.5$, when $b = 2.1$ and $d = 1$ ■

Discontinuous areas of $\frac{\partial W^c}{\partial t}$

$\frac{\partial W^c}{\partial t} = \frac{-2A(-2+b)bd(-26+(4+3b)^2d)-2(-28+4(5-b+7b^2)d+b(2+b)(4+3b)^2d^2)t}{2(-10+(8+18b+9b^2)d)^2}$

When $2(-10+(8+18b+9b^2)d)^2 = 0$ is the area where $\frac{\partial W^c}{\partial t}$ is discontinuous

The following equation is this areas:

$d = \frac{10}{8+18b+9b^2}$

Proposition 5

Claim: $W_1^s > W_2^s$ when $b \geq 4.8$, $d \geq 1$ and $A \geq 3$.

$W_1^s - W_2^s = -(-64A^2b^2d^2(448b^5d^2+64b^6d^2-56b^3d(21+40d)+b^4d(-387+656d)+b^2(568+1923d-8768d^2)-8(449-1064d+576d^2)-16b(21-607d+672d^2))+16Abd(2b^4(163-21064d)d^2+2112b^6d^3+384b^7d^3-8b^5d^2(199+300d)+b^2d(-13347+119122d-155456d^2)-4b^3d(7-10270d+30272d^2)-8(227+1106d-3784d^2+2304d^3)-4b(-834+9031d-29274d^2+22848d^3))t+(19528-(94560+141384b+102235b^2)d+4(54984+192824b+260633b^2+126184b^3+21009b^4)d^2-4(64816+351368b+721595b^2+680032b^3+315889b^4+68136b^5+$

$4960b^6)d^3 + 64(6+7b+2b^2)^2(48+232b+331b^2+102b^3+4b^4)d^4)t^2)/(128(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)^2)$

I need to prove that the $M = -(-64A^2b^2d^2(448b^5d^2+64b^6d^2-56b^3d(21+40d)+b^4d(-387+656d)+b^2(568+1923d-8768d^2)-8(449-1064d+576d^2)-16b(21-607d+672d^2))+16Abd(2b^4(163-21064d)d^2+2112b^6d^3+384b^7d^3-8b^5d^2(199+300d)+b^2d(-13347+119122d-155456d^2)-4b^3d(7-10270d+30272d^2)-8(227+1106d-3784d^2+2304d^3)-4b(-834+9031d-29274d^2+22848d^3))t+(19528-(94560+141384b+102235b^2)d+4(54984+192824b+260633b^2+126184b^3+21009b^4)d^2-4(64816+351368b+721595b^2+680032b^3+315889b^4+68136b^5+4960b^6)d^3+64(6+7b+2b^2)^2(48+232b+331b^2+102b^3+4b^4)d^4)t^2) > 0$ and $N = (128(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)^2) > 0$.

Step 1. To prove that $M(A,b,d,t) > 0$, where $A > 0$ and $0 \leq t \leq 1$. I solve the following optimization problem with constraints:

$$\underset{b,d}{\text{Min}}\ M(A,b,d,t)$$

subject to

$$b \geq 4.8, \tag{4.21}$$

$$d \geq 1, \tag{4.22}$$

$$A = 3, \tag{4.23}$$

$$t = 1 \tag{4.24}$$

The minimum value that M can attain is $10,468,000,000$, when $b = 4.8$, $d = 1$, $A = 3$ and $t = 1$.

Step 2. $\frac{\partial M(A,b,d,t)}{\partial A} = 128Ab^2d^2(448b^5d^2+64b^6d^2-56b^3d(21+40d)+b^4d(-387+656d)+b^2(568+1923d-8768d^2)-8(449-1064d+576d^2)-16b(21-607d+672d^2))-16bd(2b^4(163-21064d)d^2+2112b^6d^3+384b^7d^3-8b^5d^2(199+300d)+b^2d(-13347+119122d-155456d^2)-4b^3d(7-10270d+30272d^2)-8(227+1106d-3784d^2+2304d^3)-4b(-834+9031d-29274d^2+22848d^3))t$

To prove that $M_A(A,b,d,t) = 128Ab^2d^2(448b^5d^2 + 64b^6d^2 - 56b^3d(21+40d) + b^4d(-387+656d) + b^2(568+1923d-8768d^2) - 8(449-1064d+576d^2) - 16b(21-607d+672d^2)) - 16bd(2b^4(163-21064d)d^2 + 2112b^6d^3 + 384b^7d^3 - 8b^5d^2(199+300d) + b^2d(-13347+119122d-155456d^2) - 4b^3d(7-10270d+30272d^2) - 8(227+1106d-3784d^2+2304d^3) - 4b(-834+9031d-29274d^2+22848d^3))t > 0$, I solve the following optimization problem with constraints:

$$\underset{b,d}{\text{Min}}\ M_A(A,b,d,t)$$

subject to

$$b \geq 4.8, \tag{4.25}$$

$$d \geq 1, \tag{4.26}$$

$$A = 3, \tag{4.27}$$

$$t = 1 \tag{4.28}$$

The minimum value that M_A can attain is $13,151,100,000$, when $b = 4.8$, $d = 1$, $A = 3$ and $t = 1$.

Step 3. $\frac{\partial M_A(A,b,d,t)}{\partial A} = 128b^2d^2(448b^5d^2 + 64b^6d^2 - 56b^3d(21+40d) + b^4d(-387+656d) + b^2(568+1923d-8768d^2) - 8(449-1064d+576d^2) - 16b(21-607d+672d^2))$

To prove that $M_{AA}(b,d) = 128b^2d^2(448b^5d^2 + 64b^6d^2 - 56b^3d(21+40d) + b^4d(-387+656d) + b^2(568+1923d-8768d^2) - 8(449-1064d+576d^2) - 16b(21-607d+672d^2)) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 4.8 \\ d \geq 1}} M_{AA}(b,d)$$

The minimum value that M_{AA} can attain is $4,536,860,000$, when $b = 4.8$, $d = 1$

Step 4. $\frac{\partial M_A(A,b,d,t)}{\partial t} = -16bd(2b^4(163-21064d)d^2 + 2112b^6d^3 + 384b^7d^3 - 8b^5d^2(199+300d) + b^2d(-13347+119122d-155456d^2) - 4b^3d(7-10270d+30272d^2) - 8(227+1106d-3784d^2+2304d^3) - 4b(-834+9031d-29274d^2+22848d^3))$

To prove that $M_{At}(b,d) = -16bd(2b^4(163-21064d)d^2 + 2112b^6d^3 + 384b^7d^3 - 8b^5d^2(199+300d) + b^2d(-13347+119122d-155456d^2) - 4b^3d(7-10270d+30272d^2) - 8(227+1106d-3784d^2+2304d^3) - 4b(-834+9031d-29274d^2+22848d^3)) < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b\geq 4.8\\ d\geq 1}} M_{At}(b,d)$$

The maximum value that M_{At} can attain is $-459,540,000$, when $b = 4.8$, $d = 1$

Step 5. $\frac{\partial M(A,b,d,t)}{\partial t} = -16Abd(2b^4(163-21064d)d^2 + 2112b^6d^3 + 384b^7d^3 - 8b^5d^2(199+300d) + b^2d(-13347+119122d-155456d^2) - 4b^3d(7-10270d+30272d^2) - 8(227+1106d-3784d^2+2304d^3) - 4b(-834+9031d-29274d^2+22848d^3)) - 2(19528-(94560+141384b+102235b^2)d + 4(54984+192824b+260633b^2+126184b^3+21009b^4)d^2 - 4(64816+351368b+721595b^2+680032b^3+315889b^4+68136b^5+4960b^6)d^3 + 64(6+7b+2b^2)^2(48+232b+331b^2+102b^3+4b^4)d^4)t$

To prove that $M_t(A,b,d,t) = -16Abd(2b^4(163-21064d)d^2 + 2112b^6d^3 + 384b^7d^3 - 8b^5d^2(199+300d) + b^2d(-13347+119122d-155456d^2) - 4b^3d(7-10270d+30272d^2) - 8(227+1106d-3784d^2+2304d^3) - 4b(-834+9031d-29274d^2+22848d^3)) - 2(19528-(94560+141384b+102235b^2)d + 4(54984+192824b+260633b^2+126184b^3+21009b^4)d^2 - 4(64816+351368b+721595b^2+680032b^3+315889b^4+68136b^5+4960b^6)d^3 + 64(6+7b+2b^2)^2(48+232b+331b^2+102b^3+4b^4)d^4)t < 0$, I solve the following optimization problem with constraints:

$$\begin{aligned}
&\underset{b,d}{\text{Max}}\ M_t(A,b,d,t) \\
&\text{subject to} \\
&b \geq 4.8, && (4.29)\\
&d \geq 1, && (4.30)\\
&A = 3, && (4.31)\\
&t = 1 && (4.32)
\end{aligned}$$

The maximum value that M_t can attain is $-18,517,200,000$, when $b = 4.8$, $d = 1$, $A = 3$ and $t = 1$

Step 6. $\frac{\partial M_t(A,b,d,t)}{\partial A} = -16bd(2b^4(163 - 21064d)d^2 + 2112b^6d^3 + 384b^7d^3 - 8b^5d^2(199 + 300d) + b^2d(-13347 + 119122d - 155456d^2) - 4b^3d(7 - 10270d + 30272d^2) - 8(227 + 1106d - 3784d^2 + 2304d^3) - 4b(-834 + 9031d - 29274d^2 + 22848d^3))$

To prove that $M_{tA}(b,d) = -16bd(2b^4(163 - 21064d)d^2 + 2112b^6d^3 + 384b^7d^3 - 8b^5d^2(199 + 300d) + b^2d(-13347 + 119122d - 155456d^2) - 4b^3d(7 - 10270d + 30272d^2) - 8(227 + 1106d - 3784d^2 + 2304d^3) - 4b(-834 + 9031d - 29274d^2 + 22848d^3)) < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b \geq 4.8 \\ d \geq 1}} M_{tA}(b,d)$$

The maximum value that M_{tA} can attain is $-459,540,000$, when $b = 4.8$, $d = 1$

Step 7. $\frac{\partial M_t(A,b,d,t)}{\partial t} = -2(19528 - (94560 + 141384b + 102235b^2)d + 4(54984 + 192824b + 260633b^2 + 126184b^3 + 21009b^4)d^2 - 4(64816 + 351368b + 721595b^2 + 680032b^3 + 315889b^4 + 68136b^5 + 4960b^6)d^3 + 64(6 + 7b + 2b^2)^2(48 + 232b + 331b^2 + 102b^3 + 4b^4)d^4)$

To prove that $M_{tt}(b,d) = -2(19528 - (94560 + 141384b + 102235b^2)d + 4(54984 + 192824b + 260633b^2 + 126184b^3 + 21009b^4)d^2 - 4(64816 + 351368b + 721595b^2 + 680032b^3 + 315889b^4 +$

$68136b^5 + 4960b^6)d^3 + 64(6 + 7b + 2b^2)^2(48 + 232b + 331b^2 + 102b^3 + 4b^4)d^4) < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b \geq 4.8 \\ d \geq 1}} M_{tt}(b, d)$$

The maximum value that M_{tt} can attain is $-17,138,600,000$, when $b = 4.8$, $d = 1$

Step 8. I need to proof that $M_{AA} - M_{tA} > 0$

To prove that $M_{AA}(b, d) - M_{tA}(b, d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 4.8 \\ d \geq 1}} M_{AA}(b, d) - M_{tA}(b, d)$$

The minimum value that $M_{AA} - M_{tA}$ can attain is $4,077,320,000$, when $b = 4.8$, $d = 1$

Step 9. To prove that $N(b, d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 4.8 \\ d \geq 1}} N(b, d)$$

The minimum value that N can attain is $85,472,200,000$, when $b = 4.8$, $d = 1$ ■

Discontinuous areas of $W_1^s - W_2^s$

$W_1^s - W_2^s = -(-64A^2b^2d^2(448b^5d^2 + 64b^6d^2 - 56b^3d(21 + 40d) + b^4d(-387 + 656d) + b^2(568 + 1923d - 8768d^2) - 8(449 - 1064d + 576d^2) - 16b(21 - 607d + 672d^2)) + 16Abd(2b^4(163 - 21064d)d^2 + 2112b^6d^3 + 384b^7d^3 - 8b^5d^2(199 + 300d) + b^2d(-13347 + 119122d - 155456d^2) - 4b^3d(7 - 10270d + 30272d^2) - 8(227 + 1106d - 3784d^2 + 2304d^3) - 4b(-834 + 9031d - 29274d^2 + 22848d^3))t + (19528 - (94560 + 141384b + 102235b^2)d + 4(54984 + 192824b + 260633b^2 + 126184b^3 + 21009b^4)d^2 - 4(64816 + 351368b + 721595b^2 + 680032b^3 + 315889b^4 + 68136b^5 + 4960b^6)d^3 + 64(6 + 7b + 2b^2)^2(48 + 232b + 331b^2 + 102b^3 + 4b^4)d^4)t^2)/(128(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)^2)$

When $(128(64-(164+220b+73b^2)d+4(24+70b+69b^2+28b^3+4b^4)d^2)^2)=0$ are the areas where $W_1^s - W_2^s$ is discontinuous

The following equations are these areas:

$d = \frac{164+220b+73b^2-Sqrt[2320+480b+1688b^2+3448b^3+1233b^4]}{2(96+280b+276b^2+112b^3+16b^4)}$

$d = \frac{164+220b+73b^2+Sqrt[2320+480b+1688b^2+3448b^3+1233b^4]}{2(96+280b+276b^2+112b^3+16b^4)}$

Proposition 6

Claim: $W_1^s > W^c$ when $b \geq 7.8$, $d \geq 0.8$ and $A \geq 2$.

$W_1^s - W^c = -(-64A^2b^2d^2(38832b^9d^4 + 3600b^10d^4 + 4b^7d^3(-54223+35960d)+4b^8d^3(-7289+37801d) + 12b^5d^2(29168+69005d-297536d^2) - 3b^6d^2(-28400+141003d+276512d^2) - 8b^3d(10682+357743d-1244068d^2+949952d^3)-4b^4d(26307+39461d-1294891d^2+1707504d^3)+b^2(46368+749788d-5457664d^2+9650624d^3-5055488d^4)-96(1093-6022d+11824d^2-9984d^3+3072d^4)-64b(1723-21244d+65032d^2-74544d^3+29184d^4))+16Abd(79584b^10d^5+12672b^11d^5 - 8b^9d^4(9067+35488d) - 84b^8d^4(-633+55010d) + 2b^7d^3(54311+2750792d-10774128d^2) - 3b^6d^3(643337-11029886d+18621376d^2) + 4b^4d^2(1198949-16258769d+38936330d^2-24256416d^3)-8b^5d^2(-3382+2408037d-11841610d^2+11400160d^3)-4b^2d(1008051-9842274d+24057824d^2-22208544d^3+7031296d^4)-8b^3d(23353-2828857d+13566844d^2-19200556d^3+8332736d^4) - 32(-28627+139386d-260952d^2+235616d^3-104064d^4+18432d^5) - 16b(-7198+477345d-1853018d^2+2689216d^3-1705760d^4+403968d^5))t + (-3131168+4(3189744+2806744b+1853945b^2)d - 4(3348960+14280064b-2368324b^2-686718b^3+979579b^4)d^2-(9258752+107145600b+300592368b^2+361278816b^3+204256844b^4+495344200b^5+3098471b^6)d^3+4(6804480+58569088b+200785408b^2+359633528b^3+369917256b^4+223847068b^5+77357745b^6+13749130b^7+933699b^8)d^4-4(4586496+44910592b+190967168b^2+456392384b^3+675084060b^4+644963892b^5+402819575b^6+162018588b^7+39885008b^8+5334856b^9+282016b^10)d^5+16(24+46b+29b^2+6b^3)^2(448+3424b+10668b^2+15804b^3+10799b^4+2808b^5+176b^6)d^6)t^2)/(128(-10+(8+18b+9b^2)d)^2(64-(164+220b+73b^2)d+$

$4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)^2)$

I need to prove that the $R = -(-64A^2b^2d^2(38832b^9d^4 + 3600b^10d^4 + 4b^7d^3(-54223 + 35960d) + 4b^8d^3(-7289 + 37801d) + 12b^5d^2(29168 + 69005d - 297536d^2) - 3b^6d^2(-28400 + 141003d + 276512d^2) - 8b^3d(10682 + 357743d - 1244068d^2 + 949952d^3) - 4b^4d(26307 + 39461d - 1294891d^2 + 1707504d^3) + b^2(46368 + 749788d - 5457664d^2 + 9650624d^3 - 5055488d^4) - 96(1093 - 6022d + 11824d^2 - 9984d^3 + 3072d^4) - 64b(1723 - 21244d + 65032d^2 - 74544d^3 + 29184d^4)) + 16Abd(79584b^10d^5 + 12672b^11d^5 - 8b^9d^4(9067 + 35488d) - 84b^8d^4(-633 + 55010d) + 2b^7d^3(54311 + 2750792d - 10774128d^2) - 3b^6d^3(643337 - 11029886d + 18621376d^2) + 4b^4d^2(1198949 - 16258769d + 38936330d^2 - 24256416d^3) - 8b^5d^2(-3382 + 2408037d - 11841610d^2 + 11400160d^3) - 4b^2d(1008051 - 9842274d + 24057824d^2 - 22208544d^3 + 7031296d^4) - 8b^3d(23353 - 2828857d + 13566844d^2 - 19200556d^3 + 8332736d^4) - 32(-28627 + 139386d - 260952d^2 + 235616d^3 - 104064d^4 + 18432d^5) - 16b(-7198 + 477345d - 1853018d^2 + 2689216d^3 - 1705760d^4 + 403968d^5))t + (-3131168 + 4(3189744 + 2806744b + 1853945b^2)d - 4(3348960 + 1428064b - 2368324b^2 - 686718b^3 + 979579b^4)d^2 - (9258752 + 107145600b + 300592368b^2 + 361278816b^3 + 204256844b^4 + 49534420b^5 + 3098471b^6)d^3 + 4(6804480 + 58569088b + 200785408b^2 + 359633528b^3 + 369917256b^4 + 223847068b^5 + 77357745b^6 + 13749130b^7 + 933699b^8)d^4 - 4(4586496 + 44910592b + 190967168b^2 + 456392384b^3 + 675084060b^4 + 644963892b^5 + 402819575b^6 + 162018588b^7 + 39885008b^8 + 5334856b^9 + 282016b^10)d^5 + 16(24 + 46b + 29b^2 + 6b^3)^2(448 + 3424b + 10668b^2 + 15804b^3 + 10799b^4 + 2808b^5 + 176b^6)d^6)t^2) > 0$ and $S = (128(-10 + (8 + 18b + 9b^2)d)^2(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)^2) > 0.$

Step 1. To prove that $R(A, b, d, t) > 0$, where $A > 0$ and $0 \leq t \leq 1$. I solve the following optimization problem with constraints:

$$\underset{b,d}{\text{Min}}\ R(A,b,d,t)$$

subject to

$$b \geq 7.8, \tag{4.33}$$

$$d \geq 0.8, \tag{4.34}$$

$$A = 2, \tag{4.35}$$

$$t = 1 \tag{4.36}$$

The minimum value that R can attain is 1.67539x10^{16}, when $b = 7.8$, $d = 0.8$, $A = 2$ and $t = 1$.

Step 2. $\frac{\partial R(A,b,d,t)}{\partial A} = 128Ab^2d^2(38832b^9d^4+3600b^10d^4+4b^7d^3(-54223+35960d)+4b^8d^3(-7289+37801d) + 12b^5d^2(29168 + 69005d - 297536d^2) - 3b^6d^2(-28400 + 141003d + 276512d^2) - 8b^3d(10682+357743d-1244068d^2+949952d^3)-4b^4d(26307+39461d-1294891d^2+1707504d^3)+b^2(46368 + 749788d - 5457664d^2 + 9650624d^3 - 5055488d^4) - 96(1093 - 6022d + 11824d^2 - 9984d^3 + 3072d^4) - 64b(1723 - 21244d + 65032d^2 - 74544d^3 + 29184d^4)) - 16bd(79584b^10d^5 + 12672b^11d^5 - 8b^9d^4(9067 + 35488d) - 84b^8d^4(-633 + 55010d) + 2b^7d^3(54311 + 2750792d - 10774128d^2) - 3b^6d^3(643337 - 11029886d + 18621376d^2) + 4b^4d^2(1198949 - 16258769d + 38936330d^2-24256416d^3)-8b^5d^2(-3382+2408037d-11841610d^2+11400160d^3)-4b^2d(1008051-9842274d + 24057824d^2 - 22208544d^3 + 7031296d^4) - 8b^3d(23353 - 2828857d + 13566844d^2 - 19200556d^3+8332736d^4)-32(-28627+139386d-260952d^2+235616d^3-104064d^4+18432d^5)-16b(-7198 + 477345d - 1853018d^2 + 2689216d^3 - 1705760d^4 + 403968d^5))t$

To prove that $R_A(A,b,d,t) = 128Ab^2d^2(38832b^9d^4 + 3600b^10d^4 + 4b^7d^3(-54223 + 35960d) + 4b^8d^3(-7289 + 37801d) + 12b^5d^2(29168 + 69005d - 297536d^2) - 3b^6d^2(-28400 + 141003d + 276512d^2)-8b^3d(10682+357743d-1244068d^2+949952d^3)-4b^4d(26307+39461d-1294891d^2+1707504d^3) + b^2(46368 + 749788d - 5457664d^2 + 9650624d^3 - 5055488d^4) - 96(1093 - 6022d + 11824d^2-9984d^3+3072d^4)-64b(1723-21244d+65032d^2-74544d^3+29184d^4))-16bd(79584b^10d^5+12672b^11d^5 - 8b^9d^4(9067 + 35488d) - 84b^8d^4(-633 + 55010d) + 2b^7d^3(54311 + 2750792d -$

$10774128d^2) - 3b^6d^3(643337 - 11029886d + 18621376d^2) + 4b^4d^2(1198949 - 16258769d + 38936330d^2 - 24256416d^3) - 8b^5d^2(-3382 + 2408037d - 11841610d^2 + 11400160d^3) - 4b^2d(1008051 - 9842274d + 24057824d^2 - 22208544d^3 + 7031296d^4) - 8b^3d(23353 - 2828857d + 13566844d^2 - 19200556d^3 + 8332736d^4) - 32(-28627 + 139386d - 260952d^2 + 235616d^3 - 104064d^4 + 18432d^5) - 16b(-7198 + 477345d - 1853018d^2 + 2689216d^3 - 1705760d^4 + 403968d^5))t > 0$, I solve the following optimization problem with constraints:

$$\underset{b,d}{\text{Min}}\ R_A(A,b,d,t)$$

subject to

$$b \geq 7.8, \tag{4.37}$$

$$d \geq 0.8, \tag{4.38}$$

$$A = 3, \tag{4.39}$$

$$t = 1 \tag{4.40}$$

The minimum value that R_A can attain is $4.9445\text{x}10^{16}$, when $b = 7.8$, $d = 0.8$, $A = 3$ and $t = 1$.

Step 3. $\frac{\partial R_A(A,b,d,t)}{\partial A} = 128b^2d^2(38832b^9d^4 + 3600b^10d^4 + 4b^7d^3(-54223 + 35960d) + 4b^8d^3(-7289 + 37801d) + 12b^5d^2(29168 + 69005d - 297536d^2) - 3b^6d^2(-28400 + 141003d + 276512d^2) - 8b^3d(10682 + 357743d - 1244068d^2 + 949952d^3) - 4b^4d(26307 + 39461d - 1294891d^2 + 1707504d^3) + b^2(46368 + 749788d - 5457664d^2 + 9650624d^3 - 5055488d^4) - 96(1093 - 6022d + 11824d^2 - 9984d^3 + 3072d^4) - 64b(1723 - 21244d + 65032d^2 - 74544d^3 + 29184d^4))$

To prove that $R_{AA}(b,d) = 128b^2d^2(38832b^9d^4 + 3600b^10d^4 + 4b^7d^3(-54223 + 35960d) + 4b^8d^3(-7289 + 37801d) + 12b^5d^2(29168 + 69005d - 297536d^2) - 3b^6d^2(-28400 + 141003d + 276512d^2) - 8b^3d(10682 + 357743d - 1244068d^2 + 949952d^3) - 4b^4d(26307 + 39461d - 1294891d^2 + 1707504d^3) + b^2(46368 + 749788d - 5457664d^2 + 9650624d^3 - 5055488d^4) - 96(1093 - 6022d + 11824d^2 - 9984d^3 + 3072d^4) - 64b(1723 - 21244d + 65032d^2 - 74544d^3 + 29184d^4)) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b\geq 7.8\\ d\geq 0.8}} R_{AA}(b,d)$$

The minimum value that R_{AA} can attain is 1.66651x10^{16}, when $b = 7.8$, $d = 0.8$

Step 4. $\frac{\partial R_A(A,b,d,t)}{\partial t} = -16bd(79584b^10d^5 + 12672b^11d^5 - 8b^9d^4(9067 + 35488d) - 84b^8d^4(-633 + 55010d) + 2b^7d^3(54311 + 2750792d - 10774128d^2) - 3b^6d^3(643337 - 11029886d + 18621376d^2) + 4b^4d^2(1198949 - 16258769d + 38936330d^2 - 24256416d^3) - 8b^5d^2(-3382 + 2408037d - 11841610d^2 + 11400160d^3) - 4b^2d(1008051 - 9842274d + 24057824d^2 - 22208544d^3 + 7031296d^4) - 8b^3d(23353 - 2828857d + 13566844d^2 - 19200556d^3 + 8332736d^4) - 32(-28627 + 139386d - 260952d^2 + 235616d^3 - 104064d^4 + 18432d^5) - 16b(-7198 + 477345d - 1853018d^2 + 2689216d^3 - 1705760d^4 + 403968d^5))$

To prove that $R_{At}(b,d) = -16bd(79584b^10d^5 + 12672b^11d^5 - 8b^9d^4(9067 + 35488d) - 84b^8d^4(-633 + 55010d) + 2b^7d^3(54311 + 2750792d - 10774128d^2) - 3b^6d^3(643337 - 11029886d + 18621376d^2) + 4b^4d^2(1198949 - 16258769d + 38936330d^2 - 24256416d^3) - 8b^5d^2(-3382 + 2408037d - 11841610d^2 + 11400160d^3) - 4b^2d(1008051 - 9842274d + 24057824d^2 - 22208544d^3 + 7031296d^4) - 8b^3d(23353 - 2828857d + 13566844d^2 - 19200556d^3 + 8332736d^4) - 32(-28627 + 139386d - 260952d^2 + 235616d^3 - 104064d^4 + 18432d^5) - 16b(-7198 + 477345d - 1853018d^2 + 2689216d^3 - 1705760d^4 + 403968d^5)) < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b\geq 7.8\\ d\geq 0.8}} R_{At}(b,d)$$

The maximum value that R_{At} can attain is -5.50318x10^{14}, when $b = 7.8$, $d = 0.8$

Step 5. $\frac{\partial R(A,b,d,t)}{\partial t} = -16Abd(79584b^10d^5 + 12672b^11d^5 - 8b^9d^4(9067 + 35488d) - 84b^8d^4(-633 + 55010d) + 2b^7d^3(54311 + 2750792d - 10774128d^2) - 3b^6d^3(643337 - 11029886d + 18621376d^2) + 4b^4d^2(1198949 - 16258769d + 38936330d^2 - 24256416d^3) - 8b^5d^2(-3382 + 2408037d - 11841610d^2 + 11400160d^3) - 4b^2d(1008051 - 9842274d + 24057824d^2 - 22208544d^3 + 7031296d^4) - 8b^3d(23353 - 2828857d + 13566844d^2 - 19200556d^3 + 8332736d^4) - 32(-28627 + 139386d - 260952d^2 +$

$235616d^3-104064d^4+18432d^5)-16b(-7198+477345d-1853018d^2+2689216d^3-1705760d^4+403968d^5))-2(-3131168+4(3189744+2806744b+1853945b^2)d-4(3348960+1428064b-2368324b^2-686718b^3+979579b^4)d^2-(9258752+107145600b+300592368b^2+361278816b^3+204256844b^4+49534420b^5+3098471b^6)d^3+4(6804480+58569088b+200785408b^2+359633528b^3+369917256b^4+223847068b^5+77357745b^6+13749130b^7+933699b^8)d^4-4(4586496+44910592b+190967168b^2+456392384b^3+675084060b^4+644963892b^5+402819575b^6+162018588b^7+39885008b^8+5334856b^9+282016b^10)d^5+16(24+46b+29b^2+6b^3)^2(448+3424b+10668b^2+15804b^3+10799b^4+2808b^5+176b^6)d^6)t$

To prove that $R_t(A,b,d,t) = -16Abd(79584b^10d^5+12672b^11d^5-8b^9d^4(9067+35488d)-84b^8d^4(-633+55010d)+2b^7d^3(54311+2750792d-10774128d^2)-3b^6d^3(643337-11029886d+18621376d^2)+4b^4d^2(1198949-16258769d+38936330d^2-24256416d^3)-8b^5d^2(-3382+2408037d-11841610d^2+11400160d^3)-4b^2d(1008051-9842274d+24057824d^2-22208544d^3+7031296d^4)-8b^3d(23353-2828857d+13566844d^2-19200556d^3+8332736d^4)-32(-28627+139386d-260952d^2+235616d^3-104064d^4+18432d^5)-16b(-7198+477345d-1853018d^2+2689216d^3-1705760d^4+403968d^5))-2(-3131168+4(3189744+2806744b+1853945b^2)d-4(3348960+1428064b-2368324b^2-686718b^3+979579b^4)d^2-(9258752+107145600b+300592368b^2+361278816b^3+204256844b^4+49534420b^5+3098471b^6)d^3+4(6804480+58569088b+200785408b^2+359633528b^3+369917256b^4+223847068b^5+77357745b^6+13749130b^7+933699b^8)d^4-4(4586496+44910592b+190967168b^2+456392384b^3+675084060b^4+644963892b^5+402819575b^6+162018588b^7+39885008b^8+5334856b^9+282016b^10)d^5+16(24+46b+29b^2+6b^3)^2(448+3424b+10668b^2+15804b^3+10799b^4+2808b^5+176b^6)d^6)t < 0$, I solve the following optimization problem with constraints:

$$\underset{b,d}{\text{Max}}\ R_t(A,b,d,t)$$

subject to

$$b \geq 7.8, \tag{4.41}$$

$$d \geq 0.8, \tag{4.42}$$

$$A = 3, \tag{4.43}$$

$$t = 0 \tag{4.44}$$

The maximum value that R_t can attain is -1.65095x10^{15}, when $b = 7.8$, $d = 0.8$, $A = 3$ and $t = 0$

Step 6. $\frac{\partial R_t(A,b,d,t)}{\partial A} = -16bd(79584b^10d^5 + 12672b^11d^5 - 8b^9d^4(9067 + 35488d) - 84b^8d^4(-633 + 55010d) + 2b^7d^3(54311 + 2750792d - 10774128d^2) - 3b^6d^3(643337 - 11029886d + 18621376d^2) + 4b^4d^2(1198949 - 16258769d + 38936330d^2 - 24256416d^3) - 8b^5d^2(-3382 + 2408037d - 11841610d^2 + 11400160d^3) - 4b^2d(1008051 - 9842274d + 24057824d^2 - 22208544d^3 + 7031296d^4) - 8b^3d(23353 - 2828857d + 13566844d^2 - 19200556d^3 + 8332736d^4) - 32(-28627 + 139386d - 260952d^2 + 235616d^3 - 104064d^4 + 18432d^5) - 16b(-7198 + 477345d - 1853018d^2 + 2689216d^3 - 1705760d^4 + 403968d^5))$

To prove that $R_{tA}(b,d) = -16bd(79584b^10d^5 + 12672b^11d^5 - 8b^9d^4(9067 + 35488d) - 84b^8d^4(-633 + 55010d) + 2b^7d^3(54311 + 2750792d - 10774128d^2) - 3b^6d^3(643337 - 11029886d + 18621376d^2) + 4b^4d^2(1198949 - 16258769d + 38936330d^2 - 24256416d^3) - 8b^5d^2(-3382 + 2408037d - 11841610d^2 + 11400160d^3) - 4b^2d(1008051 - 9842274d + 24057824d^2 - 22208544d^3 + 7031296d^4) - 8b^3d(23353 - 2828857d + 13566844d^2 - 19200556d^3 + 8332736d^4) - 32(-28627 + 139386d - 260952d^2 + 235616d^3 - 104064d^4 + 18432d^5) - 16b(-7198 + 477345d - 1853018d^2 + 2689216d^3 - 1705760d^4 + 403968d^5)) < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b \geq 7.8 \\ d \geq 0.8}} R_{tA}(b,d)$$

The maximum value that R_{tA} can attain is $-5.50318\text{x}10^{14}$, when $b = 7.8$, $d = 0.8$

Step 7. $\frac{\partial R_t(A,b,d,t)}{\partial t} = -2(-3131168 + 4(3189744 + 2806744b + 1853945b^2)d - 4(3348960 + 1428064b - 2368324b^2 - 686718b^3 + 979579b^4)d^2 - (9258752 + 107145600b + 300592368b^2 + 361278816b^3 + 204256844b^4 + 49534420b^5 + 3098471b^6)d^3 + 4(6804480 + 58569088b + 200785408b^2 + 3596335283b^3 + 369917256b^4 + 223847068b^5 + 77357745b^6 + 13749130b^7 + 933699b^8)d^4 - 4(4586496 + 44910592b + 1909671168b^2 + 456392384b^3 + 675084060b^4 + 644963892b^5 + 402819575b^6 + 162018588b^7 + 39885008b^8 + 5334856b^9 + 282016b^10)d^5 + 16(24 + 46b + 29b^2 + 6b^3)^2(448 + 3424b + 10668b^2 + 15804b^3 + 10799b^4 + 2808b^5 + 176b^6)d^6)$

To prove that $R_{tt}(b,d) = -2(-3131168 + 4(3189744 + 2806744b + 1853945b^2)d - 4(3348960 + 1428064b - 2368324b^2 - 686718b^3 + 979579b^4)d^2 - (9258752 + 107145600b + 300592368b^2 + 361278816b^3 + 204256844b^4 + 49534420b^5 + 3098471b^6)d^3 + 4(6804480 + 58569088b + 200785408b^2 + 3596335283b^3 + 369917256b^4 + 223847068b^5 + 77357745b^6 + 13749130b^7 + 933699b^8)d^4 - 4(4586496 + 44910592b + 1909671168b^2 + 456392384b^3 + 675084060b^4 + 644963892b^5 + 402819575b^6 + 162018588b^7 + 39885008b^8 + 5334856b^9 + 282016b^10)d^5 + 16(24 + 46b + 29b^2 + 6b^3)^2(448 + 3424b + 10668b^2 + 15804b^3 + 10799b^4 + 2808b^5 + 176b^6)d^6) < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b \geq 7.8 \\ d \geq 0.8}} R_{tt}(b,d)$$

The maximum value that R_{tt} can attain is $-3.09514\text{x}10^{16}$, when $b = 7.8$, $d = 0.8$

Step 8. I need to proof that $R_{AA} - R_{tA} > 0$

To prove that $R_{AA}(b,d) - R_{tA}(b,d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 7.8 \\ d \geq 0.8}} R_{AA}(b,d) - R_{tA}(b,d)$$

The minimum value that $R_{AA} - R_{tA}$ can attain is $1.61148\text{x}10^{16}$, when $b = 7.8$, $d = 0.8$

Step 9. To prove that $S(b,d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b\geq 7.8\\ d\geq 0.8}} S(b,d)$$

The minimum value that S can attain 2.3962x10^{17}, when $b = 7.8$, $d = 0.8$ ■

Discontinuous areas of $W_1^s - W^c$

$W_1^s - W^c = -(-64A^2b^2d^2(38832b^9d^4 + 3600b^10d^4 + 4b^7d^3(-54223 + 35960d) + 4b^8d^3(-7289 + 37801d) + 12b^5d^2(29168 + 69005d - 297536d^2) - 3b^6d^2(-28400 + 141003d + 276512d^2) - 8b^3d(10682 + 357743d - 1244068d^2 + 949952d^3) - 4b^4d(26307 + 39461d - 1294891d^2 + 1707504d^3) + b^2(46368 + 749788d - 5457664d^2 + 9650624d^3 - 5055488d^4) - 96(1093 - 6022d + 11824d^2 - 9984d^3 + 3072d^4) - 64b(1723 - 21244d + 65032d^2 - 74544d^3 + 29184d^4)) + 16Abd(79584b^10d^5 + 12672b^11d^5 - 8b^9d^4(9067 + 35488d) - 84b^8d^4(-633 + 55010d) + 2b^7d^3(54311 + 2750792d - 10774128d^2) - 3b^6d^3(643337 - 11029886d + 18621376d^2) + 4b^4d^2(1198949 - 16258769d + 38936330d^2 - 24256416d^3) - 8b^5d^2(-3382 + 2408037d - 11841610d^2 + 11400160d^3) - 4b^2d(1008051 - 9842274d + 24057824d^2 - 22208544d^3 + 7031296d^4) - 8b^3d(23353 - 2828857d + 13566844d^2 - 19200556d^3 + 8332736d^4) - 32(-28627 + 139386d - 260952d^2 + 235616d^3 - 104064d^4 + 18432d^5) - 16b(-7198 + 477345d - 1853018d^2 + 2689216d^3 - 1705760d^4 + 403968d^5))t + (-3131168 + 4(3189744 + 2806744b + 1853945b^2)d - 4(3348960 + 14280064b - 2368324b^2 - 686718b^3 + 979579b^4)d^2 - (9258752 + 107145600b + 300592368b^2 + 361278816b^3 + 204256844b^4 + 495334420b^5 + 3098471b^6)d^3 + 4(6804480 + 58569088b + 200785408b^2 + 359633528b^3 + 369917256b^4 + 223847068b^5 + 77357745b^6 + 13749130b^7 + 933699b^8)d^4 - 4(4586496 + 44910592b + 190967168b^2 + 456392384b^3 + 675084060b^4 + 644963892b^5 + 402819575b^6 + 162018588b^7 + 39885008b^8 + 5334856b^9 + 282016b^10)d^5 + 16(24 + 46b + 29b^2 + 6b^3)^2(448 + 3424b + 10668b^2 + 15804b^3 + 10799b^4 + 2808b^5 + 176b^6)d^6)t^2)/(128(-10 + (8 + 18b + 9b^2)d)^2(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 + 4b^4)d^2)^2)$

When $(128(-10 + (8 + 18b + 9b^2)d)^2(64 - (164 + 220b + 73b^2)d + 4(24 + 70b + 69b^2 + 28b^3 +$

$4b^4)d^2)^2) = 0$ are the areas where $W_1^s - W^c$ is discontinuous

The following equations are these areas:

$d = \frac{10}{8+18b+9b^2}$

$d = \frac{164+220b+73b^2-Sqrt[2320+480b+1688b^2+3448b^3+1233b^4]}{2(96+280b+276b^2+112b^3+16b^4)}$

$d = \frac{164+220b+73b^2+Sqrt[2320+480b+1688b^2+3448b^3+1233b^4]}{2(96+280b+276b^2+112b^3+16b^4)}$

Proposition 7

Claim: $W^c > W^a$ when $b \geq 2.1$, $d \geq 0.6$ and $A \geq 3$.

$W^c - W^a = (A^2b^2d^2(-4-64d+84b^3d+45b^4d-4b^2(19+13d)-8b(-13+20d))-2A(-2+b)bd(78-(74+176b+131b^2)d+(4+11b+6b^2)^2d^2)t-(84-4(22+25b+49b^2)d+5(4-4b-20b^2-6b^3+17b^4)d^2+b(2+b)(4+11b+6b^2)^2d^3)t^2)/(2(-3+(1+2b)^2d)(-10+(8+18b+9b^2)d)^2)$

I need to prove that the $T = (A^2b^2d^2(-4-64d+84b^3d+45b^4d-4b^2(19+13d)-8b(-13+20d))-2A(-2+b)bd(78-(74+176b+131b^2)d+(4+11b+6b^2)^2d^2)t-(84-4(22+25b+49b^2)d+5(4-4b-20b^2-6b^3+17b^4)d^2+b(2+b)(4+11b+6b^2)^2d^3)t^2) > 0$ and $U = (2(-3+(1+2b)^2d)(-10+(8+18b+9b^2)d)^2) > 0$.

Step 1. To prove that $T(A,b,d,t) > 0$, where $A > 0$ and $0 \leq t \leq 1$. I solve the following optimization problem with constraints:

$$\underset{b,d}{\text{Min}}\ T(A,b,d,t)$$

subject to

$$b \geq 2.1, \tag{4.45}$$

$$d \geq 0.6, \tag{4.46}$$

$$A = 3, \tag{4.47}$$

$$t = 1 \tag{4.48}$$

The minimum value that T can attain is $1,624.86$, when $b = 2.1$, $d = 0.6$, $A = 3$ and $t = 1$.

Step 2. $\frac{\partial T(A,b,d,t)}{\partial A} = 2Ab^2d^2(-4 - 64d + 84b^3d + 45b^4d - 4b^2(19 + 13d) - 8b(-13 + 20d)) - 2(-2 + b)bd(78 - (74 + 176b + 131b^2)d + (4 + 11b + 6b^2)^2d^2)t$

To prove that $T_A(A,b,d,t) = 2Ab^2d^2(-4 - 64d + 84b^3d + 45b^4d - 4b^2(19 + 13d) - 8b(-13 + 20d)) - 2(-2 + b)bd(78 - (74 + 176b + 131b^2)d + (4 + 11b + 6b^2)^2d^2)t > 0$, I solve the following optimization problem with constraints:

$$\underset{b,d}{\text{Min}}\ T_A(A,b,d,t)$$

subject to

$$b \geq 2.1, \tag{4.49}$$

$$d \geq 0.6, \tag{4.50}$$

$$A = 3, \tag{4.51}$$

$$t = 1 \tag{4.52}$$

The minimum value that T_A can attain is $4,575.41$, when $b = 2.1$, $d = 0.6$, $A = 3$ and $t = 1$.

Step 3. $\frac{\partial T_A(A,b,d,t)}{\partial A} = 2b^2d^2(-4 - 64d + 84b^3d + 45b^4d - 4b^2(19 + 13d) - 8b(-13 + 20d))$

To prove that $T_{AA}(b, d) = 2b^2d^2(-4-64d+84b^3d+45b^4d-4b^2(19+13d)-8b(-13+20d)) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b\geq 2.1\\ d\geq 0.6}} T_{AA}(b, d)$$

The minimum value that T_{AA} can attain is $1,566.96$, when $b = 2.1$, $d = 0.6$

Step 4. $\frac{\partial T_A(A,b,d,t)}{\partial t} = -2(-2 + b)bd(78 - (74 + 176b + 131b^2)d + (4 + 11b + 6b^2)^2d^2)$

To prove that $T_{At}(b, d) = -2(-2 + b)bd(78 - (74 + 176b + 131b^2)d + (4 + 11b + 6b^2)^2d^2) < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b\geq 2.1\\ d\geq 0.6}} T_{At}(b, d)$$

The maximum value that T_{At} can attain is -125.48, when $b = 2.1$, $d = 0.6$

Step 5. $\frac{\partial T(A,b,d,t)}{\partial t} = -2A(-2 + b)bd(78 - (74 + 176b + 131b^2)d + (4 + 11b + 6b^2)^2d^2) - 2(84 - 4(22 + 25b + 49b^2)d + 5(4 - 4b - 20b^2 - 6b^3 + 17b^4)d^2 + b(2 + b)(4 + 11b + 6b^2)^2d^3)t$

To prove that $T_t(A, b, d, t) = -2A(-2+b)bd(78-(74+176b+131b^2)d+(4+11b+6b^2)^2d^2) - 2(84-4(22+25b+49b^2)d+5(4-4b-20b^2-6b^3+17b^4)d^2+b(2+b)(4+11b+6b^2)^2d^3)t < 0$, I solve the following optimization problem with constraints:

$$\underset{b,d}{\text{Max}}\ T_t(A,b,d,t)$$

subject to

$$b \geq 2.1, \tag{4.53}$$

$$d \geq 0.6, \tag{4.54}$$

$$A = 3, \tag{4.55}$$

$$t = 0 \tag{4.56}$$

The maximum value that T_t can attain is -376.44, when $b = 2.1$, $d = 0.6$, $A = 3$ and $t = 0$

Step 6. $\frac{\partial T_t(A,b,d,t)}{\partial A} = -2(-2+b)bd(78-(74+176b+131b^2)d+(4+11b+6b^2)^2d^2)$

To prove that $T_{tA}(b,d) = -2(-2+b)bd(78-(74+176b+131b^2)d+(4+11b+6b^2)^2d^2) < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b\geq 2.1\\ d\geq 0.6}} T_{tA}(b,d)$$

The maximum value that T_{tA} can attain is -125.48, when $b = 2.1$, $d = 0.6$

Step 7. $\frac{\partial T_t(A,b,d,t)}{\partial t} = -2(84-4(22+25b+49b^2)d+5(4-4b-20b^2-6b^3+17b^4)d^2+b(2+b)(4+11b+6b^2)^2d^3)$

To prove that $T_{tt}(b,d) = -2(84-4(22+25b+49b^2)d+5(4-4b-20b^2-6b^3+17b^4)d^2+b(2+b)(4+11b+6b^2)^2d^3) < 0$, I solve the following optimization problem with constraints:

$$\max_{\substack{b\geq 2.1\\ d\geq 0.6}} T_{tt}(b,d)$$

The maximum value that T_{tt} can attain is $-10,100.10$, when $b = 2.1$, $d = 0.6$

Step 8. I need to proof that $T_{AA} - T_{tA} > 0$

To prove that $T_{AA}(b,d) - T_{tA}(b,d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 2.1 \\ d \geq 0.6}} T_{AA}(b,d) - T_{tA}(b,d)$$

The minimum value that $T_{AA} - T_{tA}$ can attain is $1,441.48$, when $b = 2.1$, $d = 0.6$

Step 9. To prove that $U(b,d) > 0$, I solve the following optimization problem with constraints:

$$\min_{\substack{b \geq 2.1 \\ d \geq 0.6}} U(b,d)$$

The minimum value that U can attain is $45,099$, when $b = 2.1$, $d = 0.6$ ■

Discontinuous areas of $W^c - W^a$

$W^c - W^a = (A^2b^2d^2(-4 - 64d + 84b^3d + 45b^4d - 4b^2(19 + 13d) - 8b(-13 + 20d)) - 2A(-2 + b)bd(78 - (74 + 176b + 131b^2)d + (4 + 11b + 6b^2)^2d^2)t - (84 - 4(22 + 25b + 49b^2)d + 5(4 - 4b - 20b^2 - 6b^3 + 17b^4)d^2 + b(2 + b)(4 + 11b + 6b^2)^2d^3)t^2)/(2(-3 + (1 + 2b)^2d)(-10 + (8 + 18b + 9b^2)d)^2)$

When $(2(-3 + (1 + 2b)^2d)(-10 + (8 + 18b + 9b^2)d)^2) = 0$ are the areas where $W^c - W^a$ is discontinuous

The following equations are these areas:

$d = \frac{3}{(1+2b)^2}$

$d = \frac{10}{8+18b+9b^2}$