

## ON THE MEAN-STANDARD DEVIATION FRONTIER

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**Resumen:** En este trabajo se presenta un análisis de la Frontera de Media Desviación Estándar (MSF) en términos de dos portafolios notables, y se estudia la localización geométrica de dichos portafolios en la frontera. Se presenta un resumen de resultados ya conocidos junto con demostraciones de resultados nuevos. Se introduce una manera de medir la distancia entre dos MSF. Esta medida está relacionada con modelos de valuación de activos que suponen que los precios de los activos pueden ser representados por un factor estocástico de descuento, como el CAPM (Capital Asset Pricing Model) y el APT (Arbitrage Pricing Theory). Se da una aplicación en la que la distancia entre dos fronteras específicas se puede interpretar como una medida del error de especificación de un modelo de valuación de activos.

**Abstract:** This paper presents a characterization of the mean-standard deviation frontier (MSF) in terms of pricing and averaging securities and explores the geometry of these securities relative to the geometry of the MSF. A summary of already known results is presented along with proof of new results. A measure of the distance between two mean-standard deviation frontiers is presented here. This measure is related to asset pricing models which imply that security prices can be represented by a stochastic discount factor, such as the CAPM (Capital Asset Pricing Model) and the APT (Arbitrage Pricing Theory). An application is given in which the distance between two specific frontiers can be interpreted as a measure of model misspecification.

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## 1. Introduction

An important concept in finance is that of the mean-standard deviation frontier (MSF). A number of asset pricing models, including the Capital Asset Pricing Model (CAPM) and the exact version of the Arbitrage Pricing Theory (APT), conclude that specific portfolios are in the MSF.

This paper has two purposes: The first is to give a characterization of the MSF in terms of the *pricing security* and the *averaging security*, and to explore the geometry of these securities relative to the geometry of the MSF.

The above securities are defined as follows: Given the assumption of linearity in markets (i.e. any linear combination of two marketed securities is, itself, a marketed security, while the price of the linear combination is the linear combination of the two first prices) if the marketed space  $M$  is finite dimensional, by the Riesz Representation Theorem, there exists a unique security  $c$  in  $M$  (or  $c^*$  in  $M$ ) called the pricing security, such that for all securities  $z$  in  $M$ , its price  $\pi(z)$  can be represented as

$$\pi(z) = E(zc) \text{ (or } \pi(z) = \text{Cov}(z, c^*)).$$

Also, there exists a unique security  $m$  in  $M$  (or  $m^*$  in  $M$ ) called the averaging security, such that for all  $z$  in  $M$ , its expected value  $E(z)$  can be represented as

$$E(z) = E(zm) \text{ (or } E(z) = \text{Cov}(z, m^*)).$$

The second purpose is to suggest a definition of a measurement of a distance between mean-standard deviation frontiers. This measurement can be interpreted as a measure of model misspecification.

Section 2 sets up the Hilbert space framework (Chamberlain and Rothschild, 1983; Duffie, 1988). Section 3 deals with the concept of the MSF, (Huang and Litzenberger, 1988; Ingersoll, 1987). The pricing portfolio and the averaging portfolio are defined, and a characterization of the MSF is given in terms of these portfolios. Section 4 presents a description of geometrical properties of these securities relative to the MSF in the mean-standard deviation space. Section 5 presents a way to measure the distance between mean-standard deviation frontiers. Sec-

tion 6 gives an example in which the distance between two specific frontiers can be interpreted as a measure of model misspecification. Section 7 summarizes the results of the paper.

## 2. The Setting

In setting up the Hilbert space framework (Chamberlain and Rothschild, 1983; Duffie, 1988), there is an underlying probability space  $(\Omega, F, P)$ .  $L^2(P)$  is the vector space of real-valued random variables whose variances exist. A *security* for an economy is an element  $z$  in  $L^2(P)$ , and we may think of it as a random variable whose payoff,  $z(\omega)$  in state  $\omega \in \Omega$ , is measured in "consumption".

The *riskless security* is the random variable  $\mathbf{1} : \Omega \rightarrow R$  such that  $\mathbf{1}(\omega) = 1$  for all  $\omega \in \Omega$ . A *risky security* is any security with positive variance.

A *portfolio* is a finite linear combination of securities. That is,  $p$  is a portfolio if and only if there exists some finite subset  $\{x_1, x_2, \dots, x_N\}$  of  $L^2(P)$  and real numbers  $\alpha_1, \alpha_2, \dots, \alpha_N$  such that

$$p = \sum_{i=1}^N \alpha_i x_i.$$

In the case of incomplete markets, not any security is available "at a price". Let  $T \subseteq L^2(P)$  be the subset of traded securities. We assume that if  $x_1, x_2, \dots, x_n$  are traded securities, then any portfolio of them is also in  $T$ .

A given non-zero linear functional  $\pi$  on  $T$  assigns market values to marketed assets. For any security  $z$  in  $T$  whose market value  $\pi(z)$  is not zero, the return of  $z$  is the random variable

$$R_z = \frac{z}{\pi(z)}.$$

Then, for any portfolio  $p = \sum_{i=1}^N \alpha_i x_i$  of traded securities such that the market values  $\pi(p), \pi(x_1), \pi(x_2), \dots, \pi(x_N)$  are not zero, the return of  $p$  is the random variable

$$R_p = \sum_{i=1}^N \omega_i R_i$$

where  $R_i$  is the return of  $x_i$ , and  $\omega_i \equiv \frac{\alpha_i \pi(x_i)}{\pi(p)}$  is the proportion of the price of the portfolio  $p$  associated to security  $x_i$  and, hence,  $\sum_{i=1}^N \omega_i = 1$ .

### 3. The Mean-standard Deviation Frontier

Let  $x$  denote the vector  $(x_1, x_2, \dots, x_N)'$  of  $N$  risky securities  $x_1, x_2, \dots, x_N$  ( $N \geq 1$ ) with linearly independent returns, and let  $R$  denote the corresponding vector of returns  $(R_1, R_2, \dots, R_N)'$ . It is assumed that  $\Sigma_R$ , the covariance matrix of the vector  $R$ , is nonsingular. This implies that the riskless security **1** is not a portfolio of the returns  $R_1, R_2, \dots, R_N$ .

If  $Y = (y_{ij})$  is a  $p \times q$  random matrix, let  $E(Y)$  denote the matrix whose  $i$ - $j$ th element is  $E(y_{ij})$ , and if all the elements  $y_{ij}$  are traded securities, let  $\pi(Y)$  denote the matrix whose  $i$ - $j$ th element is  $\pi(y_{ij})$ .

The concept of mean-standard deviation frontier can be found in standard finance books such as Huang and Litzenberger (1988) and Ingersoll (1987). Given a set  $S$  of  $M$  traded securities ( $M \geq 1$ ), a portfolio of them is the *frontier portfolio* with expected return  $E$  if its return has the minimum variance among returns of portfolios (of securities in  $S$ ) that have expected return  $E$ .

The actual formulation of the minimization problem above depends mainly on whether the riskless security **1** is, or is not, a portfolio of the securities considered.

For the case in which the riskless security **1** is not a portfolio of the securities considered, the minimization problem can be formulated as follows. A portfolio  $p$  of  $x_1, x_2, \dots, x_N$  with return  $\omega'_E R$  is a frontier portfolio with expected return  $\bar{E}$  if and only if  $\omega_E$  solves the problem:

$$\begin{aligned} \min_{\omega \in \mathbf{R}^N} \quad & \omega' \Sigma_R \omega \\ \text{s.t.} \quad & \omega' E(R) = E \text{ and } \omega' \mathbf{1}_N = 1 \end{aligned}$$

where  $\Sigma_R$  is the variance-covariance matrix of the random vector  $R$ , and  $\mathbf{1}_N$  is the  $N$ -vector of ones.

For the case in which the riskless security **1** with return  $R_0 = 1 / \pi(\mathbf{1})$  is included, the minimization problem can be formulated as follows. A portfolio  $p$  of the riskless security **1** and the risky securities  $x_1, x_2, \dots, x_N$

( $N \geq 1$ ) with return  $\omega_0 R_0 + \omega'_E R$  is a frontier portfolio with expected return  $E$  if and only if  $\omega_0 \in \mathbf{R}$  and  $\omega_E \in \mathbf{R}^N$  solve the problem:

$$\begin{aligned} \min_{\omega_0 \in \mathbf{R}, \omega_E \in \mathbf{R}^N} \quad & \omega'_E \Sigma_R \omega_E \\ \text{s.t.} \quad & \omega_0 R_0 + \omega'_E E(R) = E \text{ and } \omega_0 + \omega'_E \mathbf{1}_N = 1. \end{aligned}$$

In both cases, for each  $E \in \mathbf{R}$  for which the minimization problem above has a solution, let  $\sigma^2(E) \equiv \omega'_E \Sigma_R \omega_E$  (i.e.  $\sigma^2(E)$  is the variance of the frontier portfolio with expected return  $E$ ).

The *mean-standard deviation frontier* generated by the securities considered is the set:

$$MSF \equiv \{(\sigma(E), E) \in \mathbf{R}^2 \mid \text{the minimization problem above has a solution for } E\}.$$

The mean-standard deviation frontier can be described in terms of the portfolios  $m_R, c_R, m_R^*, c_R^*$  and  $c_{1,R}$  (Chamberlain and Rothschild, 1983) defined by the following:

DEFINITION 1. *Regardless of whether the riskless security  $\mathbf{1}$  is included, or not, let  $c_R, c_R^*, m_R$  and  $m_R^*$  be the portfolios of the form  $\alpha'R$  ( $\alpha \in \mathbf{R}^N$ ) that satisfy:*

$$\mathbf{1}_N = E(Rc_R)$$

$$\mathbf{1}_N = \text{Cov}(R, c_R^*)$$

$$E(R) = E(Rm_R)$$

$$E(R) = \text{Cov}(R, m_R^*) .$$

$c_R$  and  $c_R^*$  are known as *pricing securities*.  $m_R$  and  $m_R^*$  are known as *averaging securities*: Given the assumption of linearity in markets, together with the fact that expected value is a linear operator and covariance is a bilinear operator, from the four equations above, we get that for any portfolio  $p = \alpha'x$ ,  $\pi(p) = E(pc_R) = \text{Cov}(p, c_R^*)$  and  $E(p) = E(pm_R) = \text{Cov}(p, m_R^*)$ .

When the riskless security  $\mathbf{1}$  is included, let  $c_{1,R}$  be the portfolio of the form  $\alpha_0 \mathbf{1} + \alpha' R$  ( $\alpha_0 \in \mathbf{R}$ ,  $\alpha \in \mathbf{R}^N$ ) that satisfies:

$$\begin{bmatrix} \pi(\mathbf{1}) \\ \mathbf{1}_N \end{bmatrix} = E \left( \begin{bmatrix} \mathbf{1} \\ R \end{bmatrix} c_{1,R} \right).$$

$c_{1,R}$  is known as *pricing security*: For any portfolio  $p = \gamma_0 \mathbf{1} + \gamma' x$  ( $\gamma_0 \in \mathbf{R}$ ,  $\gamma \in \mathbf{R}^N$ ),  $\pi(p) = E(p c_{1,R})$ .

Given that we are considering a finite dimensional vector  $R$  of returns, it is straightforward to show (see Hansen and Jagannathan, 1993) that:

$$\begin{aligned} c_R &\equiv \mathbf{1}'_N (E(RR'))^{-1} R \\ c_R^* &\equiv \mathbf{1}'_N \Sigma_R^{-1} R \end{aligned} \quad (1)^1$$

$$m_R \equiv E(R)' (E(RR'))^{-1} R$$

$$m_R^* \equiv E(R)' \Sigma_R^{-1} R$$

$$c_{1,R} = \frac{\pi(\mathbf{1}) - E(c_R)}{1 - E(m_R)} + \left( c_R - \frac{\pi(\mathbf{1}) - E(c_R)}{1 - E(m_R)} m_R \right). \quad (2)^2$$

Chamberlain and Rothschild (1983) show that:

$$m_R^* = \frac{m_R}{1 - E(m_R)} \quad (3)$$

and

$$c_R^* = c_R + E(c_R) \frac{m_R}{1 - E(m_R)}. \quad (4)$$

<sup>1</sup>  $E(RR') = \Sigma_R + E(R)E(R)'$ . Given that  $\Sigma_R$  is nonsingular,  $E(RR')$  is nonsingular and  $(E(RR'))^{-1} = \Sigma_R^{-1} - \Sigma_R^{-1}E(R) \frac{1}{1 + E(R)' \Sigma_R^{-1} E(R)} E(R)' \Sigma_R^{-1}$  (see Rao, 1973).

<sup>2</sup>  $E(m_R) \neq 1$ . Otherwise, given that  $\text{var}(m_R) = E(m_R)(1 - E(m_R))$ ,  $\text{var}(m_R) = 0$  and  $m_R = 1$ . This contradicts the fact that 1 is not a portfolio of  $R_1, R_2, \dots, R_N$  because it is assumed that  $\Sigma_R$  is nonsingular.

When the riskless security **1** is not included, and  $N = 1$ , the frontier consists of only one point in the  $(\sigma, E)$ -space. When the riskless security **1** is not included,  $N \geq 2$ , and the  $N$  returns are independently and identically distributed with mean  $E_0$  and variance  $\sigma_0^2$ , the frontier is the horizontal line

$$\{(\sigma, E_0) \mid \sigma \geq \frac{\sigma_0}{\sqrt{N}}\}.$$

When the riskless security **1** is not included,  $N \geq 2$ , the  $N$  returns are linearly independent, and at least two returns have different expected values, the mean-standard deviation frontier is a hyperbola with center

$$\left(0, \frac{E(c_R^*)}{\text{Var}(c_R^*)}\right),$$

vertex

$$\left(\frac{1}{\sqrt{\text{Var}(c_R^*)}}, \frac{E(c_R^*)}{\text{Var}(c_R^*)}\right)$$

and squared slope of the asymptotes  $\delta^2 = \frac{\Delta}{\text{Var}(c_R^*)}$  where

$$\Delta \equiv \text{Var}(m_R^*) \text{Var}(c_R^*) - (E(c_R^*))^2.$$

The equation of this hyperbola in terms of  $\text{Var}(m_R^*)$ ,  $\text{Var}(c_R^*)$  and  $E(c_R^*)$  is:

$$\text{Var}(c_R^*)\sigma^2 - \frac{(\text{Var}(c_R^*))^2}{\Delta} \left(E - \frac{E(c_R^*)}{\text{Var}(c_R^*)}\right)^2 = 1. \quad (5)$$

When the riskless security **1** is included,  $N \geq 1$ , and the  $N$  returns are linearly independent with at least one of the returns of the risky assets having an expected value different from the return of the riskless asset, the mean-standard deviation frontier then consists of two straight lines intersecting each other at the point  $(0, R_0)$  and with squared slopes

$$S_{MSF_{1,R}}^2 = \text{Var}(m_R^* - R_0 c_R^*) \quad (6)$$

$$= (E(R) - \mathbf{1}_N R_0)' \Sigma_R^{-1} (E(R) - \mathbf{1}_N R_0) \quad (7)$$

The equation of the mean-standard deviation frontier is:

$$E = R_0 \pm \sigma \sqrt{\text{Var}(m_R^* - R_0 c_R^*)}. \quad (8)$$

This equation can also be written in terms of  $\text{Var}(c_{1,R})$ ,  $E(c_{1,R})$  and  $R_0$  as:

$$E = R_0 \pm \sigma \frac{\sqrt{\text{Var}(c_{1,R})}}{E(c_{1,R})}. \quad (9)$$

#### 4. Geometrical Properties of the Pricing and Averaging Securities

The geometrical properties of the pricing and averaging securities are as follows:

PROPOSITION 1. *When the riskless security 1 is not included,  $N \geq 2$ , and the  $N$  returns are linearly independent with at least two returns having different expected values,  $c_R^*$  is the frontier portfolio whose return  $\frac{c_R^*}{\pi(c_R^*)}$  has the smallest variance among the returns of all frontier portfolios. This means*

$$\left( \sigma \left( \frac{c_R^*}{\pi(c_R^*)} \right), E \left( \frac{c_R^*}{\pi(c_R^*)} \right) \right) = \left( \frac{1}{\sqrt{\text{Var}(c_R^*)}}, \frac{E(c_R^*)}{\text{Var}(c_R^*)} \right)$$

*is the point in the MSF nearest to the  $E$ -axis in the  $(\sigma - E)$ -space.*

PROOF. In the last section the point

$$\left( \frac{1}{\sqrt{\text{Var}(c_R^*)}}, \frac{E(c_R^*)}{\text{Var}(c_R^*)} \right)$$

was identified as the vertex of the MSF. Given that this point is equal to

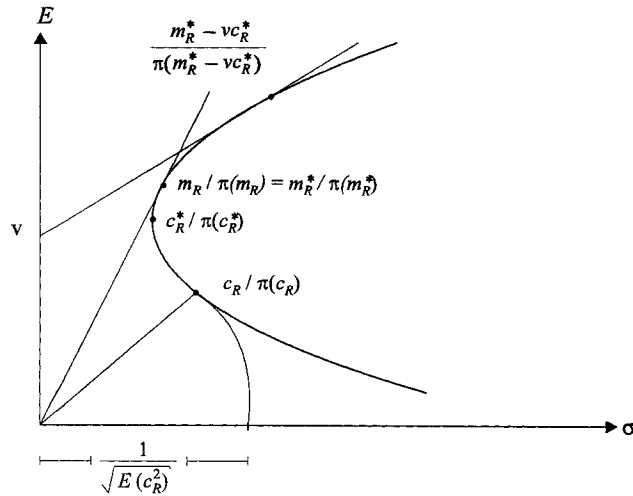
$$\left( \sigma \left( \frac{c_R^*}{\pi(c_R^*)} \right), E \left( \frac{c_R^*}{\pi(c_R^*)} \right) \right)$$



then,  $c_R^*$  is the frontier portfolio whose return  $\frac{c_R^*}{\pi(c_R^*)}$  has the smallest variance among the returns of all frontier portfolios. ■

Proposition 1 is shown in figure 1.

Figure 1



Location in the MSF, of the returns of the pricing and averaging portfolios when the riskless security 1 is not included:  $c_R^* / \pi(c_R^*)$  (Proposition 1),  $(m_R^* - v c_R^*) / \pi(m_R^* - v c_R^*)$  (Proposition 2),  $m_R / \pi(m_R) = m_R^* / \pi(m_R^*)$  (Proposition 3) and  $c_R / \pi(c_R)$ .

For all  $v \in \mathbf{R}$ , the portfolio  $m_R^* - v c_R^*$  also has a geometrical interpretation as shown in the following:

PROPOSITION 2: When the riskless security 1 is not included,  $N \geq 2$ , and the  $N$  returns are linearly independent with at least two returns having different expected values, for all  $v \in \mathbf{R}$ ,  $m_R^* - v c_R^*$  is the frontier portfolio whose return  $\frac{m_R^* - v c_R^*}{\pi(m_R^* - v c_R^*)}$  solves the problem:

$$\text{Max}_{\{x \in L^2(P) \mid x \text{ is a frontier portfolio}\}} \frac{E(R_x) - v}{\sqrt{\text{var}(R_x)}}. \quad (10)$$

Thus the point

$$\left( \sigma \left( \frac{m_R^* - \nu c_R^*}{\pi(m_R^* - \nu c_R^*)} \right), E \left( \frac{m_R^* - \nu c_R^*}{\pi(m_R^* - \nu c_R^*)} \right) \right)$$

is the tangency point of the MSF with the steepest ray passing through the point  $(0, \nu)$  that intersects the mean-standard deviation frontier. The squared slope of this tangent ray,  $S_{\nu, R}^2$  is:

$$S_{\nu, R}^2 = \text{Var}(m_R^* - \nu c_R^*) \quad (11)$$

$$= (E(R) - \mathbf{1}_N \nu)' \Sigma_R^{-1} (E(R) - \mathbf{1}_N \nu). \quad (12)$$

PROOF. The maximization problem (10) is:

$$\text{Max}_{E \in \mathbf{R}} \frac{(E - \nu)^2}{\sigma^2(E)}$$

$$\text{By (5), } \sigma^2(E) = \frac{\text{Var}(c_R^*)E^2 - 2E(c_R^*)E + \text{Var}(m_R^*)}{\Delta}.$$

The maximization problem then becomes:

$$\text{Max}_{E \in \mathbf{R}} \frac{\Delta(E - \nu)^2}{\text{Var}(c_R^*)E^2 - 2E(c_R^*)E + \text{Var}(m_R^*)}.$$

Solving the First Order Condition, and checking the Second Order Condition, we get that the maximum is obtained at:

$$E^* = \frac{\text{Var}(m_R^*) - \nu E(c_R^*)}{E(c_R^*) - \nu(\text{Var}(c_R^*))} = \frac{E(m_R^* - \nu c_R^*)}{\pi(m_R^* - \nu c_R^*)} = E(R_{m_R^* - \nu c_R^*})$$

This identifies portfolio  $m_R^* - \nu c_R^*$  as one solving the maximization problem.

By definition

$$\begin{aligned}
 S_{v,R}^2 &= \frac{(E(R_{m_R^* - v c_R^*}) - v)^2}{\text{Var}(R_{m_R^* - v c_R^*})} \\
 &= \frac{\left( E \left( \frac{m_R^* - v c_R^*}{\pi(m_R^* - v c_R^*)} \right) - v \right)^2}{\text{Var} \left( \frac{m_R^* - v c_R^*}{\pi(m_R^* - v c_R^*)} \right)} = \frac{\left( \frac{\text{Var}(m_R^*) - v E(c_R^*)}{E(c_R^*) - v(\text{Var}(c_R^*))} - v \right)^2}{\frac{\text{Var}(m_R^* - v c_R^*)}{(E(c_R^*) - v(\text{Var}(c_R^*)))^2}} \\
 &= \frac{(\text{Var}(m_R^*) - 2v E(c_R^*) + v^2 \text{Var}(c_R^*))^2}{\text{Var}(m_R^* - v c_R^*)} \\
 &= \frac{(\text{Var}(m_R^* - v c_R^*))^2}{\text{Var}(m_R^* - v c_R^*)} = \text{Var}(m_R^* - v c_R^*).
 \end{aligned}$$

Thus,  $S_{v,R}^2 = \text{Var}(m_R^* - v c_R^*)$ .

Using (1):

$$\begin{aligned}
 S_{v,R}^2 &= \text{Var} [(E(R) - \mathbf{1}_N v)' \Sigma_R^{-1} R] \\
 &= (E(R) - \mathbf{1}_N v)' \Sigma_R^{-1} (E(R) - \mathbf{1}_N v). \quad \blacksquare
 \end{aligned}$$

Proposition 2 is shown in figure 1.

When the riskless security **1** is not included,  $N \geq 2$ , and the  $N$  returns are linearly independent with at least two returns having different expected values,  $c_R$  is the frontier portfolio whose return  $\frac{c_R}{\pi(c_R)}$  has the smallest second moment among the returns of all frontier portfolios. (See Lemma 3.1 in Hansen and Richard, 1987). This means

$$\left( \sigma \left( \frac{c_R}{\pi(c_R)} \right), E \left( \frac{c_R}{\pi(c_R)} \right) \right)$$

is at the tangency point of a circle with center (0,0) and the negative sloping portion of the mean-standard deviation frontier. The radius  $r$  of this circle is  $r = \frac{1}{\sqrt{E(c_R^2)}}$ . (See figure 1).

PROPOSITION 3. *When the riskless security 1 is not included,  $N \geq 2$ , and the  $N$  returns are linearly independent with at least two returns having different expected values,  $m_R$  is the frontier portfolio whose return has the largest expected value-standard deviation ratio among the returns of all frontier portfolios. This means the point*

$$\left( \sigma \left( \frac{m_R}{\pi(m_R)} \right), E \left( \frac{m_R}{\pi(m_R)} \right) \right)$$

*is the tangency point of the MSF with the steepest ray passing through the origin that intersects the mean-standard deviation frontier. The squared slope of this tangent ray is*

$$\frac{E(m_R)}{1 - E(m_R)} = E(R)' \Sigma_R^{-1} E(R).$$

PROOF. From (3),  $\frac{m_R^*}{\pi(m_R^*)} = \frac{m_R}{\pi(m_R)}$ . The result is obtained taking  $v = 0$  in Proposition 1. ■

Proposition 3 is shown in figure 1.

When the riskless security 1 is included,  $N \geq 1$ , and the  $N$  returns are linearly independent with at least one of the returns of the risky assets having an expected value different from the return of the riskless asset,  $c_{1,R}$  is the frontier portfolio whose return has the smallest second moment among the returns of all frontier portfolios. (See Lemma 3.1 in

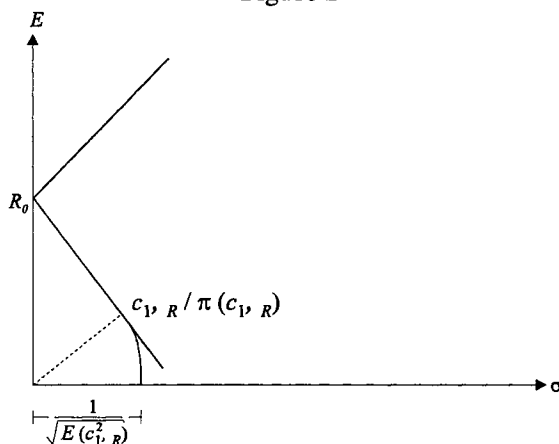
Hansen and Richard, 1987). This means  $\left( \sigma \left( \frac{c_{1,R}}{\pi(c_{1,R})} \right), E \left( \frac{c_{1,R}}{\pi(c_{1,R})} \right) \right)$  is

at the tangency point of a circle with center (0,0) and the negative sloping straight line of the MSF. The radius of this circle is

$$r = \frac{1}{\sqrt{E(c_{1,R}^2)}}$$

and the slope of the straight line is  $-S_R$  where  $S_R = \frac{1}{\pi(1)} \sigma(c_{1,R})$ .  
This result is shown in figure 2.

Figure 2



Location in the mean-standard deviation frontier of the returns of the pricing security  $c_{1,R} / \pi(c_{1,R})$  when the riskless security 1 is included.

## 5. Distance between Mean-Standard Deviation Frontiers

Suppose the  $N$ -dimensional vector of returns ( $N \geq 2$ )  $R$  is partitioned as  $R = \begin{pmatrix} g \\ R_I \end{pmatrix}$  where  $g$  is a  $K$ -dimensional vector of returns ( $1 \leq K \leq N-1$ ).

Sometimes it is of interest to compare the mean-standard deviation frontier generated by  $R$  ( $MSF_R$ ) with the MSF generated by  $g$  ( $MSF_g$ ). When the riskless security 1 is included, we would like to compare the MSF generated by 1 and  $R$  ( $MSF_{1,R}$ ) with the MSF generated by 1 and  $g$  ( $MSF_{1,g}$ ).

The motivation for doing such comparisons comes from the fact that a number of asset pricing theories can be stated in terms of mean-standard deviation frontiers. For example, the *Zero-Beta Black-Lintner*

CAPM (Capital Asset Pricing Model) can be stated as: the mean-standard deviation frontier generated by the Market portfolio  $M$  ( $MSF_M$ ) is a point in the mean-standard deviation frontier generated by the returns of the  $N$  risky securities considered ( $MSF_R$ ).

The Sharpe-Lintner CAPM (Capital Asset Pricing Model) can be stated as: the mean-standard deviation frontier generated by the riskless asset  $1$  and the Market portfolio  $M$  is equal to the mean-standard deviation frontier generated by the riskless asset  $1$  and the returns of the  $N$  risky securities considered ( $MSF_{1,R}$ ).

When the riskless security  $1$  is not included, the exact version of the APT can be stated as the mean-standard deviation frontier generated by the  $K$  factors  $f(MSF_f)$ , intersects the mean-standard deviation frontier generated by the returns of the securities considered ( $MSF_R$ ) (Chamberlain, 1983; Lehmann and Modest, 1988).

When the riskless security  $1$  is included, the exact version of the APT can be stated as the mean-standard deviation frontier generated by the riskless security  $1$  and the  $K$  factors  $f(MSF_{1,f})$ , is equal to the mean-standard deviation frontier generated by the riskless security  $1$  and the returns of the risky securities considered ( $MSF_{1,R}$ ) (Chamberlain, 1983; Lehmann and Modest, 1988).

Because  $g$  is a part of  $R$ , the frontiers  $MSF_g$  and  $MSF_{1,g}$  are contained to the right of the frontiers  $MSF_R$  and  $MSF_{1,R}$ , respectively. One may like to have a measure of how far from each other they are. One possible way of doing this is as follows:

DEFINITION 2. *Distance between mean-standard deviation frontiers. Suppose the  $N$ -dimensional vector of returns ( $N \geq 2$ )  $R$  is partitioned as  $R = \begin{pmatrix} g \\ R_I \end{pmatrix}$  where  $g$  is a  $K$ -dimensional vector ( $1 \leq K \leq N - 1$ ). When the riskless security  $1$  is included, the distance between  $MSF_{1,R}$  and  $MSF_{1,g}$  is defined as:*

$$D(MSF_{1,R}, MSF_{1,g}) \equiv S_{MSF_{1,R}}^2 - S_{MSF_{1,g}}^2$$

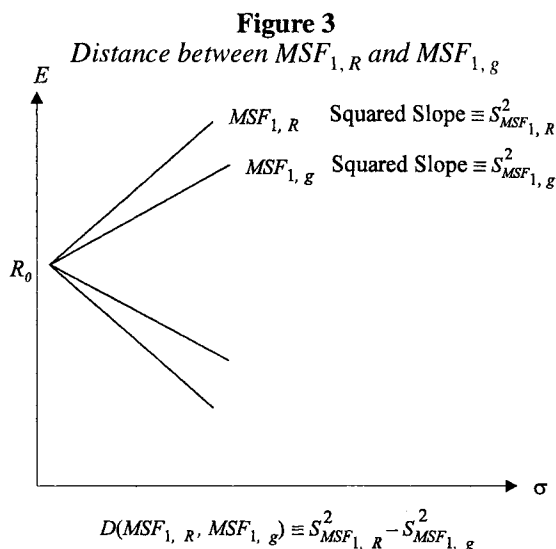
where  $S_{MSF_{1,R}}^2$  is the squared slope of the two straight lines that form the cone  $MSF_{1,R}$  and  $S_{MSF_{1,g}}^2$  is the squared slope of the two straight lines that form the cone  $MSF_{1,g}$ . Its formulas are given in (6) and (7).

When the riskless security **1** is not included, the distance between  $MSF_R$  and  $MSF_g$  is defined as:

$$D(MSF_R, MSF_g) \equiv \min_{v \in \mathbf{R}} (S_{v,R}^2 - S_{v,g}^2)$$

where  $S_{v,R}^2$  is the squared slope of the steepest ray passing through the point  $(0, v)$  tangent to  $MSF_R$ , and  $S_{v,g}^2$  is the squared slope of the steepest ray passing through the point  $(0, v)$  tangent to  $MSF_g$ .<sup>3</sup>

Definition 2 is shown in figure 3 and 4.



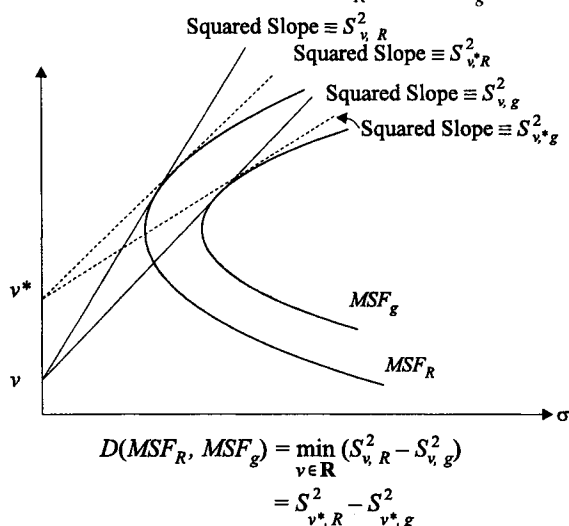
<sup>3</sup> The measure given in definition 2 has the desirable property that its probability distribution can be found when doing empirical work (see Shanken, 1985). An alternative definition which is statistically more difficult to handle in empirical work, but that lends itself better to theoretical work with some asset pricing models (see Hansen and Jagannathan, 1993) is as follows:

DEFINITION 3. When the riskless security **1** is included,

$$D_2(MSF_{1,R}, MSF_{1,g}) \equiv \frac{1}{(\pi(1))^2} (S_{MSF_{1,R}}^2 - S_{MSF_{1,g}}^2).$$

When the riskless security **1** is not included,  $D_2(MSF_R, MSF_g) \equiv \min_{v \in \mathbf{R}} \frac{1}{v^2} (S_{v,R}^2 - S_{v,g}^2)$ .

**Figure 4**  
Distance between  $MSF_R$  and  $MSF_g$



From the definition above, it is clear that  $D(MSF_{1,R}, MSF_{1,g}) = 0$  if and only if  $MSF_{1,R} = MSF_{1,g}$ , and  $D(MSF_R, MSF_g) = 0$  if and only if  $MSF_R \cap MSF_g \neq \emptyset$ .<sup>4</sup>

In terms of the distance between frontiers presented above, the conclusions of the four asset pricing models mentioned before can be written as follows:

*Zero-Beta Black-Lintner CAPM:*  $D(MSF_R, MSF_M) = 0$

*Sharpe-Lintner CAPM:*  $D(MSF_{1,R}, MSF_{1,M}) = 0$

*APT when the riskless security 1 is not included:*  $D(MSF_R, MSF_f) = 0$

*APT when the riskless security 1 is included:*  $D(MSF_{1,R}, MSF_{1,f}) = 0$ .

There exists a relationship between the difference of squared slopes  $S^2_{\nu,R} - S^2_{\nu,g}$  used above in the definition of  $D(MSF_R, MSF_g)$  and the regression of  $R_I$  on  $g$ . This is shown in the following (see Gibbons, Ross and Shanken, 1989, and Shanken, 1987):

<sup>4</sup> The same is true for the alternative definition:  $D_2(MSF_{1,R}, MSF_{1,g}) = 0$  if and only if  $MSF_{1,R} = MSF_{1,g}$ , and  $D_2(MSF_R, MSF_g) = 0$  if and only if  $MSF_R \cap MSF_g \neq \emptyset$ . This means  $D(MSF_{1,R}, MSF_{1,g}) = 0$  if and only if  $D_2(MSF_{1,R}, MSF_{1,g}) = 0$  and  $D(MSF_R, MSF_g) = 0$  if and only if  $D_2(MSF_R, MSF_g) = 0$ .



PROPOSITION 4. *The  $N$ -dimensional vector of returns ( $N \geq 2$ )  $R$  is partitioned as  $R = \begin{pmatrix} g \\ R_I \end{pmatrix}$  where  $g$  is a  $K$ -dimensional vector ( $1 \leq K \leq N-1$ ). For any  $v \in \mathbf{R}$ , consider the regression:*

$$R_I - \mathbf{1}_{N-K}v = \alpha(v) + B(g - \mathbf{1}_K v) + \varepsilon_N$$

where  $E(\varepsilon_N) = 0$ ,  $\Sigma_{\varepsilon_N} \equiv \text{var}(\varepsilon_N)$ ,  $\alpha(v) = E(R_I) - \mathbf{1}_{N-K}v - B(E(g) - \mathbf{1}_K v)$ .

Then, for all  $v \in \mathbf{R}$ ,  $S^2_{v,R} - S^2_{v,g} = (\alpha(v))' \Sigma_{\varepsilon_N}^{-1} \alpha(v)$  and

$$S^2_{MSF_{1,R}} - S^2_{MSF_{1,g}} = (\alpha(R_0))' \Sigma_{\varepsilon_N}^{-1} \alpha(R_0).$$

PROOF.

$$R = \begin{pmatrix} g \\ R_I \end{pmatrix}. \text{ Then, } \Sigma_R = \begin{pmatrix} \Sigma_g & \text{cov}(g, R_I) \\ \text{cov}(g, R_I)' & \Sigma_{R_I} \end{pmatrix} = \begin{pmatrix} \Sigma_g & \Sigma_g B' \\ B \Sigma_g & B \Sigma_g B' + \Sigma_{\varepsilon_N} \end{pmatrix}$$

$$\Sigma_R^{-1} = \begin{pmatrix} \Sigma_g^{-1} + B' \Sigma_{\varepsilon_N}^{-1} B & -B' \Sigma_{\varepsilon_N}^{-1} \\ -\Sigma_{\varepsilon_N}^{-1} B & \Sigma_{\varepsilon_N}^{-1} \end{pmatrix}.$$

By (12),

$$\begin{aligned} S^2_{v,R} - S^2_{v,g} &= (E(R) - \mathbf{1}_N v)' \Sigma_R^{-1} (E(R) - \mathbf{1}_N v) \\ &\quad - (E(g) - \mathbf{1}_K v)' \Sigma_g^{-1} (E(g) - \mathbf{1}_K v) \\ &= \left( E \begin{pmatrix} g \\ R_I \end{pmatrix} - \begin{pmatrix} \mathbf{1}_K \\ \mathbf{1}_{N-K} \end{pmatrix} v \right)' \begin{pmatrix} \Sigma_g^{-1} + B' \Sigma_{\varepsilon_N}^{-1} B & -B' \Sigma_{\varepsilon_N}^{-1} \\ -\Sigma_{\varepsilon_N}^{-1} B & \Sigma_{\varepsilon_N}^{-1} \end{pmatrix} \left( E \begin{pmatrix} g \\ R_I \end{pmatrix} - \begin{pmatrix} \mathbf{1}_K \\ \mathbf{1}_{N-K} \end{pmatrix} v \right) \\ &\quad - (E(g) - \mathbf{1}_K v)' \Sigma_g^{-1} (E(g) - \mathbf{1}_K v) \\ &= [E(R_I) - \mathbf{1}_{N-K} v - B(E(g) - \mathbf{1}_K v)]' \Sigma_{\varepsilon_N}^{-1} [E(R_I) - \mathbf{1}_{N-K} v - B(E(g) - \mathbf{1}_K v)] \\ &= (\alpha(v))' \Sigma_{\varepsilon_N}^{-1} \alpha(v). \end{aligned}$$

Similarly, by (7),

$$\begin{aligned}
 S_{MSF_{1,R}}^2 - S_{MSF_{1,g}}^2 &= (E(R) - \mathbf{1}_N R_0)' \Sigma_R^{-1} (E(R) - \mathbf{1}_N R_0) \\
 &\quad - (E(g) - \mathbf{1}_K R_0)' \Sigma_g^{-1} (E(g) - \mathbf{1}_K R_0) \\
 &= (E(R_I) - \mathbf{1}_{N-K} R_0 - B(E(g) - \mathbf{1}_K R_0))' \Sigma_{\epsilon_N}^{-1} (E(R_I) \\
 &\quad - \mathbf{1}_{N-K} R_0 - B(E(g) - \mathbf{1}_K R_0)) \\
 &= (\alpha(R_0))' \Sigma_{\epsilon_N}^{-1} \alpha(R_0). \quad \blacksquare
 \end{aligned}$$

Putting together equations (6), (11), and proposition 4, the distance between frontiers can be written as follows:

**PROPOSITION 5.** *When the riskless security 1 is included, the distance between  $MSF_{1,R}$  and  $MSF_{1,g}$  can be written as:*

$$\begin{aligned}
 D(MSF_{1,R}, MSF_{1,g}) &= \text{Var}(m_R^* - R_0 c_R^*) - \text{Var}(m_g^* - R_0 c_g^*) \\
 &= (E(R_I) - \mathbf{1}_{N-K} R_0 - B(E(g) - \mathbf{1}_K R_0))' \Sigma_{\epsilon_N}^{-1} (E(R_I) \\
 &\quad - \mathbf{1}_{N-K} R_0 - B(E(g) - \mathbf{1}_K R_0)) \quad (15) \\
 &= (\alpha(R_0))' \Sigma_{\epsilon_N}^{-1} \alpha(R_0)
 \end{aligned}$$

where  $R_0$  is the return of the riskless security 1.

*When the riskless security 1 is not included, the distance between  $MSF_R$  and  $MSF_g$  can be written as:*

$$D(MSF_R, MSF_g) = \min_{v \in \mathbf{R}} \{ \text{Var}(m_R^* - v c_R^*) - \text{Var}(m_g^* - v c_g^*) \} \quad (14)$$

$$\begin{aligned}
 &= \min_{v \in \mathbf{R}} (E(R_I) - \mathbf{1}_{N-K} v - B(E(g) - \mathbf{1}_K v))' \Sigma_{\epsilon_N}^{-1} (E(R_I) \\
 &\quad - \mathbf{1}_{N-K} v - B(E(g) - \mathbf{1}_K v)) \quad (15) \\
 &= \min_{v \in \mathbf{R}} (\alpha(v))' \Sigma_{\epsilon_N}^{-1} \alpha(v).
 \end{aligned}$$

It is easily checked that the value of  $v$  that solves minimization problem (14) is:

$$v^* = \frac{E(c_R^*) - E(c_g^*)}{\text{Var}(c_R^*) - \text{Var}(c_g^*)}.$$

Using (3) and (4), this can be written in terms of  $E(c_R)$ ,  $E(c_g)$  and  $E(m_g)$  as:

$$v^* = \frac{E(c_R)[1 - E(m_g)] - E(c_g)[1 - E(m_R)]}{[E(c_R^2) - E(c_g^2)](1 - E(m_R))(1 - E(m_g)) + (E(c_R))^2(1 - E(m_g)) - (E(c_g))^2(1 - E(m_R))}.$$

Alternatively, solving the minimization problem (15), we get:<sup>5</sup>

$$v^* = \frac{(\mathbf{1}_{N-K} - B\mathbf{1}_K)' \Sigma_{\epsilon_N}^{-1} (E(R_I) - BE(g))}{(\mathbf{1}_{N-K} - B\mathbf{1}_K)' \Sigma_{\epsilon_N}^{-1} (\mathbf{1}_{N-K} - B\mathbf{1}_K)}.$$

## 6. The Distance between Mean-Standard Deviation Frontiers as a Measure of Model Misspecification

Depending on the asset pricing model under consideration and the particular problem being analyzed, the distance between frontiers defined in this paper can have different interpretations. An example is presented in this section in which the distance between specific frontiers can be interpreted as a measure of model misspecification in similar terms to those used in Hansen and Jagannathan (1993). Let us consider an pricing security model that concludes that

<sup>5</sup> Solving the minimization problem given in the alternative definition 3, we get

$$\begin{aligned} v^{**} &= \frac{(E(R_I) - BE(g))' \Sigma_{\epsilon_N}^{-1} (E(R_I) - BE(g))}{(E(R_I) - BE(g))' \Sigma_{\epsilon_N}^{-1} (\mathbf{1}_{N-K} - B\mathbf{1}_K)} = \frac{E(m_R^*) - E(m_g^*)}{E(c_R^*) - E(c_g^*)} \\ &= \frac{E(m_R) - E(m_g)}{E(c_R)(1 - E(m_g)) - E(c_g)(1 - E(m_R))}. \end{aligned}$$

$$D(MSF_R, MSF_g) = 0 \quad (16)$$

or

$$D(MSF_{1,R}, MSF_{1,g}) = 0$$

like the four models in section 5 (see equations 13).

A common problem is that the actual vector  $g$  is not observable (for example, the Market portfolio in the case of the CAPM or the factors  $f$  in the APT are not observable), and a *proxy* for  $g$  must be used instead.

Let  $y \equiv (y_1, y_2, \dots, y_K)'$  be a  $K$ -dimensional vector of observable random variables with finite variance. Let  $P$  be the space generated by the  $N$  securities  $x_1, x_2, \dots, x_N$ , and

$$\text{Proj}(y | P) \equiv (\text{Proj}(y_1 | P), \text{Proj}(y_2 | P), \dots, \text{Proj}(y_K | P))',$$

where  $\text{Proj}(y_k | P)$  is the orthogonal projection of the random variable  $y_k$  onto the space  $P$ . Let  $\text{Proj}(y | P)$  be a *proxy* for  $g$ , let  $MSF_{\text{Proj}(y | P)}$  be the mean-standard deviation frontier generated by the returns of  $\text{Proj}(y | P)$ , and let  $MSF_{1, \text{Proj}(y | P)}$  be the mean-standard deviation frontier generated by the riskless asset  $1$  and the returns of  $\text{Proj}(y | P)$ .

Then, the distance

$$D(MSF_R, MSF_{\text{Proj}(y | P)})$$

or

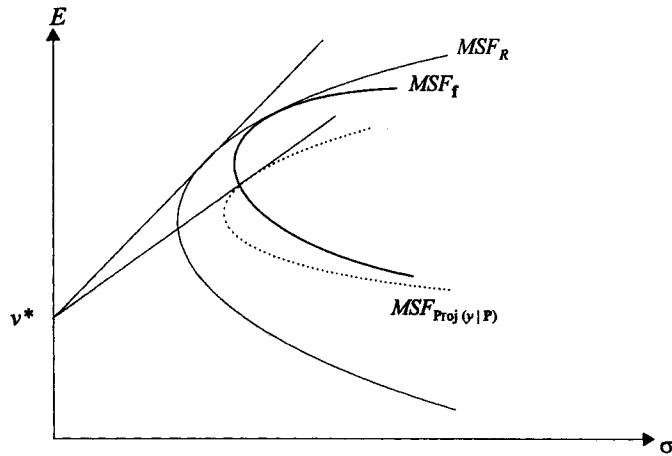
$$D(MSF_{1,R}, MSF_{1, \text{Proj}(y | P)})$$

can be seen as a measure of model misspecification arising from the fact of using the proxy  $\text{Proj}(y | P)$  instead of  $g$ . This is illustrated for the case of the APT when there is no riskless security  $1$ , in Figure 5.

## 7. Summary

The equation of the MSF is expressed in terms of the variance and/or expected values of the returns of the pricing and/or averaging securities (see equations 5, 8, and 9). This approach emphasizes the role of these securities in the formulation of the MSF.

Figure 5



Distance between frontiers as a measure of model misspecification for the case of the APT when there is no riskless security 1. According to the APT,  $D(MSF_R, MSF_f) = 0$ . The factors  $f_1, f_2, \dots, f_K$  are not observable. Instead, the *proxy*  $\text{Proj}(y|P)$  is used and  $D(MSF_R, MSF_{\text{Proj}(y|P)})$  is a measure of the model misspecification arising from the use of the proxy.

$MSF_R$  and  $MSF_f$  intersect each other.

$MSF_R$  and  $MSF_{\text{Proj}(y|P)}$  do not intersect each other. The “better” is the proxy  $\text{Proj}(y|P)$ , the smaller is the measure of model misspecification  $D(MSF_R, MSF_{\text{Proj}(y|P)})$ .

The geometrical properties of the returns of the pricing and averaging securities relative to the  $MSF$  in the  $(\sigma, E)$ -space are summarized as follows (figures 1 and 2):

When the riskless security 1 is not included,  $N \geq 2$ , and the  $N$  returns are linearly independent with at least two returns having different expected values, the pricing security  $c_R^*$  is the frontier portfolio whose return has the smallest variance among the returns of all frontier portfolios (Proposition 1), and for all  $v \in \mathbf{R}$ ,  $m_R^* - vc_R^*$  is the frontier portfolio whose return

$$\frac{m_R^* - vc_R^*}{\pi(m_R^* - vc_R^*)},$$

has the property that the point

$$\left( \sigma \left( \frac{m_R^* - v c_R^*}{\pi(m_R^* - v c_R^*)} \right), F \left( \frac{m_R^* - v c_R^*}{\pi(m_R^* - v c_R^*)} \right) \right)$$

is the tangency point of the MSF with the steepest ray passing through the point  $(0, v)$  that intersects the MSF (proposition 2).

When the riskless security **1** is not included,  $N \geq 2$ , and the  $N$  returns are linearly independent with at least two returns having different expected values, the averaging security  $m_R$  is the frontier portfolio whose return has the largest expected value-standard deviation ratio among the returns of all frontier portfolios (Proposition 3).

A method to measure the distance between frontiers is as follows:

When the riskless security **1** is included,

$$D(MSF_{1,R}, MSF_{1,f}) \equiv S_{MSF_{1,R}}^2 - S_{MSF_{1,f}}^2$$

where  $S_{MSF_{1,R}}^2$  is the squared slope of the two straight lines that form the cone  $MSF_{1,R}$ , and  $S_{MSF_{1,f}}^2$  is the squared slope of the two straight lines that form the cone  $MSF_{1,f}$ .

When the riskless security **1** is not included,

$$D(MSF_R, MSF_f) = \min_{v \in \mathbf{R}} (S_{v,R}^2 - S_{v,f}^2)$$

where  $S_{v,R}^2$  is the squared slope of the steepest ray passing through the point  $(0, v)$  tangent to  $MSF_R$ , and  $S_{v,f}^2$  is the squared slope of the steepest ray passing through the point  $(0, v)$  tangent to  $MSF_f$ .

A number of asset pricing models can be stated in terms of the distance between two frontiers in the form  $D(MSF_{1,R}, MSF_{1,g}) = 0$ , or  $D(MSF_R, MSF_g) = 0$ , where  $g$  is a random vector identified according to the pricing security model under consideration. When instead of  $g$ , a proxy  $\text{Proj}(y | \mathbf{P})$  is used, the distance  $D(MSF_{1,R}, MSF_{1, \text{Proj}(y | \mathbf{P})})$ , or  $D(MSF_R, MSF_{\text{Proj}(y | \mathbf{P})})$  can be interpreted as a measure of model misspecification.

The link between the distance between frontiers and some security pricing models will be analyzed in more detail in future work.

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