Games and Evolutionary Dynamics

Selected Theoretical and Applied Developments

Saúl Mendoza-Palacios
Alfonso Mercado
Editors

EL COLEGIO DE MÉXICO

GAMES AND EVOLUTIONARY DYNAMICS: SELECTED THEORETICAL AND APPLIED DEVELOPMENTS

Centro de Estudios Económicos
Programa de Análisis Económico de México

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PREFACE

The idea of this publication originated from the desire to cover some technical, theoretical and empirical gaps in the literature on game-based models and evolutionary dynamics. This book is about selected developments and applications of optimal control and, more particularly, evolutionary dynamics. It is mainly intended for readers in economics and mathematics but may also be useful for those in sociology, politics, health, public security and natural resources. The book contains 13 chapters that are focused on new modeling developments and applications to microeconomics and public policy. The vast majority of the chapters included have undergone expert criticism in congresses and other academic meetings, which has enriched their contributions.

Its first three chapters cover modeling developments, exploring innovative approaches and methods. Chapter I's authors find that there are minimax strategies (as game against nature) under a discounted cost criterion when holding times on states are non-observable by the controller in semi-Markov control models. Chapter II presents a new class of cooperative games where the characteristic function is defined by a set of matrices which has a vector as its image. An axiomatically characterized solution is provided. Part One ends with Chapter III, in which its authors offer original results on the state of the art of axiomatic solutions for partition function form games.

The following four chapters offer applications for microeconomics. Chapter IV studies a simultaneous game in two stages with perfect equilibrium. It finds that the shadow cost of public funds produces an important impact on the capacity choice in a mixed duopoly with product differentiation. In Chapter V, the framework of general equilibrium theory is dynamized by incorporating decisions made by firm managers. These decisions serve as an engine of economic evolution along a path of general equilibrium dynamics in the long run. Chapter VI, on the other hand, is based on two models of two economies with indivisible goods in a metric space, proposing general conditions for no emptiness of the core of both models considering the social planner's problem. Chapter VII contributes to the literature on the relationship between market size, and entry, using the case of Mexico. The authors find a positive relationship, in both newspaper and radio markets, between market size and the number of products, with higher elasticity found in the number of radio stations with respect to population, than in newspapers.

The six remaining chapters focus on policy applications. Key issues on economic development, natural resources, health, public security and politics are analyzed. Chapter VIII proposes two models of a "big push" policy for industrialization; one model is static and the other is dynamic, incorporating an evolutionary dynamics approach.

Chapter IX is based on replicator dynamics, analyzing the interaction between actors (cooperators, defectors and enforcers) and forest resources. The stability of the system is studied in order to determine a sustainable policy towards forest resources. Chapter X focuses on optimal control policies in forest management. Differing from previous works, this chapter considers a profit function depending on the timber production and forest maintenance costs, and therefore, brings a more precise overview of the optimal forest management problem. Chapter XI studies optimal control policies for specific epidemic models. In Chapter XII, a normal game model is developed and applied to public security policies. As a dominant equilibrium, municipal presidents seek their private interests and police officers obey them, receiving monetary payoffs for corruption and for obedience. Finally, Chapter XIII compares three fiscal strategies to prevent the majority of a population from favoring secession from their country to form an independent one. An optimal concession is found to keep a country unified.

Saul Mendoza-Palacios and Alfonso Mercado

I. SEMI-MARKOV CONTROL MODELS AND GAMES AGAINST NATURE

Fernando Luque-Vásquez, J. Adolfo Minjárez-Sosa and Luz del Carmen Rosas-Rosas

Abstract

We deal with a class of semi-Markov control models with Borel state and control spaces and possibly unbounded costs, where the holding times on states are non-observable and their distributions are unknown by the controller. The system is modeled as a game against nature, which is a particular case of a minimax control system. The objective is to show the existence of minimax strategies under a discounted cost criterion.

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1. Introduction

This chapter deals with a class of semi-Markov control models with Borel state and control spaces, possibly unbounded costs, and unknown holding time distributions under a discounted optimality criterion. Specifically, we assume that the holding times are nonnegative and non-observable random variables (r.v.'s) whose unknown distributions may change from decision epoch to decision epoch.

This work is a sequel to Luque-Vásquez and Minjárez-Sosa [13] where we have studied the case when the holding times are observable, independent and identically distributed (i.i.d.) r.v.'s with a common density function which is independent of the state-action pairs. Under such an assumption of observability, it was possible to implement statistical estimation methods of the density, together with optimization procedures, to construct nearly optimal policies. Instead, in this work, we assume that the only information the controller has is that the holding time distributions belong to an appropriate set of probability measures Θ which depends on the state-action pairs. In this scenario, the semi-Markov optimal control problem is studied as a minimax control problem known as *game against nature*, which, roughly speaking, can be formulated as follows: the controller has an opponent, namely, "nature," that, at each decision epoch, selects a distribution from the set Θ for the corresponding holding time. Then, the controller will try to minimize its cost in the worst scenario imposed by nature. The controller's objective is to choose actions directed to minimize the maximum discounted cost generated on the set Θ. Therefore, our main objective in this paper is to show the existence of minimax strategies.

As is well known, a condition to ensure the existence of measurable selectors in minimax control problems (see, e.g., González-

Trejo et al. [5], Jaskiewicz and Nowak [9], and Luque-Vásquez et al. [15]) is the compactness (or $\sigma-$compactness) of the opponent control set. At first glance, we assume such a condition in our problem could be too strong, since the holding times are nonnegative r.v.'s, and, therefore, the opponent control set would be formed by probability measures on $[0,\infty)$, which is not necessarily a $\sigma-$compact set. However, by imposing an additional condition, compactness is achieved. Specifically, we take $\Theta := \mathbb{P}_0[0,\infty)$, where $\mathbb{P}_0[0,\infty)$ is the set of probability measures on $[0,\infty)$ with finite expectation, and whose $\sigma-$compactness is shown in Luque-Vásquez and Minjárez-Sosa [14].

Minimax *Markov* control systems and their applications have been widely studied for both the discrete and continuous time cases (see, e.g., Altman and Hordijk [1], Coraluppi and Marcus [4], and González-Trejo et al. [5][1]). It is worth remarking on the difference between the Markovian case and our problem. Typically, minimax Markov control systems are applied to study control problems when the random disturbance process, that is, the driving process, is a sequence of independent and non-observable r.v.'s with unknown distributions. In this case, the opponent, at each stage t, selects a probability measure θ_t which determines the transition law to the next state as well as the mean one-stage cost. Then, it is easy to see that the Markovian performance index depends explicitly on the sequence $\{\theta_t\}$ collected throughout the evolution of the system according to the opponent strategies selected, and then it is possible to obtain a minimax equation. Hence, the minimax control problem is stated in a standard way: *to find a strategy that minimizes the maximum cost over all admissible*

[1]See also Hordijk et al. [7], Jagannathan [8], Jaskiewicz and Nowak [9], Kalyanasundaram et al. [10], Küenle [11], Kurano [12], Milliken et al. [16], Savkin and Peterson [18], and Yu and Guo [20].

strategies for the opponent. Instead, for the minimax semi-Markov control problem we are concerned with, because the opponent selects the holding time distribution θ_t, we first need to state a representation of the performance index in terms of the sequence $\{\theta_t\} \subset \mathbb{P}_0[0,\infty)$, and next obtain the corresponding minimax equation. However, since, for general strategies, the holding times are not necessarily conditionally independent, such a representation is only possible when the controller as well as the opponent are restricted to applying Markovian strategies. According to this fact, before showing the existence of minimax strategies, we need to prove the sufficiency of the Markov strategies to solve the minimax semi-Markov control problem. To the best of our knowledge, minimax semi-Markov control systems have not been studied under our context.

The chapter is organized as follows. In Section 2, we present the semi-Markov control problem, whereas in Section 3, we define the strategies. Then, in Section 4, we describe the minimax discounted optimality criterion and the assumptions we will be dealing with. Next, in Section 5, we present the reduction of the minimax problem to the class of Markov strategies. Finally, Sections 6 and 7 contain the preliminary and the main results, respectively.

2. The semi-Markov control model

Notation. Given a Borel space X (that is, a Borel subset of a complete and separable metric space), its Borel σ-algebra is denoted by $\mathcal{B}(X)$, and "measurable" for either sets or functions, which means "Borel measurable." $\mathbb{P}(X)$ denotes the set of probability measures on X. Let X and Y be Borel spaces. Then a stochastic kernel $Q(dx \mid y)$ on X given Y is a function such that $Q(\cdot \mid y)$ is a probability measure on X for each fixed $y \in Y$, and $Q(B \mid \cdot)$

is a measurable function on Y for each fixed $B \in \mathcal{B}(X)$. We denote by $\mathbb{N}$ ($\mathbb{N}_0$) the set of positive (nonnegative) integers; $\mathbb{R}$ ($\mathbb{R}_+$) denotes the set of real (nonnegative real) numbers.

Minimax semi-Markov control model. We consider the following *minimax semi-Markov control model*:

$$MSMC = (X, A, \Theta, \mathbb{K}_A, \mathbb{K}, Q, D, d) \tag{2.1}$$

where X is the *state space*, A represents the *controller's action space* and Θ is the *opponent's action space* or *nature space*. We assume that X and A are Borel spaces and Θ is the set of probability measures on $[0, \infty)$ with finite expectation; that is,

$$\Theta = \mathbb{P}_0[0, \infty) := \left\{ \theta \in \mathbb{P}[0, \infty) : \int_0^\infty s\theta(ds) < \infty \right\}.$$

The set $\mathbb{K}_A \in \mathcal{B}(X \times A)$ is the *constraint set for the controller*, which induces a multifunction $x \longmapsto A(x)$ from X to A where

$$A(x) := \{a \in A : (x, a) \in \mathbb{K}_A\}$$

represents the set of admissible actions (or controls) for the controller when the state is $x \in X$. We assume that $\mathbb{K}_A$ contains the graph of a measurable function from X to A, or equivalently, the multifunction $x \longmapsto A(x)$ has a measurable selector, that is, a measurable function $f : X \to A$ such that $f(x) \in A(x)$ for all $x \in X$, which holds under Assumption 4.2 below.

The set $\mathbb{K} \in \mathcal{B}(X \times A \times \Theta)$ is the *constraint set for the opponent* and

$$\Theta(x, a) := \{\theta \in \Theta : (x, a, \theta) \in \mathbb{K}\}$$

is the set of admissible actions for the opponent when the state is $x \in X$ and the controller chooses the action $a \in A(x)$. We assume that $\mathbb{K}$ contains the graph of a measurable function from $\mathbb{K}_A$ to Θ. Again, this condition holds under Assumption 4.2.

The *transition law* $Q(\cdot \mid \cdot)$ is a stochastic kernel on X given $\mathbb{K}_A$, and the *cost functions* D and d are (possibly unbounded) nonnegative measurable functions on $\mathbb{K}_A$.

Interpretation. At the time of the nth decision epoch T_n $(n = 0, 1, ...)$, the system is in the state $x_n = x \in X$ and the controller chooses an action $a_n = a \in A(x)$. Then the opponent (nature) picks a probability distribution $\theta_n = \theta \in \Theta\,(x, a)$ and the system remains in the state x during a nonnegative random time δ_{n+1} with distribution θ. Next, the following happens:

1) an immediate cost $D\,(x, a)$ is incurred;

2) the system jumps to a new state $x_{n+1} = y$ according to a transition law $Q\,(\cdot \mid x, a)$; and

3) a cost rate $d\,(x, a)$ is imposed until the transition occurs. Once the transition to state y occurs, the process is repeated.

The random variable δ_n is called the nth holding time, and it is observed that the decision epochs $\{T_n\}$ are $T_0 = 0$ and $T_n = T_{n-1} + \delta_n$ for $n = 1, 2, ...$

In particular, we assume that the costs are continuously discounted with a discount factor $\alpha > 0$, that is: a cost C incurred at time t is equivalent to a cost $C \exp(-\alpha t)$ at time 0. Then, the *one-stage cost function* $c : \mathbb{K} \to \mathbb{R}$ is given by

$$c\,(x, a, \theta) := D(x, a) + d(x, a) \int_0^\infty \int_0^t \exp(-\alpha s) ds \theta\,(dt)\,. \quad (2.2)$$

Therefore, because of the dependence of the one-stage cost on the distribution $\theta \in \Theta\,(x, a)$ selected by the opponent, the goal of the controller is to minimize the maximum cost imposed by nature. Thus, the controller must select actions guaranteeing the best performance in the worst possible situation.

Remark 2.1. (a) We suppose that Θ is endowed with a weak topology. Hence, $\theta_n \overset{w}{\to} \theta$ if and only if $\int u d\theta_n \to \int u d\theta$ for each bounded and continuous function u en $[0, \infty)$.

(b) Furthermore, from Luque-Vásquez and Minjárez-Sosa [14] we have that Θ is a σ-compact space.

3. Strategies

The actions or controls applied by the controller as well as his/her opponent at the decision epochs are selected according to rules known as control strategies defined as follows:

Let $\mathbb{H}_0 := X$, $\mathbb{H}'_0 := \mathbb{K}_A$, and for $n \in \mathbb{N}$ let $\mathbb{H}_n := (\mathbb{K})^n \times X$ and $\mathbb{H}'_n := (\mathbb{K})^n \times \mathbb{K}_A$. Hence, generic elements of $\mathbb{H}_n$ and $\mathbb{H}'_n$ called "histories" are of the form

$$h_n = (x_0, a_0, \theta_0, ..., x_{n-1}, a_{n-1}, \theta_{n-1}, x_n)$$

and $h'_n = (h_n, a_n)$, respectively.

A strategy for the controller $\pi = \{\pi_n\}$ is a sequence of stochastic kernels on A given $\mathbb{H}_n$ such that $\pi_n(A(x_n)|h_n) = 1$ for all $h_n \in \mathbb{H}_n$ and $n \in \mathbb{N}_0$. If there exists a sequence $\{\varphi_n\}$ of stochastic kernels on A given X such that $\pi_n(\cdot|h_n) = \varphi_n(\cdot|x_n)$, then π is called a Markov strategy for the controller. We denote by Π the set of all strategies and Φ is the set of Markov strategies for the controller. A strategy π is said to be a deterministic Markov strategy if there exists a sequence $\{f_n\}$ of functions in the set

$$\mathbb{F}_A := \{f : X \to A \mid f \text{ is measurable,} \\ \text{and } f(x) \in A(x) \ \forall x \in X\}$$

such that for each $n = 0, 1, 2, ...$ $\pi_n(\cdot|h_n)$ is concentrated at $f_n(x_n)$. If, in addition, $f_n \equiv f \in \mathbb{F}_A$, then π is said to be a stationary strategy for the controller.

A strategy for the opponent is a sequence $\gamma = \{\gamma_n\}$ of stochastic kernels on Θ given $\mathbb{H}'_n$ such that $\gamma_n(\Theta(x_n, a_n)|h'_n) = 1$ for all $h'_n \in \mathbb{H}'_n$ and $n \in \mathbb{N}_0$. If there exists a sequence $\{\psi_n\}$ of stochastic kernels on Θ given $X \times A$ such that $\gamma_n(\cdot|h'_n) = \psi_n(\cdot|x_n, a_n)$, then γ is called a Markov strategy for the opponent. The set of all strategies for the opponent is denoted by Γ and Ψ is the set of all Markov strategies for the opponent. A strategy γ is said to be a deterministic Markov strategy for the opponent if there exists a sequence $\{g_n\}$ of functions in the set

$$\mathbb{F}_\Theta := \{g : X \times A \to \Theta \mid g \text{ is measurable}, \\ g(x,a) \in \Theta(x,a)\ \forall (x,a) \in \mathbb{K}_A\}$$

such that $\gamma_n(\cdot|h'_n)$ is concentrated at $g_n(x_n, a_n)$. If $g_n \equiv g \in \mathbb{F}_\Theta$ for all n, then γ is said to be stationary strategy for the opponent. As usual, every stationary strategy for the controller (opponent) is identified with the corresponding function f (g).

Let $(\Omega, \mathcal{F})$ be the (canonical) measurable space consisting of the sample space $\Omega = (X \times A \times \Theta)^\infty$ and the corresponding product σ-algebra $\mathcal{F}$. Then, by a theorem of Ionescu Tulcea (see, e.g., Bertsekas and Shreve [2]), for each pair of strategies $(\pi, \gamma) \in \Pi \times \Gamma$ and $x \in X$, there is a probability measure $P_x^{\pi\gamma}$ and a stochastic process $\{(x_n, a_n, \theta_n),\ n = 0, 1, ...\}$ where $x_n,\ a_n$ and θ_n represent the state and the actions of the controller and the opponent, respectively, at the time of the nth decision epoch. This process satisfies: for $Y \in \mathcal{B}(X)$,

$$P_x^{\pi\gamma}(x_0 \in Y) = \delta_x(Y);$$

$$P_x^{\pi\gamma}(x_{n+1} \in Y \mid h'_n, \theta_n) = Q(Y \mid x_n, a_n);$$

$$P_x^{\pi\gamma}(a_n \in A' \mid h_n) = \pi_n(A' \mid h_n), \quad A' \in \mathcal{B}(A);$$

$$P_x^{\pi\gamma}(\theta_n \in B' \mid h'_n) = \gamma_n(B' \mid h'_n), \quad B' \in \mathcal{B}(\Theta);$$

and

$$P_x^{\pi\gamma}(\delta_{n+1} \leq t \mid h'_n, \theta_n) = \theta_n\left([0,t]\right), \quad t > 0.$$

4. Minimax discounted optimality criterion

For a fixed $\alpha > 0$, an initial state x in X and each pair of strategies $(\pi, \gamma) \in \Pi \times \Gamma$, the *total expected discounted cost* is defined as

$$V(\pi, \gamma, x) := E_x^{\pi\gamma} \left[\sum_{n=0}^{\infty} \exp(-\alpha T_n) c(x_n, a_n, \theta_n) \right] \tag{4.3}$$

where $E_x^{\pi\gamma}$ denotes the expectation operator with respect to the probability measure $P_x^{\pi\gamma}$.

Let

$$V'(\pi, x) := \sup_{\gamma \in \Gamma} V(\pi, \gamma, x), \quad x \in X, \ \pi \in \Pi. \tag{4.4}$$

Then, the *minimax control problem* associated with the control model $MSMC$ is to find a strategy $\pi^* \in \Pi$ such that

$$V'(\pi^*, x) = \inf_{\pi \in \Pi} V'(\pi, x) = \inf_{\pi \in \Pi} \sup_{\gamma \in \Gamma} V(\pi, \gamma, x) =: V^*(x). \tag{4.5}$$

For $(x, a) \in \mathbb{K}_A$ and $\theta \in \Theta(x, a)$ we define:

$$\Delta_\alpha(\theta) := \int_0^\infty \exp(-\alpha s)\, \theta(ds) \tag{4.6}$$

and

$$\tau_\alpha(\theta) := \frac{1 - \Delta_\alpha(\theta)}{\alpha}. \tag{4.7}$$

Then the one-stage-cost function can be rewritten as,

$$c(x, a, \theta) = D(x, a) + \tau_\alpha(\theta) d(x, a), \quad (x, a, \theta) \in \mathbb{K}. \tag{4.8}$$

To conclude this section, we now introduce two sets of conditions on the model $MSMC$ (2.1). Assumption 4.1 is a (uniform) regularity condition and ensures that in a bounded time interval there are, at most, a finite number of transitions of the process, while in Assumption 4.2, we impose continuity and compactness conditions to ensure the existence of minimax selectors.

Assumption 4.1. There exist $\eta > 0$ and $\varepsilon > 0$ such that for all $(x,a) \in \mathbb{K}_A$ and $\theta \in \Theta(x,a)$

$$\theta([0,\eta]) < 1 - \varepsilon.$$

From Assumption 4.1 we have that for all $(x,a) \in \mathbb{K}_A$ and $\theta \in \Theta(x,a)$,

$$\begin{aligned} \int_0^\infty e^{-\alpha s}\theta(ds) &\leq \theta([0,\eta]) + e^{-\alpha\eta}(1 - \theta([0,\eta]) \\ &< 1 - \varepsilon(1 - e^{-\alpha\eta}) < 1, \end{aligned}$$

which implies,

$$\lambda := \sup_{(x,a)\in\mathbb{K}_A} \sup_{\theta\in\Theta(x,a)} \Delta_\alpha(\theta) < 1. \tag{4.9}$$

Assumption 4.2. (a) The cost functions $D(x,a)$ and $d(x,a)$ are lower semicontinuous (l.s.c.) on $\mathbb{K}_A$. Moreover, there exist a continuous function $W : X \to [1,\infty)$ and positive constants $\bar{c}_1$, $\bar{c}_2$, and β such that

$$1 \leq \beta < \lambda^{-1}, \tag{4.10}$$

$$\sup_{a\in A(x)} D(x,a) \leq \bar{c}_1 W(x), \qquad \sup_{a\in A(x)} d(x,a) \leq \bar{c}_2 W(x), \tag{4.11}$$

and

$$\int_X W(y)\, Q(dy \mid x,a) \leq \beta W(x) \qquad \forall\, (x,a) \in \mathbb{K}_A. \tag{4.12}$$

(b) For each bounded and continuous function $u : X \to \mathbb{R}$, the function

$$(x,a) \longmapsto \int_X u(y)\, Q(dy \mid x,a)$$

is continuous on $\mathbb{K}_A$.
(c) The function

$$(x,a) \longmapsto \int_X W(y)\, Q(dy \mid x,a)$$

is continuous on $\mathbb{K}_A$.
(d) The set $A(x)$ is compact for each $x \in X$. In addition, the set valued mapping $x \to A(x)$ is upper semi-continuous (u.s.c.).
(e) The set $\Theta(x,a)$ is σ-compact for each $(x,a) \in \mathbb{K}_A$ and the set valued mapping $(x,a) \longmapsto \Theta(x,a)$ is l.s.c.

Remark 4.3. (a) From Assumption 4.2*(b), if* $u : X \to \mathbb{R}$ is l.s.c., then the function

$$(x,a) \longmapsto \int_X u(y)\, Q(dy \mid x,a)$$

is l.s.c. *on* $\mathbb{K}_A$.

(b) From (4.11) in Assumption 4.2(a), $c(x,a,\theta) \leq \bar{c}W(x)$ $\forall (x,a,\theta) \in \mathbb{K}$*, where* $\bar{c} := \bar{c}_1 + \bar{c}_2$*.*

(c) For each $(x,a) \in \mathbb{K}_A$, $\Delta_\alpha(\theta)$, $\tau_\alpha(\theta)$ and $c(x,a,\cdot)$ are continuous on $\Theta(x,a)$. Indeed, if $\{\theta_n\}$ is a sequence in $\Theta(x,a)$ such that $\theta_n \xrightarrow{w} \theta$, then

$$\begin{aligned}\lim_{n\to\infty} \Delta_\alpha(\theta_n) &= \lim_{n\to\infty} \int_0^\infty \exp(-\alpha s)\, \theta_n(ds) \\ &= \int_0^\infty \exp(-\alpha s)\, \theta(ds) = \Delta_\alpha(\theta),\end{aligned}$$

which, in turn, yields

$$\lim_{n\to\infty} \tau_\alpha(\theta_n) = \tau_\alpha(\theta) \tag{4.13}$$

and

$$\lim_{n\to\infty} c(x,a,\theta_n) = c(x,a,\theta).$$

(d) From Assumption *4.2(a), (4.8), and (4.13), it follows that* $c(\cdot,\cdot,\cdot)$ is l.s.c. *on* $\mathbb{K}$.

(e) *Assumption 4.2(d), (e) implies the existence of measurable selectors for the set valued functions* $x \to A(x)$ and $(x,a) \longmapsto \Theta(x,a)$ (see Brown and Purves [3]). Moreover, since Θ is σ-compact [see Remark 2.1(b)], the σ-compactness condition in Assumption *4.2(e) can be replaced by the assumption that* $\Theta(x,a)$ be closed for all $(x,a) \in \mathbb{K}_A$.

We denote by $\mathbb{B}_W$ the normed linear space of all measurable functions $u : X \to \mathbb{R}$ with norm

$$\|u\|_W := \sup_{x\in X} \frac{|u(x)|}{W(x)} < \infty, \tag{4.14}$$

and by $\mathbb{L}_W$ the subspace of l.s.c. functions in $\mathbb{B}_W$.

5. Reduction of the minimax problem

In this section we will prove that for each pair $(\pi,\gamma) \in \Pi \times \Gamma$ there exists a pair $(\sigma,\rho) \in \Phi \times \Psi$ such that for each $x \in X$,

$$V(\pi,\gamma,x) = V(\sigma,\rho,x).$$

Furthermore, we obtain an alternative representation of the performance index (4.3) of the family of markovian strategies with which we can get a more tractable minimax equation in the next sections.

For $(\pi, \gamma) \in \Pi \times \Gamma$, $x \in X$ and $n = 0, 1, \ldots$ we define the finite measures $M_{x,n}^{\pi\gamma}$ on $X \times A \times \Theta$ and $m_{x,n}^{\pi\gamma}$ on X by

$$M_{x,n}^{\pi\gamma}(D) := E_x^{\pi\gamma} e^{-\alpha T_n} I\{(x_n, a_n, \theta_n) \in D\} \tag{5.15}$$

and

$$m_{x,n}^{\pi\gamma}(Y) := E_x^{\pi\gamma} e^{-\alpha T_n} I\{x_n \in Y\}$$

where $D \in \mathcal{B}(X \times A \times \Theta)$, $Y \in \mathcal{B}(X)$ and $I\{\cdot\}$ is the indicator function.

Since $m_{x,n}^{\pi\gamma}$ is the marginal of $M_{x,n}^{\pi\gamma}$ on X, by Corollary 7.27.2 in Bertsekas and Shreve [2], there exists a stochastic kernel ν_n on $X \times \Theta$ given X such that for $Y \in \mathcal{B}(X)$, $A' \in \mathcal{B}(A)$ and $B' \in \mathcal{B}(\Theta)$,

$$M_{x,n}^{\pi\gamma}(Y \times A' \times B') = \int_Y \nu_n(A' \times B' \mid y) m_{x,n}^{\pi\gamma}(dy).$$

Then, by Corollary 7.27.1 in Bertsekas and Shreve [2], there exist stochastic kernels σ_n on A given X and ρ_n on Θ given $X \times A$ such that

$$M_{x,n}^{\pi\gamma}(Y \times A' \times B') = \int_Y \int_{A'} \rho_n(B' \mid y, a)\sigma_n(da \mid y) m_{x,n}^{\pi\gamma}(dy). \tag{5.16}$$

As $M_{x,n}^{\pi\gamma}$ is concentrated on $\mathbb{K}$, then for every $n = 0, 1, 2, \ldots$ we can select versions of σ_n and ρ_n such that for every $y \in X$, $\sigma_n(A(y) \mid y) = 1$ and $\rho_n(\Theta(y, a) \mid y, a) = 1$ for every $(y, a) \in \mathbb{K}_A$. Hence, $\sigma := \{\sigma_n\}$ is a Markov strategy for the controller and $\rho := \{\rho_n\}$ is a Markov strategy for the opponent.

Lemma 5.1. *Let $g : X \times A \times \Theta \to \mathbb{R}$ and $h : X \to \mathbb{R}$ be nonnegative measurable functions. Then for every $n = 0, 1, 2, \ldots$*

$$E_x^{\pi\gamma} e^{-\alpha T_n} g(x_n, a_n, \theta_n) = \int_{X \times A \times \Theta} g(y, a, \theta) M_{x,n}^{\pi\gamma}(d(y, a, \theta)) \tag{5.17}$$

and

$$E_x^{\pi\gamma} e^{-\alpha T_n} h(x_n) = \int_X h(y) m_{x,n}^{\pi\gamma}(dy). \tag{5.18}$$

Proof. If $D \in \mathcal{B}(X \times A \times \Theta)$ and $g = I_D$, then (5.17) follows directly from (5.15). Thus, by linearity and the monotone convergence theorem, we obtain the result for a nonnegative measurable function g. The proof of (5.18) is analogous. □

Lemma 5.2. *Let (π, γ) be a pair of strategies in $\Pi \times \Gamma$, and $\sigma = \{\sigma_n\},\ \rho = \{\rho_n\}$ the corresponding Markov strategies defined in (5.16). Then for $x \in X$ and $n = 0, 1, 2, ...$*

$$M_{x,n}^{\pi\gamma} = M_{x,n}^{\sigma\rho} \tag{5.19}$$

and $V(\pi, \gamma, x) = V(\sigma, \rho, x)$.

Proof. We will prove (5.19) by induction. First, observe that for $Y \in \mathcal{B}(X)$,

$$m_{x,0}^{\pi\gamma}(Y) := E_x^{\pi\gamma} I\{x_0 \in Y\} = \delta_x(Y) = m_{x,0}^{\sigma\rho}(Y).$$

Moreover, from (5.15) and (5.16) it follows that for $Y \in \mathcal{B}(X)$, $A' \in \mathcal{B}(A)$ and $B' \in \mathcal{B}(\Theta)$,

$$M_{x,0}^{\pi\gamma}(Y \times A' \times B') = M_{x,0}^{\sigma\rho}(Y \times A' \times B').$$

Suppose that (5.19) holds for some $n \in \mathbb{N}_0$. Then

$$
\begin{aligned}
&m^{\pi\gamma}_{x,n+1}(Y)\\
&= E^{\pi\gamma}_x e^{-\alpha T_{n+1}} I\{x_{n+1} \in Y\}\\
&= E^{\pi\gamma}_x E^{\pi\gamma}_x [e^{-\alpha(T_n+\delta_{n+1})} I\{x_{n+1} \in Y\} \mid T_n, x_n, a_n, \theta_n]\\
&= E^{\pi\gamma}_x e^{-\alpha T_n} E^{\pi\gamma}_x \left[e^{-\alpha\delta_{n+1}} I\{x_{n+1} \in Y\} \mid T_n, x_n, a_n, \theta_n\right]\\
&= E^{\pi\gamma}_x e^{-\alpha T_n} Q(Y \mid x_n, a_n) \int_0^\infty e^{-\alpha s}\theta_n(ds) \qquad (5.20)\\
&= \int_{X\times A\times\Theta} g(y,a,\theta) M^{\pi\gamma}_{x,n}(d(y,a,\theta))\\
&= \int_{X\times A\times\Theta} g(y,a,\theta) M^{\sigma\rho}_{x,n}(d(y,a,\theta)),
\end{aligned}
$$

where $g(y,a,\theta) = Q(Y \mid y,a) \int_0^\infty e^{-\alpha s}\theta(ds)$. By setting $\sigma = \pi$ and $\rho = \gamma$ and repeating the process in (5.20), we obtain

$$
m^{\pi\gamma}_{x,n+1}(Y) = m^{\sigma\rho}_{x,n+1}(Y). \qquad (5.21)
$$

Now, we prove that

$$
M^{\pi\gamma}_{x,n+1} = M^{\sigma\rho}_{x,n+1}. \qquad (5.22)
$$

For this, first observe that from (5.16) and (5.21),

$$
\begin{aligned}
&M^{\pi\gamma}_{x,n+1}(Y \times A' \times B')\\
&= \int_Y \int_{A'} \rho_{n+1}(B' \mid y,a)\sigma_{n+1}(da \mid y) m^{\pi\gamma}_{x,n+1}(dy)\\
&= \int_Y \int_{A'} \rho_{n+1}(B' \mid y,a)\sigma_{n+1}(da \mid y) m^{\sigma\rho}_{x,n+1}(dy). \qquad (5.23)
\end{aligned}
$$

On the other hand,

$$\begin{aligned}
&M^{\sigma\rho}_{x,n+1}(Y \times A' \times B') \\
&= E^{\sigma\rho}_x e^{-\alpha T_{n+1}} I\{x_{n+1} \in Y, a_{n+1} \in A', \theta_{n+1} \in B'\} \\
&= E^{\sigma\rho}_x E^{\sigma\rho}_x [e^{-\alpha T_{n+1}} I\{x_{n+1} \in Y, a_{n+1} \in A', \theta_{n+1} \in B'\} \mid T_{n+1}, x_{n+1}] \\
&= E^{\sigma\rho}_x e^{-\alpha T_{n+1}} I\{x_{n+1} \in Y\} E^{\sigma\rho}_x \left[I\{a_{n+1} \in A', \theta_{n+1} \in B'\} \mid T_{n+1}, x_{n+1}\right] \\
&= E^{\sigma\rho}_x e^{-\alpha T_{n+1}} I\{x_{n+1} \in Y\} \textstyle\int_{A'} \rho_{n+1}(B' \mid x_{n+1}, a_{n+1}) \sigma_{n+1}(da_{n+1} \mid x_{n+1}) \\
&= \int_X I\{y \in Y\} \int_{A'} \rho_{n+1}(B' \mid y, a) \sigma_{n+1}(da \mid y) m^{\sigma\rho}_{x,n+1}(dy) \\
&= \int_Y \int_{A'} \rho_{n+1}(B' \mid y, a) \sigma_{n+1}(da \mid y) m^{\sigma\rho}_{x,n+1}(dy), \qquad (5.24)
\end{aligned}$$

where we have used (5.18) with

$$h(y) = I\{y \in Y\} \int_{A'} \rho_{n+1}(B' \mid y, a) \sigma_{n+1}(da \mid y).$$

Then (5.22) follows from (5.23) and (5.24). From (5.17) (with $g = c$) and (5.19) we obtain

$$V(\pi, \gamma, x) = V(\sigma, \rho, x).$$

□

Hence, by Lemma 5.2, the minimax problem can be reduced in the following manner: find a Markov strategy $\sigma^* \in \Phi$ such that

$$V'(\sigma^*, x) = \inf_{\sigma \in \Phi} V'(\sigma, x) = \inf_{\sigma \in \Phi} \sup_{\rho \in \Psi} V(\sigma, \rho, x).$$

We conclude this section introducing an alternative representation of the performance index (4.3).

Proposition 5.3. *For each $x \in X$ and a pair $(\sigma, \rho) \in \Phi \times \Psi$,*

$$V(\sigma, \rho, x) := E^{\sigma\rho}_x \left[c(x_0, a_0, \theta_0) + \sum_{j=1}^{\infty} \prod_{k=0}^{j-1} \Delta_\alpha(\theta_k)\, c(x_j, a_j, \theta_j) \right].$$

Proof. Note that for $x \in X$ and $(\sigma, \rho) \in \Phi \times \Psi$, it follows from the construction of $P_x^{\sigma\gamma}$ that

$$
\begin{aligned}
&E_x^{\sigma\rho} e^{-\alpha T_1} c(x_1, a_1, \theta_1) \\
&= \int_A \sigma_0(da_0 \mid x) \int_\Theta \int_0^\infty e^{-\alpha t_1} \theta_0(dt_1) \rho_0(d\theta_0 \mid x, a_0) \\
&\int_X Q(dx_1 \mid x, a_0) \int_A \sigma_1(da_1 \mid x_1) \int_\Theta c(x_1, a_1, \theta_1) \rho_1(d\theta_1 \mid x_1, a_1) \\
&= \int_A \sigma_0(da_0 \mid x) \int_\Theta \Delta_\alpha(\theta_0) \rho_0(d\theta_0 \mid x, a_0) \\
&\int_X Q(dx_1 \mid x, a_0) \int_A \sigma_1(da_1 \mid x_1) \int_\Theta c(x_1, a_1, \theta_1) \rho_1(d\theta_1 \mid x_1, a_1) \\
&= E_x^{\sigma\rho} \Delta_\alpha(\theta_0) c(x_1, a_1, \theta_1),
\end{aligned}
$$

and for $n = 2, 3, \ldots$

$$
E_x^{\sigma\rho} e^{-\alpha T_n} c(x_n, a_n, \theta_n) = E_x^{\sigma\rho} \Delta_\alpha(\theta_0) \cdots \Delta_\alpha(\theta_{n-1}) c(x_n, a_n, \theta_n).
$$

Then we can write

$$
V(\sigma, \rho, x) := E_x^{\sigma\rho} \left[c(x_0, a_0, \theta_0) + \sum_{j=1}^{\infty} \prod_{k=0}^{j-1} \Delta_\alpha(\theta_k)\, c(x_j, a_j, \theta_j) \right].
$$

□

6. Preliminary results

For a function $u \in \mathbb{B}_W$ and $(x, a, \theta) \in \mathbb{K}$, we define

$$
H(u, x, a, \theta) := c(x, a, \theta) + \Delta_\alpha(\theta) \int_X u(y)\, Q(dy \mid x, a), \tag{6.25}
$$

and

$$T_\alpha u(x) := \inf_{a\in A(x)} \sup_{\theta\in\Theta(x,a)} H(u,x,a,\theta). \tag{6.26}$$

A first consequence from previous assumptions is the following:

Lemma 6.1. *If Assumptions 4.1 and 4.2 hold, then:*
(a) The operator T_α maps $\mathbb{L}_W$ into itself.
(b) For each $u \in \mathbb{L}_W$ there exists $f^* \in \mathbb{F}_A$ such that

$$T_\alpha u(x) = \sup_{\theta\in\Theta(x,f^*)} H(u,x,f^*,\theta), \quad x \in X. \tag{6.27}$$

Proof. *(a)* Note that from Assumption 4.2*(a)* and Remark 4.3, for every $u \in \mathbb{L}_W$, $H(u,\cdot,\cdot,\cdot)$ is l.s.c. on $\mathbb{K}$ and

$$\begin{aligned} H(u,x,a,\theta) &\leq \bar{c}W(x) + \lambda \|u\|_W \int_X W(y)\, Q(dy \mid x,a) \\ &\leq \bar{c}W(x) + \lambda\beta \|u\|_W W(x) = KW(x), \end{aligned}$$

where $K := \bar{c} + \lambda\beta\|u\|_W < \infty$.

On the other hand, let $\{(x_n,a_n)\}$ be a sequence in $\mathbb{K}_A$ converging to $(x,a) \in \mathbb{K}_A$, and $\theta_0 \in \Theta(x,a)$ be arbitrary. Then, since $(x,a) \longmapsto \Theta(x,a)$ is l.s.c., there exists $\theta_n \in \Theta(x_n,a_n)$ such that $\theta_n \to \theta_0$ (see Proposition D.2 in Hernández-Lerma and Lasserre [6]). Hence,

$$\begin{aligned} \liminf_{n\to\infty} \sup_{\theta\in\Theta(x_n,a_n)} H(u,x_n,a_n,\theta) &\geq \liminf_{n\to\infty} H(u,x_n,a_n,\theta_n) \\ &\geq H(u,x,a,\theta_0), \end{aligned}$$

where the last inequality follows from the lower semicontinuity of $H(u,\cdot,\cdot,\cdot)$ on $\mathbb{K}$. Since θ_0 is arbitrary, we obtain

$$\liminf_{n\to\infty} \sup_{\theta\in\Theta(x_n,a_n)} H(u,x_n,a_n,\theta) \geq \sup_{\theta\in\Theta(x,a)} H(u,x,a,\theta).$$

Therefore, the function

$$(x,a) \longmapsto \sup_{\theta\in\Theta(x,a)} H(u,x,a,\theta) \tag{6.28}$$

is l.s.c. on $\mathbb{K}_A$. Then, from a well known result in Schäl [19] (see, also, Proposition D.5 in Hernández-Lerma and Lasserre [6]), we have that

$$T_\alpha u(x) = \inf_{a\in A(x)} \sup_{\theta\in\Theta(x,a)} H(u,x,a,\theta) \tag{6.29}$$

is l.s.c. on X and

$$|T_\alpha u(x)| \leq KW(x). \tag{6.30}$$

Hence, from (6.29) and (6.30), $T_\alpha u(\cdot) \in \mathbb{L}_W$ for all $u \in \mathbb{L}_W$.

(b) Since the function defined in (6.28) is nonnegative and l.s.c. on $\mathbb{K}_A$, standard arguments on the existence of minimizers (see for instance Rieder [17] or Schäl [19]) imply the existence of $f^* \in \mathbb{F}_A$ such that (6.27) holds. □

Lemma 6.2. *If Assumptions 4.1 and 4.2(a) hold, T_α is a contraction operator on $\mathbb{L}_W$ with modulus $\lambda\beta < 1$, i.e.,*

$$\|T_\alpha u - T_\alpha u'\|_W \leq \lambda\beta \|u-u'\|_W \quad \forall u,u' \in \mathbb{L}_W.$$

Proof. Let u,$u' \in \mathbb{L}_W$. Then,

$$\begin{aligned}
&|T_\alpha u(x) - T_\alpha u'(x)| \\
\leq\ & \left| \inf_{a\in A(x)} \sup_{\theta\in\Theta(x,a)} H(u,x,a,\theta) - \inf_{a\in A(x)} \sup_{\theta\in\Theta(x,a)} H(u',x,a,\theta) \right| \\
\leq\ & \sup_{a\in A(x)} \sup_{\theta\in\Theta(x,a)} \Delta_\alpha(\theta) \int_X |u(y) - u'(y)| \, Q(dy \mid x,a) \\
\leq\ & \lambda \|u-u'\|_W \int_X W(y)\, Q(dy \mid x,a) \\
\leq\ & \lambda\beta W(x) \|u-u'\|_W .
\end{aligned}$$

Thus,

$$\|T_\alpha u - T_\alpha u'\|_W \leq \lambda\beta \|u - u'\|_W \quad \forall u, u' \in \mathbb{L}_W.$$

□

For each $n \in \mathbb{N}$, $x \in X$ and $(\sigma, \rho) \in \Phi \times \Psi$ we define the n-stage expected discounted cost as,

$$V^n(\sigma, \rho, x) := \begin{cases} E_x^{\sigma\rho}\left[c(x_0, a_0, \theta_0)\right] & n = 1 \\ E_x^{\sigma\rho}\left[c(x_0, a_0, \theta_0) + \sum_{j=1}^{n-1}\prod_{k=0}^{j-1}\Delta_\alpha(\theta_k)\, c(x_j, a_j, \theta_j)\right] & \geq 2. \end{cases}$$

In addition, we define the sequence $\{v_n\}$ in $\mathbb{L}_W$ as $v_0 \equiv 0$, and for $n \in \mathbb{N}$,

$$v_n(x) = T_\alpha v_{n-1}(x), \quad x \in X. \tag{6.31}$$

Hence, for all $x \in X$ and $(\sigma, \rho) \in \Phi \times \Psi$,

$$V^n(\sigma, \rho, x) \nearrow V(\sigma, \rho, x) \tag{6.32}$$

and

$$v_n(x) \leq \sup_{\gamma \in \Gamma} V^n(\sigma, \gamma, x) \quad \forall n \in \mathbb{N}. \tag{6.33}$$

7. Main results

Theorem 7.1. *Under Assumptions 4.1 and 4.2, there exist a function $\hat{v} : X \to \mathbb{R}$ and a policy $f^* \in \mathbb{F}_A$ such that:*

(a) $\hat{v}$ is the unique function in $\mathbb{L}_W$ satisfying

$$\hat{v} = T_\alpha \hat{v}, \tag{7.34}$$

$$\|v_n - \hat{v}\|_W \leq \frac{\bar{c}\,(\lambda\beta)^n}{1-\lambda\beta} \quad \forall n \in \mathbb{N}, \tag{7.35}$$

and

$$\lim_{n\to\infty} \lambda^n E_x^{\sigma\rho}\left[\hat{v}\,(x_n)\right] = 0 \;\; \forall \sigma \in \Phi, \rho \in \Psi, x \in X; \tag{7.36}$$

(b) for all $x \in X$*,*

$$\hat{v}\,(x) = \sup_{\theta \in \Theta(x,f^*)} \left\{ c\,(x, f^*, \theta) + \Delta_\alpha\,(\theta) \int_X \hat{v}\,(y)\, Q\,(dy \mid x, f^*) \right\}.$$

Proof. *(a)* Since T_α is a contraction operator which maps $\mathbb{L}_W$ into itself (see Lemma 6.1) and $\mathbb{L}_W \subset \mathbb{B}_W$ is complete, from Banach's Fixed Point Theorem, there exists a unique function $\hat{v} \in \mathbb{L}_W$ such that (7.34) holds and

$$\begin{aligned} \|v_n - \hat{v}\|_W &\leq (\lambda\beta)^n \|v_0 - \hat{v}\|_W \\ &= (\lambda\beta)^n \|\hat{v}\|_W \qquad \forall n \in \mathbb{N}. \end{aligned} \tag{7.37}$$

Moreover, since

$$\|\hat{v}\|_W = \|T_\alpha \hat{v}\|_W \leq \bar{c} + (\lambda\beta)^n \|\hat{v}\|_W\,,$$

by iteration we obtain

$$\|\hat{v}\|_W \leq \frac{\bar{c}}{1-\lambda\beta}. \tag{7.38}$$

Hence, (7.35) follows from (7.37) and (7.38).

Now, from Assumption 4.2(a), for $\sigma \in \Phi, \rho \in \Psi, x \in X, n \in \mathbb{N}_0$,

$$E_x^{\sigma\rho}\left[W\,(x_{n+1})\right] \leq \beta E_x^{\sigma\rho}\left[W\,(x_n)\right],$$

which implies

$$E_x^{\sigma\rho}\left[W\left(x_n\right)\right] \leq \beta^n W\left(x\right),$$

or

$$\lambda^n E_x^{\sigma\rho}\left[W\left(x_n\right)\right] \leq \left(\lambda\beta\right)^n W\left(x\right)$$

and therefore,

$$\lim_{n\to\infty} \lambda^n E_x^{\sigma\rho}\left[W\left(x_n\right)\right] = 0.$$

This yields

$$\lim_{n\to\infty} \lambda^n E_x^{\sigma\rho}\left[\hat{v}\left(x_n\right)\right] = 0,$$

which concludes the proof of *(a)*.

The proof of part *(b)* follows directly from Lemma 6.1*(b)*. □

Theorem 7.2. *If Assumption 4.2 holds, then:*

(a) $\hat{v} = V^*$.

(b) There exists a minimax strategy $f^* \in \mathbb{F}_A$*, that is,*

$$V^*\left(x\right) = \inf_{\sigma\in\Phi} \sup_{\rho\in\Psi} V\left(\sigma,\rho,x\right) = \sup_{\rho\in\Psi} V\left(f^*,\rho,x\right).$$

Proof. *(a)* Let $f^* \in \mathbb{F}_A$ be a stationary policy such that for all $x \in X$,

$$\hat{v}\left(x\right) = \sup_{\theta\in\Theta(x,f^*)} \left\{ c\left(x,f^*,\theta\right) + \Delta_\alpha\left(\theta\right) \int_X \hat{v}\left(y\right) Q\left(dy \mid x, f^*\right) \right\}.$$

Then for all $x \in X$ and $\theta \in \Theta(x, f^*(x))$

$$\hat{v}\left(x\right) \geq c\left(x,f^*,\theta\right) + \Delta_\alpha\left(\theta\right) \int_X \hat{v}\left(y\right) Q\left(dy \mid x, f^*\right). \quad (7.39)$$

Now, let $\rho' \in \Psi$ be an arbitrary Markov strategy for the opponent. Iterating the inequality (7.39), a straightforward calculation yields

$$\begin{aligned}\hat{v}(x) \quad \geq \quad & E_x^{f^*\rho'}\left[c(x_0, f^*, \theta_0) + \sum_{j=1}^{n-1}\prod_{k=0}^{j-1}\Delta_\alpha(\theta_k)\, c(x_j, f^*, \theta_j)\right] \\ & + E_x^{f^*\rho'}\left[\prod_{k=0}^{j-1}\Delta_\alpha(\theta_k)\,\hat{v}(x_n)\right],\end{aligned}$$

which in turn implies

$$\hat{v}(x) \geq V^n(f^*, \rho', x).$$

Letting $n \to \infty$ we obtain

$$\hat{v}(x) \geq V(f^*, \rho', x) \quad \forall x \in X, \rho' \in \Psi$$

and this yields

$$\hat{v}(x) \geq \sup_{\rho\in\Psi} V(f^*, \rho, x) \quad \forall x \in X. \tag{7.40}$$

Therefore,

$$\hat{v}(x) \geq V^*(x).$$

To prove the reverse inequality, observe that for all $\sigma \in \Phi$ and $n \in \mathbb{N}$, from (6.33) we have

$$v_n(x) \leq \sup_{\rho\in\Psi} V^n(\sigma, \rho, x).$$

Now, since $v_n \to \hat{v}$ as $n \to \infty$, from (6.32) we obtain

$$\hat{v}(x) \leq \sup_{\rho\in\Psi} V(\sigma, \rho, x),$$

which implies

$$\hat{v}(x) \leq \inf_{\sigma \in \Phi} \sup_{\rho \in \Psi} V(\sigma, \rho, x) = V^*(x) \quad \forall x \in X,$$

and therefore,

$$\hat{v}(x) = V^*(x).$$

(b) From Theorem 7.1*(b)* and part *(a)* there exists $f^* \in \mathbb{F}_A$ such that

$$V^*(x) = \sup_{\theta \in \Theta(x, f^*)} \left\{ c(x, f^*, \theta) + \Delta_\alpha(\theta) \int_X V^*(y) Q(dy \mid x, f^*) \right\}.$$

Then, from (7.40) with V^* instead of $\hat{v}$, we have

$$V^*(x) \geq \sup_{\rho \in \Psi} V(f^*, \rho, x) \geq \inf_{\sigma \in \Phi} \sup_{\rho \in \Psi} V(\sigma, \rho, x),$$

which implies

$$V^*(x) = \inf_{\pi \in \Pi} \sup_{\gamma \in \Gamma} V(\pi, \gamma, x) = \sup_{\gamma \in \Gamma} V(f^*, \gamma, x).$$

Therefore, f^* is a minimax strategy. □

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II. VECTORIAL COOPERATIVE GAMES WITH MATRIX CHARACTERISTIC FUNCTIONS

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Abstract

In this work we show a new class of vectorial cooperative games where the characteristic function is defined by a set of matrices. Also, we show some situations where this new modeling can be applied for solving allocation problems. In the same way, we establish a relation between this new class of games and the transferable utility games on characteristic function form as well as multi-choice games. Finally, we provide an axiomatically characterized solution for the model proposed.

1. Introduction

From the very beginning, with the seminal work of Morgenstern and Von Neumann [5], game theory has arisen as a helpful methodology for solving problems related to agents and utilities as a result

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of their interactions. One of the branches of game theory, non-cooperative game theory, assumes that the decisions taken by an agent (or a player), in addition to the decisions of the other agents, involve an individual utility for every agent, and it is the perfect model for studying competing situations. There are tremendous advances, results and variations on this branch, as the results from Nash [7], Myerson [6], Harsanyi and Selten [3] demonstrate, just to cite a few works. On the other hand, cooperative game theory is perfect to deal with situations where there exist interactions among subsets of agents. So, each subset of players has associated a joint amount of payoff, and it is interesting to find *fair* ways for allocating that kind of profit amongst the players. Some of the most important works about cooperative game theory are from Shapley [9], Owen [8] and Aumann and Maschler [1]. Game theory has been one of the fastest growing mathematical fields in the last decades.

One interesting problem in several associations and societies is to allocate an amount among a set of agents taking into account their individual performance in each one of several activities. For example, in educational and research institutions, professors and researchers receive bonuses according to their performance in activities such as teaching, publishing papers, directing theses, etc. So, at the end of a given period of time, a given committee valuates the performance of the whole institution and assigns the total amount of bonuses (in monetary units) for the institution which must be distributed among the agents *in some way*, but taking into account their individual performances (for example, it makes sense that a professor without scientific publications and teaching should not receive any kind of bonus).

Another example arises in societies where the people must work generating basic services for the community, such as tap water, education and electricity. According to the skills of each person, or the time expended on each task related to those services, an amount of electricity is produced, a certain quantity of tap water is provided and some educational infrastructure is created. So, how does a society allocate these services among the agents who produce them? These kinds of problems can be studied using vectorial cooperative games with matrix characteristic functions.

In a general framework, in these games there is a finite set of agents and their performance in a set of activities is known (so, it makes sense to use a matrix characteristic function) and then, there is an amount associated for each activity related to the performance of all the agents. In this work, we assume that there exists a maximum available performance for each activity. In the example of the previous paragraph related to the production of services, the time that each person spent generating electricity, producing tap water and creating educational infrastructure has associated a certain amount of electricity, tap water and educational infrastructure (for example, if nobody spent time generating electricity, it would not be possible to produce tap water but there would be a lot of electricity available to the community). That is the reason to assume that the image of our characteristic function is a vector. Vectorial cooperative games were introduced by Fernández-García et al. [2], but this is the first time they are being studied jointly with matrix characteristic functions. We believe that this is an innovative proposal to the cooperative game theory field. So, after introducing the general model, we relate it to the classical notion of transferable utility games as well as the multichoice cooperative games (introduced by Hsiao and Raghavan [4]), where we notice that both models are particular cases of our modeling, which strengthens it. Finally, we show a solution for this new class of games characterized axiomatically.

This paper is organized as follows: in Section 2, we show the definitions and notation used in the work; in the same section, we show that some models in cooperative game theory are special cases of our model. In Section 3, we show a solution axiomatically characterized for our model, showing its equivalency with a model based on orders. Finally, we show some conclusions and ideas for future work.

2. Definitions and notation

Let $N := \{1, 2, \dots, n\}$ and $M := \{1, 2, \dots, m\}$ be a finite set of agents and a finite set of categories, respectively. We assume that each agent has a measurable performance $h \in \mathbb{Z}_+$ in each category, and fixing a number $k \in \mathbb{N}$, we assume that $h \leq k$. So, the set of performances of the problem is denoted by $\mathbb{Z}_k := \{0, \dots, k\}$. Thus, the set of matrices with n rows and m columns with entries on $\mathbb{Z}_k$ is denoted by $\mathbb{M}^n_m(\mathbb{Z}_k)$. Henceforth, we denote by $\mathbf{K}$ the matrix in $\mathbb{M}^n_m(\mathbb{Z}_k)$ with all entries equal to k, by $\mathbf{0}$ the matrix in $\mathbb{M}^n_m(\mathbb{Z}_k)$ with all entries equal to 0 and by $\overline{\mathbf{0}}$ the m-dimensional vector with all its entries equal to 0.

So, given a set of n agents, a set of m categories and a number $k \in \mathbb{N}$, a vectorial cooperative game with a matrix characteristic function (henceforth, a $M - game$), is a mapping

$$v : \mathbb{M}^n_m(\mathbb{Z}_k) \to \mathbb{R}^m, \qquad \text{with } v(\mathbf{0}) = \overline{\mathbf{0}}.$$

The idea behind these kinds of games is to model situations where the performance of each agent in every category impacts the amount obtained in each category for all in the society. The last condition of the previous equation assures that if there is no performance by anyone in any category, then there is no amount generated by the society in any category.

The set of M-games with set of n agents, m categories and $k \in \mathbb{N}$ is denoted by

$$G^n_m(k) := \{v \mid v : \mathbb{M}^n_m(\mathbb{Z}_k) \to \mathbb{R}^m, \text{ with } v(\mathbf{0}) = \overline{\mathbf{0}}\}$$

where it is straightforward to verify that $G^n_m(k)$ is a vector space under the usual operations:

- $v(\alpha S) = \alpha\, v(S)$ with $\alpha \in \mathbb{R}$ and for all $S \in \mathbb{M}^n_m(\mathbb{Z}_k)$
- $v(S + T) = v(S) + v(T)$ for all $S, T \in \mathbb{M}^n_m(\mathbb{Z}_k)$.

Example 2.1. Consider a society with three people. Each person has 5 available working periods of time for each activity to work in a water purifying plant and in a power plant generating electricity. The distribution of time each person works determines the total purified water produced and the electricity generated. For example, if we represent the people as rows and the activities as columns (the first column represents the working periods assigned to the water purifying plant, and the second, to the power plant), we could have

$$\begin{bmatrix} 3 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \Rightarrow (3,4), \qquad \text{and} \qquad \begin{bmatrix} 3 & 2 \\ 1 & 5 \\ 1 & 4 \end{bmatrix} \Rightarrow (4,5).$$

According to the first distribution, Person 1 assigns three periods for working in the water purifying plant and two periods in the power plant, and so on. Considering the assignments of the other persons, the society obtains three units of purified water and four units of electricity. In the second distribution, Person 2 now assigns 5 working periods to working in the power plant and then, because there is more generated electricity, it could be possible to obtain more purified water, too. These are the situations that can be adequately modeled using M-games. The problem is how to divide the total production among all the activities and players when all of them assign their total available time.

A solution for an M-game is a mapping

$$\varphi : G^n_m(k) \to \mathbb{M}^n_m(\mathbb{R})$$

where $\varphi_{ij}(v)$ denotes the payoff of agent $i \in N$ in the category $j \in M$ according to $v \in G_M^N(k)$.

When $k = 1$ and $m = 1$, then $G_M^N(k) \equiv G^N$, the set of cooperative games with transferable utility (TU-games). It is easy to prove this, because, in this case, the M-game is defined by n-dimensional column vectors with entries equal to one or zero. So, the value associated to each vector in the M-game is related to the value of a coalition where the players in the coalition match the entries on the column vector with entries equal to one.

Example 2.2. We show an example of the equivalency of an M-game with a TU-game, where $n = 4$, $k = 1$ and $m = 1$.

$$v\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = 10 \qquad \Rightarrow \qquad v(\{1, 4\}) = 10.$$

Also, it is possible to verify that if $k \geq 1$ and $m = 1$, an M-game is equal to a multi-choice cooperative game with maximum level of activity given by k. Then, we assure that the M-games are a very general kind of game, because the classical TU-games and the multichoice cooperative games are particular cases of this model.

Henceforth, we assume that the number of agents and the number of categories are fixed, so we are not going to use n and m any more unless it is strictly necessary. For simplification, we establish the next equivalency in the notation:

$$\mathbb{M}_k \equiv \mathbb{M}_m^n(\mathbb{Z}_k).$$

3. Characterizing solutions

In this section we show an axiomatic characterization of a solution for M-games. The axioms have similar versions in other kinds of games, for example, the classical TU-games.

Axiom 3.1. (Additivity) A solution $\varphi : G^n_m(k) \to \mathbb{M}^n_m(\mathbb{R})$ satisfies the additivity axiom if

$$\varphi(v) + \varphi(w) = \varphi(v + w),$$

for every pair of problems $v, w \in G^n_m(k)$.

According to this axiom, if we can divide a problem into subproblems and apply an additive solution to the original problem, the result must be equal to the sum of applying the additive solution to the subproblems.

Axiom 3.2. (Efficiency by categories) A solution $\varphi : G^n_m(k) \to \mathbb{M}^n_m(\mathbb{R})$ satisfies the efficiency by categories axiom if

$$\sum_{i \in N} \varphi_{ij}(v) = v_j(\mathbf{K}), \qquad \forall j \in M.$$

This axiom establishes that the amount to allocate in each category must be the amount generated by the agents when all of them are performing at their maximum capacity in each category. So, to accept this axiom implies that each agent is capable of performing as well as the maximum in each category.

We say that category $j \in M$ is a null category in the problem $v \in G^n_m(k)$ if $v_j(A) = 0$ for every $A \in \mathbb{M}_k$.

Axiom 3.3. (Null category) A solution $\varphi : G^n_m(k) \to \mathbb{M}^n_m(\mathbb{R})$ satisfies the null category axiom if

$$\varphi_{ij}(v) = 0$$

for each agent $i \in N$ in every null category $j \in M$ in the problem $v \in G^n_m(k)$.

A category is null when it does not matter what the agents do: the amount obtained in that category is equal to zero always. So, in these kinds of situations, it makes sense to have a payoff equal to zero for each player in a null category.

We define the minimum performance of agent i in the category j in the M- game $v \in G^n_n(k)$, m^v_{ij}, as the number in $\mathbb{Z}_k$ such that for every $A \in \mathbb{M}_k$ with $A_{ij} \leq m_{ij} - 1$ we have $v(A) = \overline{\mathbf{0}}$.

Axiom 3.4. (Proportionality according to the minimum performance) A solution $\varphi : G^n_m(k) \to \mathbb{M}^n_m(\mathbb{R})$ satisfies the proportionality according to the minimum performance axiom if

$$\frac{\varphi_{ik}(v)}{\varphi_{jk}(v)} = \frac{m^v_{ik}}{m^v_{jk}}$$

for every pair of agents $i, j \in N$ with minimum performance levels m^v_{ik} y m^v_{jk} in the category k.

The idea behind this axiom is to assign proportional payoffs to players that have a minimum required performance in a given category: the payoffs must be proportional to those required levels.

We denote by $\mathbf{e}^j_i \in \mathbb{M}_k$ the matrix with all its entries equal to zero except the ij-th entry, which value is equal to one. In the same way, $\overline{e}_i \in \mathbb{Z}^m_k$ denotes the m-dimensional vector with all its entries equal to zero except the i-th entry, that is equal to one. Likewise, for a matrix A, A_{ij} denotes the value of the matrix in the ij-th entry.

Theorem 3.5. *There exists a unique solution $\overline{\varphi} : G^n_m(k) \to \mathbb{M}^n_m(\mathbb{R})$ satisfying the additivity, efficiency by categories, null category and proportionality according to the minimum performance* *axioms. Even more, that solution is given by*

$$\overline{\varphi}_{ij}(v) = \sum_{S\in\mathbb{M}_k}\sum_{h=1}^{m}\sum_{r=1}^{S_{ij}} c_S[v_j(S) - v_j(S - r\mathbf{e}_i^h)],\ \forall i\in N, j\in M, \tag{3.1}$$

where

- $c_S = \displaystyle\sum_{h=0}^{S^c} \frac{(S^+ + h - 1)!(k\cdot m\cdot n - S^+ - h)!}{(k\cdot m\cdot n)!}\cdot\binom{S^c}{h}.$
- $S^+ = \displaystyle\sum_{\substack{i\in N\\ j\in M}} S_{ij}.$
- $\overline{S} = |(i,j) \mid S_{ij} \neq k,\ \forall i\in N, j\in M\}|.$
- $S^c = k\cdot n\cdot m - S^+ - \overline{S}.$

To prove the previous theorem, it is mandatory to define a special set of M-games that we call *unanimity* M-games as follows: for every pair of matrices $S, T\in\mathbb{M}_k$ and for every $\ell\in M$:

$$u_S^\ell(T) = \begin{cases} \overline{e}_\ell, & \text{if } T_{ij}\geq S_{ij}\ \forall i\in N, j\in M;\\ \overline{0}, & \text{otherwise.}\end{cases}$$

As an additional notation, we say that two matrices $T, S\in\mathbb{M}_k$, $T\leq S$ if $T_{ij}\leq S_{ij}$ for every $i\in N$, $j\in M$. Similarly, we define other order relationships in the set of matrices.

Lemma 3.6. *The set of unanimity M-games $\{u_S^\ell \mid \forall S\in\mathbb{M}_k, S\neq \mathbf{0}, \ell\in M\}$ is a basis for $G_m^n(k)$ and then, any M-game v could be written as follows:*

$$v = \sum_{\ell\in M}\sum_{\substack{S\in\mathbb{M}_k\\ S\neq\mathbf{0}}} c_v^\ell(S)u_S^\ell$$

where

$$c_v^\ell(S) = v_\ell(S) - \sum_{\substack{T \leq S \\ T \neq S}} c_v^l(T).$$

It is easy to see that the previous lemma is true, because the structure of the unanimity of M-games is the same as the unanimity of TU-games with characteristic function forms and, in our model, we have the same notion by category. Please refer to Shapley [9] regarding this issue.

Proof. (Proof of Theorem 3.5) First, we are going to prove uniqueness. As the solution φ is additive, we have that every M-game $v \in G_m^n(k)$

$$\varphi_{ij}(v) = \sum_{\ell \in M} \sum_{\substack{S \in \mathbb{M}_k \\ S \neq \mathbf{0}}} \varphi_{ij}(c_v^\ell(S) u_S^\ell),$$

and then, it is enough to prove uniqueness for the M-games $c_v^\ell(S) u_S^\ell$. Notice that every unanimity M-game assigns a zero value to the categories not equal to ℓ. Because of the null category axiom, we have

$$\varphi_{ij}(c_v^\ell(S) u_S^\ell) = \begin{cases} r, & \text{if } j = \ell; \\ 0, & \text{otherwise} \end{cases}$$

for some $r \in \mathbb{R}$. Notice that all the entries related to S for u_S^ℓ are minimum performance levels. So, because of the efficiency by categories and proportionality according to the minimum performance axiom, we have

$$\varphi_{ij}(c_v^\ell(S) u_S^\ell) = \begin{cases} c_v^\ell(S) \cdot \dfrac{S_{ij}}{\displaystyle\sum_{i \in N} S_{ij}}, & \text{if } j = \ell; \\ 0, & \text{otherwise} \end{cases}$$

and then, we finish the proof that a solution satisfying the axioms of the theorem defines a solution uniquely.

Now, we need to prove that expression (3.1) satisfies the axioms of the theorem. To prove that Solution (3.1) is additive and that satisfies the null category and proportionality regarding the minimum performance is straigthforward (for the latter, it is enough to see that for a unanimity of M-games u_S^ℓ each entry related to S is a minimum performance level). To show that the solution (3.1) satisfies the efficiency by category axiom, we are going to use orders of agents and, consequently, we are going to provide an alternative interpretation for the solution (3.1) based on how each category is performed by the agents.

Given a M-game $v \in G_m^n(k)$, we define the extended set of agents as follows:

$$D = \{i_j^h \mid i \in N, j \in M, h \in \{1, \ldots, k\}\}.$$

This set represents every unit of possible performance in each category of a single player. For example, if we have two players and two possible categories with a maximum performance of three units ($n = 2$, $m = 2$ and $k = 3$), then we have a set of twelve extended players. So, for example, the player 2_1^3 is representing the third activity of player 2 in the category 1. Thus, a permutation on the symmetric group S_D models the order in which the activities are done, from the beginning, where anybody has done any activity, until everybody has done all their activities. According to the previous example, the order

$$\sigma = (1_2^1, 2_2^1, 1_1^3, 1_2^2, 2_1^3, 1_1^2, 1_2^3, 2_2^2, 2_1^2, 1_1^1, 2_2^3, 2_1^1)$$

can be interpreted in several ways:

1. First, Player 1 does one activity from category 2. After that, Player 2 does an activity from category 2. In third place,

Player 1 makes an activity from category one, and so on. Finally, the last activity is done by Player 2 from category 1. According to this interpretation, it is important who does each activity from which category, where the number of the activity done, does not matter. So, under this interpretation, there is the assumption that increasing the number of activities in one unit in any category implies the same effort. A scheme about how to fill the matrix following the previous order is as follows:

$$\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 2\\1 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}2 & 2\\1 & 1\end{bmatrix}$$

$$\begin{bmatrix}2 & 3\\1 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}2 & 3\\1 & 2\end{bmatrix} \Rightarrow \begin{bmatrix}2 & 3\\2 & 2\end{bmatrix} \Rightarrow \begin{bmatrix}3 & 3\\2 & 2\end{bmatrix} \Rightarrow \begin{bmatrix}3 & 3\\2 & 3\end{bmatrix} \Rightarrow \begin{bmatrix}3 & 3\\3 & 3\end{bmatrix}$$

2. Second, it could be interesting to distinguish between the individual efforts in each activity in each category. So, for validating an activity, it is mandatory to do the previous activities for every category. That is, under this second approach, the extended player 1_1^3 is taken into account (or, for taking into account the third activity of Player 1 in the category 1); it is mandatory for Player 1 to have previously done the activities 1 and 2 in the category 1. So, under this approach, a scheme about how to fill the corresponding matrix of activities is as follows:

$$\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}0 & 2\\0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}0 & 2\\0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}0 & 2\\0 & 1\end{bmatrix}$$

$$\begin{bmatrix}0 & 3\\0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}0 & 3\\0 & 2\end{bmatrix} \Rightarrow \begin{bmatrix}0 & 3\\0 & 2\end{bmatrix} \Rightarrow \begin{bmatrix}3 & 3\\0 & 2\end{bmatrix} \Rightarrow \begin{bmatrix}3 & 3\\0 & 3\end{bmatrix} \Rightarrow \begin{bmatrix}3 & 3\\3 & 3\end{bmatrix}$$

If we want to use these two approaches for filling the activities matrix adding unitary activities, then it is affordable to pay to each extended player his marginal contribution for each category in respect to each order and to consider the payoff for Player i in the category j according to a given order as the sum of the payoffs of the extended players i_j^h for $h \in \{1, \ldots, k\}$. This procedure must be applied to every possible order. So, assuming that all the orders have the same importance, the idea is to take the arithmetic mean over these marginal contributions over all the orders. If we apply this procedure under the first approach, the result is equal to the Shapley value (Shapley [9]) for each extended player and then we must add the payoffs for the corresponding extended players for Player i in the category j. Under the second approach, the result is equal to formulation (3.1). Let us show it with the next example: Let v be an M-game with $n = 2$, $m = 2$ and $k = 3$, and we want to calculate the payoff for Player 1 in the category j. So, under this second approach based on orders, one of the marginal contributions to be considered in the calculation is

$$v_j\left(\left[\begin{array}{cc} 2 & 3 \\ 1 & 0 \end{array}\right]\right) - v_j\left(\left[\begin{array}{cc} 2 & 1 \\ 1 & 0 \end{array}\right]\right).$$

This marginal contribution indicates how the amount changes in category j when Player 1 increases his activity in category 2 from 1 to 3. There exist $(k \cdot m \cdot n)!$ possible orders of the extended set of players and then, it will be enough to count how many of these player 1_1^2 produce that contribution (the sub-index of the extended player of interest depends on the change in the position in the matrix, in this example, from 3 to 1). Notice that it is mandatory that the extended players $1_1^1, 1_1^2, 1_2^1, 1_2^3$ and 2_1^1 are sited in some position at the left of the player 1_1^2 (not necessarily in that order) to take into account that marginal contribution, and the extended players $1_1^3, 2_1^2, 2_1^1$ must be positioned in some place at the right. The players $2_1^3, 2_2^2$ and 2_2^3 can occupy any position. So, these are the orders producing the previous marginal contribution:

$$\begin{aligned}\sigma_1 &= (1_1^2, 1_2^3, 2_2^2, 2_1^1, 1_1^1, 2_2^3, 1_2^1, 1_1^2, 2_1^2, 2_2^1, 2_1^3, 1_1^3),\\ \sigma_2 &= (2_2^3, 1_1^1, 2_1^1, 1_1^2, 1_2^1, 1_2^3, 1_1^2, 1_1^3, 2_2^2, 2_1^3, 2_1^2, 2_2^1).\end{aligned}$$

It is easy to verify that the total number of orders satisfying the previous conditions for any matrix $S \in \mathbb{M}_k$ is precisely c_S, which includes the total number of possible orders (which we need for calculating the arithmetic mean). Thus, because the sum of the marginal contributions of each extended player in a given order is equal to the value of the matrix of maximum activities in the category j, Solution (3.1) is efficient by categories. □

4. Conclusions and future work

The main contribution of this work is to show a new class of cooperative games, where the characteristic function is defined using matrices and the image is a vector. These kinds of games are useful when the joint performance of the players in certain categories matters in the amount gained in each category, and it is mandatory to allocate an amount in some way. We show that this new model is a general case of classical transferable utility cooperative games or even multi-choice games. So, the resulting interest in this new model is justified because several results for this new model have an equivalency in other kinds of games. Moreover, we show a solution axiomatically characterized for our model using orders, where the main difference with other solutions based on orders is to regard each activity in each category of each player as having an importance related to the order in which it is done. Classical base- on-orders solutions just take into account the number of activities done.

For future work, it would be useful to provide additional ways for providing the solution given by (3.5), because the proportionality regarding the minimum performance axiom can be controversial. Furthermore, solutions based on extended players could have several interpretations regarding the order in which the mandatory activities are done. So, it could be interesting to characterize more solutions using alternative marginal contributions. Finally, because of the equivalency between our model and other models in cooperative game theory, it could be interesting to show the solution equivalent to 3.1 in other contexts.

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III. AXIOMATIC SOLUTIONS FOR GAMES IN PARTITION FUNCTION FORM

Joss Sánchez-Pérez

Abstract

In this work, we present the state-of-the-art and a systematic overview of solutions for games in partition function form characterized in an axiomatic way. This chapter contains some humble ideas not elaborated in other publications, such as concepts and approaches that use definitions unsupported by mathematical theorems. It also contains some of our own original results from publications on the topic, providing interpretations and comparisons, although it avoids taking up subjective positions when comparing competing models.

1. Introduction

Economic activities, both on the macro and micro level, often entail wide-spread externalities. This, in turn, leads to disputes regarding the appropriate compensation levels to be allocated to the various parties affected. The problem of how to fairly divide a surplus obtained through cooperation is one of the most fundamental issues studied in coalitional game theory and it is relevant to a wide range of economic and social situations. These

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issues are often difficult to resolve, especially in environments with externalities where the benefits of a group depend not only on its members, but also on the arrangement of agents outside the group. This is the general problem to which this paper contributes. In this line, such a problem was effectively modelled in Lucas and Thrall [7] by the concept of partition function form games: a partition function assigns a value to each pair consisting of a coalition and a coalition structure which includes that coalition. The advantage of this model is that it takes into account both internal factors (coalition itself) and external factors (coalition structure) that may affect cooperative outcomes and allows one to go deeper into cooperation problems. Thus, it is closer to real-life situations but more complex to analyze.

Since the introduction of the value by Shapley [17], much work has been devoted to the fair distribution of the earnings of cooperation among cooperating players. With a considerable delay, researchers turned to a more general and more realistic setting where the cooperation of groups or coalitions affects third parties, that is, when there are externalities.

Shapley's [17] result has been remarkable in many ways. It is not only a well-defined, nonempty point-solution, but it is fully characterized by a small number of elementary properties: axioms.

An axiomatization is the full characterization of a concept by widely accepted elementary ideas, or the (brief) list of axioms that uniquely determine the solution, in this case, the allocation of the value of the grand coalition.

Solution concepts may be complicated, and it is not expected that anyone outside the narrower field should be able to compare complex formulae. Axioms, on the other hand, reveal the true nature of these models. Breaking down the problem into simple properties, axioms should be understandable to people well beyond the experts of the field and it may be far easier to get decision makers to agree on such elementary principles than on a complex formula. Once there is an agreement on the principles, an axiomatization result provides us with a well-defined formula to apply those principles unless these decision makers drastically change their opinion on the fundamental principles once the implied allocation is presented.

This logic has worked for characteristic function form games and it seems far too easy to generalize these properties to the partition function form setting and obtain the generalization of the Shapley value to the partition function form setting. Unfortunately, the extensions turn out to be difficult and even controversial.

The purpose of this chapter is to present the state-of-the-art and a comprehensive survey of solutions for partition function form games characterized in an axiomatic method. This chapter is intended to be accessible to readers with little previous knowledge of cooperative game theory. The concepts and notation used are formally introduced and the results are presented without proofs (the reader is generally referred to the original publications). This chapter is not an original research article but contains some humble ideas not elaborated in other publications, such as concepts and approaches that use definitions unsupported by mathematical theorems. It also contains some of our own original results from publications on the topic, providing interpretations and comparisons, although it avoids taking up subjective positions in comparing competing models.[1]

We present a list of axioms used in the characterization of extended Shapley values, as well as an adaptation of axioms for

[1] A good reference for the theory of partition function form games is Kóczy [6], in which he considers the two fundamental approaches to solve partition function form games (stability and fairness).

extended Solidarity values[2]. The multiplicity of such extensions is, in part, the result of the multiplicity of extensions of axioms for characteristic function form games. Partition function form games are, of course, far more complex, but such a multiplicity of axioms is confusing and the different variations undermine the self-evidence of these properties. So first, we will focus on the most basic and widely-accepted and used axioms. Then we move on to alternatives to marginality, followed by alternatives to the null player property, and finally, we look at a set of lesser-known, alternative properties.

Briefly, the structure of the chapter is as follows. We first recall the main features of games with externalities in the next section. In Section 3, we formally introduce the most common axioms found in related literature that are needed to characterize concept solutions. In Section 4, we provide several axiomatic characterizations of solutions related to extensions of the Shapley value and Solidarity value. Finally, in Section 5, we discuss several axiomatizations of classes of solutions involving some axioms presented in Section 3.

2. Framework

Let $N = \{1, 2, ..., n\}$ be a fixed nonempty finite set, and let the members of N be interpreted as players in some game situation. Given $S \subseteq N$, $PT(S)$ denotes the set of partitions of S, so

$$\{S_1, S_2, ..., S_m\} \in PT(S) \text{ iff } \bigcup_{i=1}^{m} S_i = S,\ S_j \cap S_k = \varnothing\ \forall j \neq k.$$

For simplicity of notation, $PT(N) = PT$ and by convention, $\{\varnothing\} \in Q$ for every $Q \in PT$. For $S \subseteq N$, $P(S)$ denotes the set

[2]Precise definitions will be provided in Sect. 3.

of partitions that contains coalition S; i.e.,

$$P(S) = \{Q \in PT \mid S \in Q\} .$$

Also, let $EC = \{(S,Q) \mid S \in Q \in PT\}$ be the set of *embedded coalitions*, that is the set of coalitions together with specifications as to how the other players are aligned. The embedded coalition (S,Q) is called nontrivial if $S \neq \varnothing$.

Definition 2.1. A partition function form game[3] is a mapping

$$w : EC \to \mathbb{R}$$

with the property that $w(\varnothing, Q) = 0$ for every $Q \in PT$.

The set of games with externalities with player set N is denoted by G, i.e.,

$$G = \{w : EC \to \mathbb{R} \mid w(\varnothing, Q) = 0 \; \forall Q \in PT\}$$

and in what follows, we will only use the term 'game' to refer to a partition function form game.

The value $w(S,Q)$ represents the payoff of coalition S, given the coalition structure Q forms. In these kinds of games, the worth of some coalition depends not only on what the players of such a coalition can jointly obtain, but also on the way the other players are organized. We assume that in any game situation, the universal coalition N (embedded in $\{N\}$) will actually form, so that the players will have $w(N, \{N\})$ to divide among themselves. But we also anticipate that the actual allocation of this worth will depend on all the other potential worths $w(S,Q)$, as they influence the relative bargaining strengths of the players.

[3] Also known as "game with externalities."

For any $S \subseteq N$, let $[S]$ denote the partition of S which consists of the singleton elements of S, i.e., $[S] = \{\{j\} \mid j \in S\}$. For $Q \in PT$, $S \in Q$ and $i, k \in N$, we define $Q_{-S} = Q \backslash \{S\}$, $S_{-k} = S \backslash \{k\}$, $S_{+k} = S \cup \{k\}$ and Q^i denotes the member of Q to which i belongs. Additionally, we will denote the cardinality of a set by its corresponding lower-case letter, for instance $n = |N|$, $s = |S|$, $q = |Q|$ and so on.

Given $w_1, w_2 \in G$ and $c \in \mathbb{R}$, we define the sum $w_1 + w_2$ and the product cw_1 in G in the usual form, i.e.,

$$(w_1 + w_2)(S, Q) = w_1(S, Q) + w_2(S, Q)$$

and

$$(cw_1)(S, Q) = cw_1(S, Q),$$

respectively. It is easy to verify that G is a vector space with these operations and, also, that $\dim G = |EC|$.

Definition 2.2. A *solution on* G is a function $\varphi : G \to \mathbb{R}^n$ that assigns a real vector for each partition function form game.

If φ is a solution and $w \in G$, then we can interpret $\varphi_i(w)$ as the utility payoff which player i should expect from the game w.

Now, the group of permutations of N, $S_n = \{\theta : N \to N \mid \theta$ is bijective$\}$, acts on 2^N and on EC in the natural way; i.e., for $\theta \in S_n$:

$$\theta(S) = \{\theta(i) \mid i \in S\}$$

$$\theta(S_1, \{S_1, S_2, ..., S_l\}) = (\theta(S_1), \{\theta(S_1), \theta(S_2), ..., \theta(S_l)\}).$$

And also, S_n acts on the space of payoff vectors, $\mathbb{R}^n$:

$$\theta(x_1, x_2, ..., x_n) = (x_{\theta^{-1}(1)}, x_{\theta^{-1}(2)}, ..., x_{\theta^{-1}(n)}).$$

Definition 2.3. Let $(S, Q) \in EC$ and $i \in S$. Consider a partition Q' obtained by moving player i from S to some other (possible empty) member T of Q. The mapping $\alpha_{iT} : Q \to Q'$ defined by

$$\begin{aligned} &\alpha_{iT}(S) = S_{-i} \\ &\alpha_{iT}(T) = T_{+i} \\ &\alpha_{iT}(S') = S' \quad \text{for } S' \in Q_{-S,-T} \end{aligned}$$

is called a move for player i. Notice that $\alpha_{iT}(Q) = \{S_{-i}, T_{+i}\} \cup Q_{-S,-T}$.

On the other hand, a game is *with no externalities*[4] if and only if the payoff that the players in a coalition S can jointly obtain if this coalition is formed, is independent of the way the other players are organized. This means that in a game with no externalities, the characteristic function satisfies $w(S, Q) = w(S, Q')$ for any two partitions $Q, Q' \in PT$ and any coalition S which belongs both to Q and Q'. Hence, the worth of a coalition S can be written without reference to the organization of the remaining players, $w(S) := w(S, Q)$ for all $Q \ni S, Q \in PT$.

3. The axioms

In the cooperative game theory framework, axiomatization is an important approach to get a better understanding of cooperative solution concepts. Over the years, many different values and other concepts have been fully characterized by axioms. Some of these axiomatizations are elegant and some are a little forced. Some axioms are widely accepted and others are more controversial. These axioms are used widely and we present them divided into groups.

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[4]Also known as “game in characteristic function form.”

3.1. Efficiency

Values are used to find a fair allocation of benefits or a distribution of costs. Focusing on the case with externalities, we look for a similar tool, and therefore, it is useful to first clarify what precisely needs to be shared.

Axiom 3.1 (Efficiency)**.** For all $w \in G$:

$$\sum_{i \in N} \varphi_i(w) = w(N, \{N\}).$$

Efficiency simply states that we are looking at values that distribute the value of the grand coalition and only the value of the grand coalition. This axiom is an uncontroversial generalisation, as this is a direct translation of one of Shapley's [17] axioms into the partition function form setting. Since the value of the grand coalition is not subject to externalities, it is well-defined in partition function form games, too.

On the other hand, the efficiency axiom presupposes that players will want to form the grand coalition. This is a justified assumption if the game is cohesive, but cohesiveness has nothing to do with externalities. Even in games in characteristic function form, there may be legal or personal reasons for why players may not wish to cooperate.

3.2. Anonymity

While efficiency fixes what should be allocated, the following axioms discuss how it should be allocated. The first property states that all of the players are equal.

Axiom 3.2 (Anonymity)**.** For every $\theta \in S_n$ and $w \in G$:

$$\varphi(\theta \cdot w) = \theta \cdot \varphi(w)$$

where the game $\theta \cdot w$ is defined as $(\theta \cdot w)(S, Q) = w[\theta^{-1}(S, Q)]$.

Anonymity means that player's payoffs do not depend on their names and are only derived from their influence on the worth of the coalitions. This is natural for those living in democracies and rules out biased allocations, such as in dictatorships.

On the other hand, one can define an anonymity axiom with respect to the embedded coalitions EC. According to the next axiom, what are worthwhile are the worths of different embedded coalitions and not which embedded coalitions correspond to those worths.

To this end, let $S \subseteq N$ and θ_S a bijection on $\{(T, Q) \in EC \mid T = S\}$. For each $w \in G$, let $\theta_S \cdot w$ be the game defined by

$$(\theta_S \cdot w)(T, Q) = \begin{cases} w(T, Q) & \text{if } T \neq S \\ w(\theta_S(S, Q)) & \text{if } T = S \end{cases}.$$

Axiom 3.3 (Embedded coalition anonymity)**.** Let $S \subseteq N$. Given a bijection θ_S on $\{(T, Q) \in EC \mid T = S\}$ for all $w \in G$ and $i \in N$, it holds that

$$\varphi_i(\theta_S \cdot w) = \varphi_i(w).$$

The next axiom strengthens the anonymity axiom by requiring that the payoff of a player should not change after permutations in the set of players in $N \backslash S$, for any embedded coalition structure (S, Q).

Formally, given an embedded coalition (S, Q), we denote by $\theta_{S,Q}Q$ a new partition such that $S \in \theta_{S,Q}Q$, and the other coalitions result from a permutation of the set $N \backslash S$ applied to Q_{-S}.[5]

[5]That is, in the partition $\theta_{S,Q}Q$, the players in $N \backslash S$ are reorganized in sets whose size distribution is the same as in Q_{-S}.

Now, given the permutation $\theta_{S,Q}$, the permutation game $\theta_{S,Q} \cdot w$ is defined by $(\theta_{S,Q} \cdot w)(S,Q) = w(S, \theta_{S,Q}Q)$, $(\theta_{S,Q} \cdot w)(S, \theta_{S,Q}Q) = w(S,Q)$ and $(\theta_{S,Q} \cdot w)(T,P) = w(T,P)$

$$\forall (T,P) \in EC \backslash \{(S,Q), (S, \theta_{S,Q}Q)\}.$$

Axiom 3.4 (Strong anonymity)**.** A solution $\varphi : G \to \mathbb{R}^n$ satisfies the strong anonymity axiom[6] if

a) for every $\theta \in S_n$, $\varphi(\theta \cdot w) = \theta \cdot \varphi(w)$

b) for every $(S,Q) \in EC$ and every permutation $\theta_{S,Q}$,

$$\varphi\left(\theta_{S,Q} \cdot w\right) = \varphi(w).$$

The strong anonymity axiom, naturally, implies anonymity and imposes, in addition to equal treatment of individual players, the equal treatment of "externalities" generated by players in a given embedded coalition structure. Exchanging the names of the players inducing the same externality does not affect the payoff of any player.

3.3. Symmetry

The next property, symmetry, deals with players who can be freely exchanged. For its definition, we need some additional notation.

Let $\theta \in S_n$ and $i, j \in N$. We denote by $\theta^{ij} \in S_n$ the permutation that exchanges players i and j: $\theta^{ij}(i) = j$, $\theta^{ij}(j) = i$, and $\theta^{ij}(k) = k$ for all $k \notin \{i, j\}$.

Unlike anonymity, that looks at different players at the same position, symmetry deals with players in different positions but that have identical roles in the game.

[6]Other authors use the term 'strong symmetry' instead of 'strong anonymity' (e.g., Macho-Stadler et al. [8]).

Definition 3.5 (Symmetric players)**.** Given a game $w \in G$, players i and j are symmetric in w if $w(S, Q) = w\left[\theta^{ij}(S), \theta^{ij}(Q)\right]$.

Since symmetric players have exchangeable roles in the game, it is natural to require that they have the same value, too.

Axiom 3.6 (Symmetry)**.** Given a game $w \in G$ and symmetric players $i; j \in N$, then

$$\varphi_i(w) = \varphi_j(w).$$

For some authors, symmetry is also known as the Equal Treatment Property. Unlike anonymity that addresses the labelling of the players, symmetry compares the values of different players. Incidentally, exchanging these players has no effect on the game, so this property could be seen as a weaker version of anonymity restricted to symmetric players.

3.4. Carriers and null payers

The following set of axioms are weak conditions only describing who should, and who should not, get a share (based on marginal contributions of players). These axioms establish the difference between important and unimportant, ad absurdum, useless players. The fair share of useless players, who contribute nothing, is zero. Does anything qualify as something? The answer is less obvious than one would think.

Here, for any $Q, \overline{Q} \in PT$, we define:

$$Q \wedge \overline{Q} = \left\{ S \cap \overline{S} \mid S \in Q, \overline{S} \in \overline{Q}, S \cap \overline{S} \neq \varnothing \right\}.$$

Definition 3.7 (Carrier set)**.** Given a game $w \in G$, the set $\overline{S}$ is a carrier of w if for any embedded coalition $(S, Q) \in EC$,

$$w(S, Q) = w\left(S \cap \overline{S}, Q \wedge \{\overline{S}, N \backslash \overline{S}\}\right).$$

In other words, when we want to determine the worth of a coalition, the only thing that matters is who the members from the carrier $\overline{S}$ are and how those members are partitioned. However, note that the way players outside of the carrier are partitioned, may influence $w(S, Q)$, and therefore, they may influence the game via the externalities.

Axiom 3.8 (Carrier). For all $w \in G$ and all $S \in 2^N \backslash \{\varnothing\}$, if $\overline{S}$ is a carrier of w, then

$$\sum_{i \in \overline{S}} \varphi_i(w) = w(N, \{N\}).$$

The previous axiom states that the worth of the grand coalition must be divided among the members of a carrier. Since only the carrier generates value in this game, and its members share the entire payoff, those outside of the carrier, the players who contribute nothing, also do not get anything.

Now we turn to players who contribute nothing to the game. Such players are sometimes referred to as null players and sometimes as dummies; the literature on partition function form games uses the two terms interchangeably. Null players are players who have no influence on the game at all in the following sense:

Definition 3.9 (Null player). Let $w \in G$ and $i \in N$. Player i is a null player in the game w, if for all $P \in PT(N_{-i})$ and all $S \in P$,

$$w(S, P \cup \{\{i\}\}) = w(S_{+i}, (P_{-S}) \cup \{S_{+i}\}).$$

Once the definition of a null player has been clarified, we can give the nullity axiom which states that players who contribute nothing, do not get anything.

Axiom 3.10 (Null player property). If $i \in N$ is a null player in the game $w \in G$, then $\varphi_i(w) = 0$.

The following concept, null player in the strong sense, is originally introduced by Bolger [2] as dummy; others refer to it as weak dummy player, linking it to the weak dummy axiom (weak null player axiom). In this work, we prefer to follow de Clippel and Serrano [3] and call it a null player in the strong sense.

Definition 3.11 (Null player in the strong sense). $i \in N$ is called a null player in the strong sense in the game w if

$$w(S, Q) = w\left(S_{-i}, \alpha_{iT}(Q)\right) \tag{3.1}$$

for each embedded coalition (S, Q) such that $i \in S$ and each $T \in Q_{-S}$.[7]

Thus, the worth of a coalition is not changed if a null player in the strong sense is transferred to another coalition in the partition.

Axiom 3.12 (Weak null player property). Let $i \in N$ and let $w \in G$. If i is a null player in the strong sense in w, then $\varphi_i(w) = 0$.

Notice that for a player to be a null player in the strong sense, it must be the case that he alone receives zero for any organization of the other players and has no effect on the worth of any coalition S. The weak null player property only makes sure that a player with absolutely no influence on the gains that any coalition can obtain, should not receive, nor pay anything.

The following property considers null players with slightly weaker conditions where only the aggregate of the equations given by (3.1) with $S \ni i$ is considered, while for some partitions, a null player may actually contribute something. There are alternative names for this concept, too. Hu and Yang [5] simply call

[7] This definition of a null player agrees with the definition presented in Bolger [2] and Macho-Stadler et al. [8], and it is different than the one considered in Myerson [9] and Albizuri et al. [1].

such players dummy as opposed to Bolger's "weak dummy" (in our terminology, null player in the strong sense).

Definition 3.13 (Null player in the weak sense). $i \in N$ is called a null player in the weak sense in the game w if

$$\sum_{T \in Q, T \not\ni i} \left[w(Q^i, Q) - w\left(Q^i_{-i}, \alpha_{iT}(Q)\right)\right] = 0.$$

Axiom 3.14 (Strong null player property). Let $i \in N$ and let $w \in G$. If i is a null player in the weak sense in w, then $\varphi_i(w) = 0$.

Besides the previous versions of null player properties, one can think about other alternatives to deal with the issue of defining players that are not important according to a principle of "average marginal contributions."

For instance, one way of defining an average marginal contribution is the following: for any $(S, Q) \in EC$ such that $S \neq \varnothing$ and any $w \in G$:

$$C^w(S,Q) = \frac{1}{s} \sum_{j \in S} \left[w(S,Q) - w(S_{-j}, \{S_{-j}, \{j\}\} \cup Q_{-S}) \right]. \tag{3.2}$$

This amount, $C^w(S,Q)$, captures the notion of an average marginal contribution of a member of S (given the coalition structure Q), with a specific restriction of the organization of players after player j leaves coalition S. In this respect, player j simply leaves S and then he/she acts alone, whereas the other coalitions remain unchanged.

However, it is not the only way to consider this. For every $(S,Q) \in EC$ such that $S \neq \varnothing$ and every $w \in G$:

$$A^w(S,Q) = \frac{1}{s} \sum_{j \in S} \frac{1}{|P(S_{-j})|} \sum_{Q' \in P(S_{-j})} \left[w(S,Q) - w(S_{-j}, Q') \right]. \tag{3.3}$$

As expected, we can interpret $A^w(S, Q)$ as an average marginal contribution of a member of S, given the coalition structure Q. In the computation of $w(S, Q) - w(S_{-j}, Q')$, notice that Q' is a partition of N that reflects the dynamics of coalition formation (among players in $N \backslash S_{-j}$) once player j has left the main coalition S. Even when players in $N \backslash S_{-j}$ form a coalitional structure completely different from the original one (Q), the essence of the idea of marginal contribution remains. The reason relies on the fact that other agents' behavior could be a consequence of the departure of j from S; thus, this possibility must be taken into account.

For both definitions ((3.2) and (3.3)), note that if S contains only one player, then $A^w(S, Q) = C^w(S, Q) = w(S, Q)$.

Once we have considered this notion of average marginal contributions, we can reformulate a null player property in the following sense:

Definition 3.15. $i \in N$ is called a null player in the game w if $\overline{A}^w(S, Q) = 0$ for every $(S, Q) \in EC$ such that $i \in S$.

Now, we give the following version of a nullity axiom for games with externalities:

Axiom 3.16 ($C-$Average nullity). If $i \in N$ is a player for which $C^w(S, Q) = 0$ for every $(S, Q) \in EC$ such that $i \in S$, then $\varphi_i(w) = 0$.

Axiom 3.17 ($A-$Average nullity). If $i \in N$ is a player for which $A^w(S, Q) = 0$ for every $(S, Q) \in EC$ such that $i \in S$, then $\varphi_i(w) = 0$.

3.5. *Oligarchy*

The oligarchy axiom, introduced by Albizuri et al. [1], states that there is a specific coalition (the oligarchic coalition) with whom the worth of the grand coalition $(N, \{N\})$ is obtained. Furthermore, in these games, this specific coalition is needed to obtain a non null worth. This axiom requires the worth of N to be divided among players that belong to that coalition.

Axiom 3.18 (Oligarchy)**.** Let $w \in G$. If there exists $T \subseteq N$ such that

$$w(S, Q) = \begin{cases} w(N, \{N\}) & \text{if } S \supseteq T \\ 0 & \text{otherwise} \end{cases}$$

then, $\sum\limits_{i \in T} \varphi_i(w) = w(N, \{N\})$.

3.6. *Additivity and linearity*

While the previous axioms focused on the comparison of players within a given game, the following axioms tell us how values in different games are related and these properties are important in the characterization proofs.

Axiom 3.19 (Additivity)**.** For all $w_1, w_2 \in G$:

$$\varphi(w_1 + w_2) = \varphi(w_1) + \varphi(w_2).$$

The axiom of additivity means that when a group of players shares the benefits (or costs) stemming from two different issues, how much each player obtains does not depend on whether they consider the two issues together, or one by one. Hence, the agenda does not affect the final outcome.

Axiom 3.20 (Linearity)**.** For all $w_1, w_2 \in G$ and $c \in \mathbb{R}$:

$$\varphi(w_1 + w_2) = \varphi(w_1) + \varphi(w_2) \quad \text{and} \quad \varphi(cw_1) = c\varphi(w_1).$$

Linearity clearly implies additivity: it generalises it to arbitrary combinations. Also, sharing does not depend on the unit used to measure the benefits.

3.7. Marginality

In games with no externalities, the computation of the Shapley value is a weighted average of marginal contributions. However, marginality is not mentioned in any of the original axioms. It is Young [18] who first uses this property as one of the axioms for partition function form games, Bolger [2] uses it first and presents an extension of the Shapley value using such an axiomatization.

Axiom 3.21 (Marginality)**.** Let $w_1, w_2 \in G$. If for each partition $Q \in PT$,

$$\begin{aligned} & \sum_{S \in Q, S \neq Q^i} \left[w_1(Q^i, Q) - w_1\left(Q^i_{-i}, \alpha_{iS}(Q)\right)\right] \\ = & \sum_{S \in Q, S \neq Q^i} \left[w_2(Q^i, Q) - w_2\left(Q^i_{-i}, \alpha_{iS}(Q)\right)\right], \end{aligned}$$

then $\varphi(w_1) = \varphi(w_2)$.

Following the analysis of Bolger [2], consider an embedded coalition $(S_{-i}, \alpha_{iT}(Q))$ obtained from (S, Q) by a move for player i (from S to $T \in Q_{-S}$). Such a move is called a pivot move if S wins with respect to (S, Q) and S_{-i} loses with respect to $(S_{-i}, \alpha_{iT}(Q))$. The marginality axiom states that for simple games (the worth of any coalition is either one or zero), a player i obtains the same payoff in two games w_1 and w_2 if he has the same number of pivot moves in both games.

3.8. Similar influence

There is an axiom that addresses the issue that similar environments should lead to similar payoffs for the players. To understand the motivation for this axiom, consider the following example.

Example 3.22. Take $N = \{1, 2, 3\}$ and consider the games w_1 and w_2 defined by the worths:

$(S, Q) \in EC$	$w_1(S, Q)$	$w_2(S, Q)$
$\{1\}, \{2\}, \{3\}$	0 0 0	1 0 0
$\{1, 2\}, \{3\}$	0 0	0 0
$\{1, 3\}, \{2\}$	0 0	0 0
$\{2, 3\}, \{1\}$	0 1	0 0
$\{1, 2, 3\}$	1	1

The two games are very similar. In both, only player 1 can produce some benefits alone. The only difference is that in the first game players 2 and 3 should be together for the benefits to player 1 to be realized, while in the second game, players 2 and 3 should be separated. The payoff of players 2 and 3 (hence, the payoff of player 1, as well) can differ very much depending on whether they influence the worth of player 1 by staying together, or by separating. However, we think that this influence is very similar and therefore, it is sensible that players 2 and 3 should receive the same payoff in both games. This idea leads to the next axiom.

To introduce the similar influence axiom, we first define the notion of "similar influence."

Definition 3.23 (Players with similar influence)**.** We say that a pair of players $\{i, j\} \subseteq N$, $i \neq j$, has similar influence in games $w_1, w_2 \in G$ if $w_1(S, Q) = w_2(S, Q)$ for all $(S, Q) \in EC \backslash \{(T, P), (T, P')\}$, $w_1(T, P) = w_2(T, P')$, and $w_1(T, P') =$

$w_2(T, P)$; where the only difference between the partitions P and P' is that $\{i\}, \{j\} \in P_{-T}$, while $\{i, j\} \in P'_{-T}$.

Axiom 3.24 (Similar influence)**.** For any two games $w_1, w_2 \in G$ and for any pair of players $\{i, j\} \subseteq N$ that has similar influence in those games,

$$\varphi_i(w_1) = \varphi_i(w_2) \quad \text{and} \quad \varphi_j(w_1) = \varphi_j(w_2).$$

4. Axiomatic solutions

To study solutions for the problem of the fair distribution of the surplus generated by a group of people who are willing to cooperate with one another, one can take a normative approach (called axiomatic solutions or values). Given the coalitions and their sets of feasible payoffs as primitives, the question tackled is the identification of final payoffs awarded to each agent. An axiomatic solution provides us with a well-defined formula to apply those principles, unless these decision makers drastically change their opinion on the fundamental principles once the implied allocation is presented. In this section, we offer several values, starting with the first proposal in the literature.

Myerson [9] proceeds axiomatically and proposes a value that extends the well-known Shapley value, which is defined for games with no externalities. The three axioms that uniquely characterize the Myerson's extension are additivity, anonymity, and a carrier axiom requiring that the surplus is shared only among the members of the carrier.

Theorem 4.1 (Myerson [9])**.** *There exists a unique solution φ^M that satisfies the axioms of anonymity, carrier, and additivity. More-*

over, the function φ^M is given as:

$$\varphi_i^M(w) = \sum_{(S,Q)\in EC} (-1)^{q-1}(q-1)! \left(\frac{1}{n} - \sum_{T\in Q_{-S}, i\notin T} \frac{1}{(q-1)(n-t)} \right) w(S,Q).$$

Modern axiomatizations split the carrier axiom into two properties: the first, efficiency, is hardly controversial, but the extension of the second, null player property, to partition function form games, is sensitive to the behavioural assumptions used. A null player is a player who has no value by himself, and contributes nothing to other coalitions. Both depend on what the rest of the players do, and variations of these assumptions lead to remarkably different values.

The axiom of marginality is a powerful axiom for solutions in games with no externalities. For partition function form games, it is not sufficient with efficiency and anonymity alone to fully characterize a solution[8]. Bolger [2] needs both additivity and the weak null player property to characterize his value.

Theorem 4.2 (Bolger [2]). *There is a unique value $\varphi^B : G \to \mathbb{R}^n$ satisfying the weak null player property, anonymity, additivity, marginality, and efficiency.*

Unfortunately, there is no closed form expression for this value, but it is computed in a recursive way. As an example, consider the following case:

[8] For instance, Clippel and Serrano [3] present two values that satisfy efficiency, anonymity, and a weaker version of marginality; while Sánchez-Pérez [15] provides an analysis of a solution that can be represented as a linear combination of marginal contributions of players.

Example 4.3. For $N = \{i, j, k\}$, the Bolger value for player i is:

$$\begin{aligned}\varphi_i^B(w) = &\frac{w(N, \{N\})}{3} \\ &+ \frac{1}{12}\begin{bmatrix} 2w(\{i\}, \{\{i\}, \{j\}, \{k\}\}) - w(\{j\}, \{\{i\}, \{j\}, \{k\}\}) \\ -w(\{k\}, \{\{i\}, \{j\}, \{k\}\}) \end{bmatrix} \\ &+ \frac{1}{12}\begin{bmatrix} 2w(\{i\}, \{\{i\}, \{j, k\}\}) - w(\{j\}, \{\{j\}, \{i, k\}\}) \\ -w(\{k\}, \{\{k\}, \{i, j\}\}) \end{bmatrix} \\ &+ \frac{1}{6}\begin{bmatrix} w(\{i, j\}, \{\{k\}, \{i, j\}\}) + w(\{i, k\}, \{\{j\}, \{i, k\}\}) \\ 2w(\{j, k\}, \{\{i\}, \{j, k\}\}) \end{bmatrix}.\end{aligned}$$

In 2005, Albizuri et al. explore a different way to obtain a generalization of the Shapley value. They obtain, by means of an axiomatic approach, another extension for the Shapley value for games in partition function form. In fact, such a solution can be obtained as the Shapley value of an expected game.

Theorem 4.4 (Albizuri et al. [1]). *There exists a unique solution $\varphi^{AAR} : G \to \mathbb{R}^n$ that satisfies the axioms of efficiency, additivity, anonymity, embedded coalition, and the oligarchy axiom. Moreover, such a solution is given by*

$$\varphi_i^{AAR}(w) = \sum_{\substack{(S,Q)\in EC \\ S\ni i}} \frac{(s-1)!(n-s)!}{n!P(S,N)} w(S,Q) - \sum_{\substack{(S,Q)\in EC \\ S\not\ni i}} \frac{s!(n-s-1)!}{n!P(S,N)} w(S,Q) \tag{4.4}$$

where $P(S, N) = |\{(T, Q) \in EC \mid T = S\}|$.

As mentioned before, the solution given in (4.4) can also be computed as

$$\varphi_i^{AAR}(w) = Sh_i(v^w) \tag{4.5}$$

for all $i \in N$ and all $w \in G$, where v^w is a game with no externalities (associated to w) defined as

$$v^w(S) = \frac{1}{P(S,N)} \sum_{\substack{(T,Q)\in EC \\ T=S}} w(T,Q) \quad \text{for all } S \subseteq N$$

and Sh denotes the Shapley value defined by

$$Sh_i(v^w) = \sum_{\{S\subseteq N: i\notin S\}} \frac{s!(n-s-1)!}{n!} \left[v^w(S\cup\{i\}) - v^w(S)\right].$$

This approach of constructing a value through the Shapley value of a game with no externalities generated from a game in partition fuction form, is called "the average approach."(See Macho-Stadler et al. [8]). The average approach does not specify a particular complementary behaviour, but rather, it assumes that the coalitional value is calculated as a combination (an average) of the values of this coalition when embedded in a partition. Once these values are established for each coalition, they yield a game with no externalities.

In fact, these authors show the relationship between the average approach and the strong anonymity axiom and describe the precise restrictions stemming from the weak null player property. They basically provide an entire family of solutions that satisfies the axioms of linearity, strong anonymity and the weak null player property.

In the following theorem, one more axiom, similar influence, is added to ensure uniqueness.

Theorem 4.5 (Macho-Stadler et al. [8]). *There is a unique solution φ^{MPW} that satisfies linearity, strong anonymity, the weak null* *player property and similar influence; and it is given by*

$$\varphi_i^{MPW}(w) = \sum_{\substack{(S,Q)\in EC\\ S\ni i}} \frac{(s-1)! \prod_{T\in Q_{-S}} (t-1)!}{n!} w(S,Q) - \sum_{\substack{(S,Q)\in EC\\ S\not\ni i}} \frac{s! \prod_{T\in Q_{-S}} (t-1)!}{(n-s)n!} w(S,Q).$$

In the same sense (generalization of the Shapley value), Pham Do and Norde [11] provide an extension for games in partition function form, which is generated with the Shapley value of a game with no externalities.

Theorem 4.6 (Pham Do and Norde [11])**.** *There exists a unique solution φ^{PN} that satisfies the axioms of additivity, symmetry, efficiency and null player property. This solution is given by:*

$$\varphi_i^{PN}(w) = Sh_i(v^w) \tag{4.6}$$

for all $i \in N$ and all $w \in G$, where v^w is a game with no externalities defined by

$$v^w(S) = w(S, \{S, [N\backslash S]\}) \quad \textit{for all } S \subseteq N.$$

Notice that the solution given by (4.6) depends only on a function that gives the worth of each coalition, independently of the partition structure. Of course, this completely ignores externalities; in fact, it only takes a small part of the partition function into account.

The following solution is characterized using standard axioms, such as efficiency, symmetry, additivity, and a property that makes sure useless players get nothing. The same axioms and principles were used to characterize the solution φ^{PN} and, yet, the two values are drastically different. While the value of Pham Do and Norde [11] is based on a characteristic function generated under the assumption that players break all links of cooperation and end up as singletons when a coalition forms, Hu and Yang [5] take just the opposite approach and consider a property where the remaining players do not react at all. This shows just how sensitive such values are to the details of the definition of an axiom.

Theorem 4.7 (Hu and Yang [5])**.** *There exists a unique solution φ^{HY} satisfying the axioms of efficiency, symmetry, additivity and the strong null player property. Such solution is given by*

$$\varphi_i^{HY}(w) = \sum_{\substack{T \in Q_{-S} \\ Q \in EC}} \frac{(s-1)!(n-s)!}{|EC|\, n!} \left[w(S,Q) - w\left(S_{-i}, \alpha_{iT}(Q)\right)\right].$$

As an alternative to the Shapley value for games with no externalities, Nowak and Radzik [10] developed another coalitional value (called the Solidarity value) for games without externalities. With this, the rule followed for sharing the benefits among the players is less competitive[9] than the rule used in the Shapley value and it reflects some social behavior of players in coalitions. Such a value is based on the assumption that if a coalition S forms, then the players who contribution to S more than the average marginal contribution, support, in some sense, their weaker partners in S.

Those ideas were adapted to partition function form games, and as final examples of full characterizations of solutions, we

[9]It is very easy to find real-life examples where the groups formed seek to protect their weaker members by giving them a share of the gains obtained by the group.

present two extensions of the Solidarity value that allow us to enrich the analysis from a theoretical point of view and from a practical perspective. At the theoretical level it would be possible to formulate (considering the influence of coalition structures) new useful axioms to characterize solutions under the principle of average marginal contribution. In a practical sense, such an extension responds to situations in which, in addition to externalities, social and collective interests predominate.

Theorem 4.8 (Hernández-Lamoneda et al. [4]). *The solution* $\varphi^{HSS} : G \to \mathbb{R}^n$ *given by*

$$\varphi_i^{HSS}(w) = \sum_{\substack{(S,Q)\in EC \\ i\in S}} \frac{(n-s)!(s-1)!}{n!} \cdot C^w(S,Q) \qquad (4.7)$$

for every $i \in N$ *and every* $w \in G$*; is the unique solution satisfying the axioms of linearity, anonymity, efficiency and* $C-$*average nullity.*

Theorem 4.9 (Rodríguez-Segura and Sánchez-Pérez [12]). *The solution* $\varphi^{RS} : G \to \mathbb{R}^n$ *given by*

$$\varphi_i^{RS}(w) = \sum_{\substack{(S,Q)\in EC \\ i\in S}} \frac{(n-s)!(s-1)!}{n! \cdot |P(S)|} \cdot A^w(S,Q) \qquad (4.8)$$

for each $i \in N$ *and each* $w \in G$*, is the unique solution satisfying the axioms of linearity, anonymity, efficiency and* $A-$*average nullity.*

Recall that the amounts $C^w(S,Q)$ and $A^w(S,Q)$ are computed via (3.2) and (3.3), respectively.

Some examples of scenarios where it would be appropriate to apply a Solidarity value (for environments where externalities are present) are: humanitarian aid to territories affected by a natural disaster, where the formation of blocks exogenous to them could damage support for the victims; situations in which the distribution of some social benefit or resource indispensable for the quality of life tends to favor certain alliances for political convenience; family saving funds where the way in which investors are organized outside of an alliance affects the profits of the same and where the savers assume a position of solidarity by not expecting that some of their relatives lose their patrimony.

Applying some extensions of the Shapley value to situations similar to those mentioned above may not be the best alternative since agents receive payoffs according to their productivity and so, some of them may be excluded from any benefit or vital resource, whereas, the solution proposed in this article is even more robust, as it considers all possible ways in which external agents can organize themselves and assumes a social behavior of the agents.

5. Families of solutions

In this section, we provide four classes of axiomatic solutions. The families of solutions are the following: i) the class of all linear anonymous solutions; ii) the class of linear, anonymous and efficient solutions; iii) the class of solutions that satisfies the axioms of linearity, anonymity, efficiency and weak null player property; and iv) a class of solutions satisfying linearity, strong anonymity and the weak null player property.

In order to present such families of solutions, it is necessary to introduce some definitions and notation related to integer partitions.

5.1. Partitions of integers

A partition of a nonnegative integer is a way of expressing it as the unordered sum of other positive integers, and it is often written in tuple notation. Formally,

Definition 5.1. $\lambda = [\lambda_1, \lambda_2, ..., \lambda_l]$ is a partition of n (denoted as $\lambda \vdash n$) iff $\lambda_1, \lambda_2, ..., \lambda_l$ are positive integers and $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. Two partitions, which only differ in the order of their elements, are considered to be the same partition.

The set of all partitions of n will be denoted by $\Pi(n)$.

For example, the partitions of $n = 4$ are $[1, 1, 1, 1]$, $[2, 1, 1]$, $[2, 2]$, $[3, 1]$ and $[4]$. We will abbreviate this notation by dropping the commas, so $[2, 1, 1]$ becomes $[211]$.

If $Q \in PT$, there is a unique partition $\lambda_Q \vdash n$ associated with Q where the elements of λ_Q are exactly the cardinalities of the elements of Q. In other words, if $Q = \{S_1, S_2, ..., S_m\} \in PT$, then $\lambda_Q = [s_1, s_2, ..., s_m]$.

For a given $\lambda \vdash n$, we represent by λ^o the set of numbers determined by the λ_i's, and by $m^\lambda_{\lambda_j}$ the multiplicity of λ_j in λ. So, if $\lambda = [4, 4, 2, 1, 1, 1]$, then $\lambda^o = \{1, 2, 4\}$, $m^\lambda_1 = 3$, $m^\lambda_2 = 1$ and $m^\lambda_4 = 2$. By convention, $m^\lambda_0 = 1$ for every $\lambda \in \Pi(n)$.

Additionally, if $[\lambda_1, \lambda_2, ..., \lambda_l] \vdash n$, for $l > k \geq 1$ we define $[\lambda_1, \lambda_2, ..., \lambda_l] - [\lambda_1, \lambda_2, ..., \lambda_k] = [\lambda_{k+1}, \lambda_{k+2}, ..., \lambda_l]$. For example, $[4, 3, 2, 1, 1, 1] - [3, 1, 1] = [4, 2, 1]$.

If $\lambda \in \Pi(n)$ and $\lambda' \in \Pi(m)$, then we can form a partition $\lambda + \lambda'$ in $\Pi(n + m)$ by combining all elements of such partitions. For example, $[4, 3, 2, 1, 1, 1] + [3, 1, 1] = [4, 3, 3, 2, 1, 1, 1, 1, 1]$.

Finally, for $\lambda \in \Pi(n)$ and $z, r \in \lambda^\circ$ ($r \neq 1$), we define $\lambda^r_z = \lambda - [r, z] + [r - 1, z + 1]$.

Now, we need to define certain sets which are used in the sequel.

Definition 5.2. For a positive integer n, let A_n be a set of pairs associated with all partitions $\lambda \vdash n$ and its elements, i.e.,

$$A_n = \{(\lambda, s) \mid \lambda \in \Pi(n), s \in \lambda^\circ\}.$$

Also, define the set of triples

$$B_n = \{(\lambda, s, t) \mid \lambda \in \Pi(n) \backslash \{[n]\}, s \in \lambda^\circ, t \in (\lambda - [s])^\circ\}$$

and similarly, let E_n be a set of pairs

$$E_n = \{(\lambda, s) \mid \lambda \in \Pi(n), s \in \lambda^\circ \backslash \{1, n\}\}.$$

Example 5.3. If $n = 4$, then

$$\begin{aligned} A_4 = \{&([1111], 1), ([211], 1), ([211], 2), ([22], 2), \\ &([31], 1), ([31], 3), ([4], 4)\}, \\ B_4 = \{&([1111], 1, 1), ([211], 1, 1), ([211], 1, 2), \\ &([211], 2, 1), ([22], 2, 2), ([31], 1, 3), ([31], 3, 1)\} \end{aligned}$$

and

$$E_4 = \{([211], 2), ([22], 2), ([31], 3)\}.$$

5.2. *Characterizations*

Theorem 5.4 (Hernández-Lamoneda et al. [4])**.** *If the solution $\varphi : G \to \mathbb{R}^n$ satisfies the linearity and anonymity axioms, then there exist unique real numbers $\{\alpha_{(\lambda,s)} \mid (\lambda, s) \in A_n\} \cup \{\beta_{(\lambda,s,t)} \mid (\lambda, s, t) \in B_n\}$ such that*

$$\begin{aligned} \varphi_i(w) = &\sum_{(\lambda,s)\in A_n} \sum_{\substack{(S,Q)\in EC \\ S\ni i, |S|=s \\ \lambda_Q=\lambda}} \alpha_{(\lambda,s)} w(S, Q) \\ &+ \sum_{(\lambda,s,t)\in B_n} \sum_{\substack{(S,Q)\in EC \\ S\not\ni i, |S|=s \\ \lambda_Q=\lambda, |Q^i|=t}} \beta_{(\lambda,s,t)} w(S, Q). \end{aligned} \tag{5.9}$$

Conversely, for any real numbers $\{\alpha_{(\lambda,s)} \mid (\lambda, s) \in A_n\} \cup \{\beta_{(\lambda,s,t)} \mid (\lambda, s, t) \in B_n\}$, *the solution given by (5.9) is linear and symmetric.*

Remark 5.5. It is worthwhile to mention that a complete analysis of linear anonymous solutions for partition function form games is treated in Sánchez-Pérez [15] and Sánchez-Pérez [16], where representation theory techniques are applied to the study of such solutions.

Theorem 5.6 (Hernández-Lamoneda et al. [4]). *The solution* $\varphi : G \to \mathbb{R}^n$ *satisfies linearity, anonymity and efficiency axioms if and only if it is of the form*

$$\varphi_i(w) = \frac{w(N, \{N\})}{n} + \sum_{(\lambda,s,t) \in B_n} \beta_{(\lambda,s,t)} \left[\sum_{\substack{(S,Q) \in EC \\ S \ni i, |S|=s \\ \lambda_Q = \lambda}} \sum_{\substack{T \in Q_{-S} \\ |T|=t}} t w(S, Q) - \sum_{\substack{(S,Q) \in EC \\ S \not\ni i, |S|=s \\ \lambda_Q = \lambda, \left|Q^i\right|=t}} s w(S, Q) \right] \tag{5.10}$$

for some real numbers $\{\beta_{(\lambda,s,t)} \mid (\lambda, s, t) \in B_n\}$. *Moreover, such representation is unique.*

The expression given by (5.10) can be interpreted as follows. We start by giving $\frac{w(N,\{N\})}{n}$ to each player. For each $(S, Q) \in EC$, we keep going with one transfer from T to S for each $T \in Q_{-S}$; every player in T pays $s\beta_{(\lambda_Q,s,t)} w(S, Q)$ and every player in S receives $t\beta_{(\lambda_Q,s,t)} w(S, Q)$. At the end, the player i has the amount $\varphi_i(w)$ given by the above formula.

Theorem 5.7 (Sánchez-Pérez [13]). *The solution* $\varphi : G \to \mathbb{R}^n$ *satisfies the axioms of linearity, anonymity, efficiency and weak* *null player property if and only if it is of the form (5.10) for real numbers* $\{\beta_{(\lambda,s,t)} \mid (\lambda, s, t) \in B_n\}$ *such that*

i)

$$\beta_{([n-1,1],n-1,1)} = \frac{1}{n(n-1)} \tag{5.11}$$

and

ii) *for every* $(\lambda, r) \in E_n$*:*

$$\left(1 - m_r^\lambda\right) \left[r\beta_{(\lambda,r,r)} - (r-1)\beta_{(\lambda_r^r,r-1,r+1)}\right] = \sum_{\substack{z\in\lambda^\circ\cup\{0\} \\ z\neq r}} \left[zm_z^\lambda\beta_{(\lambda,r,z)} - (r-1)m_z^\lambda\beta_{(\lambda_z^r,r-1,z+1)}\right]. \tag{5.12}$$

Remark 5.8. The intuition behind the characterization derived in Theorem 5.7 has an interpretation as a bargaining process:

1. We allocate $\frac{w(N,\{N\})}{n}$ to each player.
2. For each $(S, Q) \in EC$ and each $T \in Q_{-S}$, we keep going with one transfer from T to S:

   i) Every player in S receives (from every player in T) the fraction $\beta_{(\lambda_Q,s,t)}$ of the worth $w(S, Q)$:

   $$t\beta_{(\lambda_Q,s,t)}w(S, Q).$$

   ii) Every player in T pays (to every player in S) a fraction $\beta_{(\lambda_Q,s,t)}$ of the worth $w(S, Q)$:

   $$s\beta_{(\lambda_Q,s,t)}w(S, Q).$$

3. Finally, these transfers must satisfy $\beta_{([n-1,1],n-1,1)} = \frac{1}{n(n-1)}$ and

$$\sum_{T\in Q_{-S}} t\beta_{(\lambda_Q,s,t)} = \sum_{T\in Q_{-S}} (s-1)\,\beta_{(\lambda_{\alpha_{iT}(Q)},s-1,t+1)}$$

   for every $(S,Q) \in EC$ such that $S \ni i$, $s \neq 1$ and $s \neq n$. Here, $\beta_{(\lambda_{\alpha_{iT}(Q)},s-1,t+1)}$ represents the fraction of the worth $w\left(S_{-i}, \alpha_{iT}(Q)\right)$ that receives each player in S_{-i} from player i.

Remark 5.9. To gain more intuition, as noted by Sánchez-Pérez [14], it is not difficult to show that the solutions characterized in Theorem 5.7 can be written as a linear combination of marginal contributions. For games with no externalities, the marginal contribution of a player i within a coalition S is defined as the loss incurred by the other members of S if i leaves the group.

For partition function form games, this number could depend on the organization of the players not in S. It is natural, therefore, to define the marginal contribution of a player within each embedded coalition. We consider the general case where a player may join another coalition T after leaving S. For this purpose, we use the idea behind the definition of a null player in the strong sense.

Formally, let i be a player, let $(S,Q) \in EC$ such that $S \ni i$ and let $T \in Q_{-S}$. Then the marginal contribution of i to (S,Q) when i joins T is given by

$$MC_{i,(S,Q),T}(w) = w(S,Q) - w\left(S_{-i}, \alpha_{iT}(Q)\right). \tag{5.13}$$

Notice that under this concept of marginal contribution for partition function form games (given by (5.13)), a player $i \in N$ is a null player in the strong sense if $MC_{i,(S,Q),T}(w) = 0$ for every $(S,Q) \in EC$ such that $S \ni i$ and every $T \in Q_{-S}$.

Proposition 5.10 (Sánchez-Pérez [14]). *Every solution satisfying the axioms of linearity, anonymity, efficiency and weak null player property is a linear combination of marginal contributions:*

$$\begin{aligned}\varphi_i(w) \quad = \quad & \sum_{\substack{(S,Q)\in EC\\ S\ni i, s\neq 1}} \sum_{T\in Q_{-S}} (s-1)\,\beta_{(\lambda_{\alpha_{iT}(Q)},s-1,t+1)} MC_{i,(S,Q),T}(w) \\ & + \sum_{\substack{(S,Q)\in EC\\ S\ni i, s=1}} \sum_{T\in Q_{-S}} t\beta_{(\lambda_Q,s,t)} MC_{i,(S,Q),T}(w).\end{aligned}$$

As mentioned in Section 4, Macho-Stadler et al. [8] provide an approach for constructing a value through the Shapley value for a game with no externalities generated from a game in partition function form (known as "the average approach"). In an environment with externalities, the worth of a group of players is influenced by the way the outside players are organized. What worth should then be "assigned" to that group of players? An obvious candidate is to take an average of the different worths of this group for all the possible organizations of the other players. Repeating this process for all groups leads to an "average"game with no externalities. A focal candidate now for a value for the original game with externalities is the Shapley value for the average game.

More formally, the "average approach" consists of, first, constructing an average game v^w associated with the partition function game w by assigning to each coalition $S \subseteq N$ the average worth

$$v^w(S) = \sum_{(S,Q)\in EC} \gamma(S,Q)\cdot w(S,Q)$$

with $\sum_{(S,Q)\in EC} \gamma(S,Q) = 1$. The authors refer to $\gamma(S,Q)$ as the "weight" of the partition Q in the computation of the value of coalition $S \in Q$. Second, the average approach constructs a value

φ for the partition function game w by taking the Shapley value of the game v^w.

The following theorem shows the intimate relation between the average approach and strong anonymity, revealing an entire class of solutions satisfying the axioms of linearity, strong anonymity and the weak null player property.

Theorem 5.11 (Macho-Stadler et al. [8]). *Consider a solution φ that satisfies the axioms of linearity and the weak null player property. Then φ also satisfies strong anonymity if and only if it can be constructed through the average approach and with symmetric weights also satisfying*

$$\gamma(S,Q) = \sum_{T \in Q_{-S}} \gamma\left(S_{-i}, \alpha_{iT}(Q)\right)$$

for all $i \in S$ and for all $(S,Q) \in EC$ such that $s > 1$.

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IV. CAPACITY CHOICE IN A MIXED DUOPOLY AND THE SHADOW COST OF PUBLIC FUNDS

Jorge Fernández-Ruiz

Abstract

We study the effect of the existence of a shadow cost of public funds on the capacity choices of firms in a mixed duopoly in a market with product differentiation. We find that it affects the choice of the public firm when products are substitutes, but it affects neither this firm's choice when products are complements, nor the choice of the private firm.

1. Introduction

The idea that firms may hold excessive capacity for strategic reasons has been widely studied in the literature of pure private oligopolies, examples being Dixit [4], Brander and Spencer [2] and Horiba and Tsutsui [6]. This issue has also been analyzed in the context of mixed oligopolies, where private firms compete with public firms, as in, for instance, Wen and Sasaki [16], Nishimori and Ogawa [12], Lu and Poddar [9], Ogawa [13], and Barcena-Ruiz and Garzón [1]. Our paper contributes to this line of research by

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adding the shadow cost of public funds to the analysis of capacity choices in a mixed duopoly.

As a background for our work, let us mention the papers by Nishimori and Ogawa [12] and Ogawa [13][1]. Nishimori and Ogawa [12] study a mixed duopoly where firms first choose their capacities and then compete in quantities in a homogeneous product market. They show that while the private firm chooses to hold excess capacity, the public firm chooses under-capacity, instead. Ogawa [13] considers the same framework as Nishimori and Ogawa [12], but allowing for product differentiation, and finds that the result of the private firm continues to hold since it chooses over-capacity. But the public firm chooses over-capacity if products are complements, and under-capacity, if they are substitutes.

In this paper we extend the above models by considering the existence of a shadow cost of public funds. We assume, following Matsumura and Tomaru [11][2], that there is a shadow cost of public funds, or an excess burden of taxation of $\alpha \geq 0$. In this framework, the social value of public firms' profits is higher than that of private firms' profits because they help to decrease the deadweight loss of taxation.

We look for the changes that the existence of a shadow cost of public funds causes on capacity choices in a mixed duopoly. We find that it changes the capacity choice of the public firm when products are substitutes –but it does not change this choice when they are complements– and this does not affect the choices of

[1] See also Fernández-Ruiz [5], who considers the possibility that the private firm is (at least partially) owned by foreign investors.

[2] As Matsumura and Tomaru [11] mention, this approach is based on the work on contract theory by Laffont and Tirole [8], Olsen and Osmundsen [14] and Laffont and Pouyet [7], and is similar to the approach by Capuano and De Feo [3] and Matsumura and Tomaru [10] on mixed oligopolies.

the private firm. The rest of this paper is organized as follows. Section 2 presents the model. Section 3 develops the results and Section 4 concludes.

2. The model

We extend the model studied in Ogawa [13] that considers a mixed duopoly with differentiated products to incorporate the shadow costs of public funds.

There is a market where firm 1 (a private firm) and firm 2 (a public firm) produce differentiated products. Let q_i be firm i's output. The representative consumer has preferences that can be represented by the following utility function:

$$U(q_1, q_2) = a(q_1 + q_2) - \frac{(q_1^2 + 2bq_1q_2 + q_2^2)}{2}, \qquad (2.1)$$

which is quadratic, strictly concave and symmetric in q_1 and q_2, as in Ogawa [13] and Barcena-Ruiz and Garzón [1]. Maximization of this utility function yields the following inverse demand functions:

$$p_i = a - q_i - bq_j \quad i = 1, 2; j = 3 - i \qquad (2.2)$$

where p_i represents firm i's price. If $b \in (-1, 0)$, the products are complements and if $b \in (0, 1)$, they are substitutes. Using the utility function in (2.1) and the demand functions in (2.2), we obtain the following consumer surplus function:

$$CS = \frac{q_1^2 + q_2^2 + 2bq_1q_2}{2}. \qquad (2.3)$$

We assume that firm i's cost function is a function of both its output q_i and its production capacity x_i and takes the functional form[3]:

$$C_i(q_i, x_i) = mq_i + (q_i - x_i)^2 \tag{2.4}$$

as in Vives [15], Horiba and Tsutsui [6], Nishimori and Ogawa [12] and Lu and Poddar [9]. This functional form implies that it is inefficient to have undercapacity $(x_i < q_i)$ or overcapacity $(x_i > q_i)$, and costs are minimized when productive capacity is exactly equal to output.

It follows from equations (2.2) and (2.4) that firm i's profits are given by

$$\Pi_i = (a - q_i - bq_j)q_i - mq_i - (q_i - x_i)^2 \quad i = 1, 2; j = 3 - i. \tag{2.5}$$

Social welfare is given by:

$$SW = CS + \Pi_1 + (1 + \alpha)\Pi_2 \tag{2.6}$$

where $\alpha \geq 0$ represents the shadow costs of public funds.

Firm 1 maximizes its profits as given in equation (2.5). Firm 2 maximizes social welfare as given in equation (2.6).

The game runs as follows: in the first stage firms simultaneously choose their productive capacities. In the second stage, knowing the two firms' first-stage decisions, they simultaneously choose their outputs.

[3]With $m < a$.

3. Results

Our solution concept will be subgame-perfect equilibrium. We start by examining the second stage of the game. In this stage, the productive capacities x_1, x_2 have already been set and thus, each firm i chooses only its output q_i to maximize its objective function, firm 1's profits for firm 1 and social welfare for firm 2.

Maximization of firm 1's profits with respect to q_1 yields:

$$q_1 = \frac{2x_1 - bq_2 - m + a}{4}. \tag{3.7}$$

Maximization of social welfare with respect to q_2 yields:

$$q_2 = \frac{(2x_2 - bq_1 - m + a)\alpha + 2x_2 - bq_1 - m + a}{4\alpha + 3}. \tag{3.8}$$

Solving for q_1 and q_2 in (3.7) and (3.8) yields:

$$q_1 = \frac{(2bx_2 - 8x_1 + (4-b)m + ab - 4a)\alpha + 2bx_2 - 6x_1 + (3-b)m + ab - 3a}{(b^2-16)\alpha + b^2 - 12} \tag{3.9}$$

$$q_2 = \frac{-(8x_2 - 2bx_1 - (4-b)m - ab + 4a)(1+\alpha)}{(b^2-16)\alpha + b^2 - 12}. \tag{3.10}$$

In the first stage of the game, each firm i chooses x_i, anticipating the choices q_1 and q_2 given in equations (3.9) and (3.10).

Firm 1's choice of x_i to maximize its profits leads to:

$$x_1 = \frac{-4(4\alpha+3)\Big((2bx_2 + (4-b)m + ab - 4a)\alpha + 2bx_2 + (3-b)(m-a)\Big)}{(b^4 - 32b^2 + 128)\alpha^2 + (2b^4 - 56b^2 + 192)\alpha + b^4 - 24b^2 + 72}. \tag{3.11}$$

Firm 2's choice of x_2 to maximize social welfare leads to:

$$x_2 = \frac{-\Big(F\alpha^2+\Big((64b-2b^3)x_1+(112-32b-3b^2+b^3)(m-a)\Big)\alpha\Big)}{(b^4-32b^2+128)\alpha^2+(2b^4-50b^2+160)\alpha+b^4-18b^2+48} \tag{3.12}$$

with

$$F = 32bx_1 + (64-16b)(m-a).$$

We can now solve for x_1 and x_2 in (3.11) and (3.12) to obtain:

$$x_1 = \frac{4(4\alpha+3)(m-a)G}{D} \tag{3.13}$$

with

$$\begin{aligned} G =&(b^3-4b^2-16b+32)\alpha^2+(2b^3-5b^2-28b+40)\alpha \\ &+b^3-b^2-12b+12 \end{aligned}$$

and

$$D = (b^6-48b^4+512b^2-1024)\alpha^3+(3b^6-126b^4+1216b^2-2048)\alpha^2$$
$$+(3b^6-108b^4+944b^2-1344)\alpha+b^6-30b^4+240b^2-288,$$

while

$$x_2 = \frac{(m-a)H}{D} \tag{3.14}$$

with

$$H = (16b^3-64b^2-256b+512)\alpha^3+(-b^5+3b^4+64b^3-192b^2-704b+1280)\alpha^2$$
$$+(-2b^5+6b^4+75b^3-188b^2-624b+1056)\alpha-b^5+3b^4+27b^3-60b^2-180b+288.$$

Replacing x_1 and x_2 from 3.13 and 3.14 into q_1, and q_2 in 3.9 and 3.10, we obtain:

$$q_1 = \frac{-(m-a)J}{D} \tag{3.15}$$

with

$$J = (\alpha b^2 + b^2 - 16\alpha - 12)((b^3 - 4b^2 - 16b + 32)\alpha^2$$
$$+(2b^3 - 5b^2 - 28b + 40)\alpha + b^3 - b^2 - 12b + 12)$$

and

$$q_2 = \frac{-(m-a)K}{D}, \tag{3.16}$$

with

$$K = (\alpha+1)(b^2\alpha - 16\alpha + b^2 - 12)(b^3\alpha - 4b^2\alpha - 16b\alpha + 32\alpha + b^3 - 4b^2 - 14b + 24).$$

We can obtain $x_1 - q_1$ and $x_2 - q_2$ from equations (3.13) to (3.16), as follows:

$$x_1 - q_1 = \frac{b^2(m-a)(\alpha+1)L}{D} \tag{3.17}$$

with

$$L = (b^3 - 4b^2 - 16b + 32)\alpha^2 + (2b^3 - 5b^2 - 28b + 40)\alpha$$
$$+ b^3 - b^2 - 12b + 12$$

and

$$x_2 - q_2 = \frac{b(m-a)M}{D} \tag{3.18}$$

with

$$M = (b^4 - 4b^3 - 16b^2 + 32b)\alpha^3 + (2b^4 - 9b^3 - 26b^2 + 72b - 32)\alpha^2$$
$$+(b^4 - 6b^3 - 9b^2 + 52b - 40)\alpha + (1-b)(b^2 - 12).$$

We then have:

Proposition 3.1. *In the mixed duopoly with a shadow cost of public funds equal to* $\alpha > 0$*:*

$i)$ *the public firm chooses over-capacity* $(x_2 - q_2 > 0)$ *if the products are complements. If the products are substitutes, there exists* $\alpha^* > 0$ *such that it chooses under-capacity* $(x_2 - q_2 < 0)$ *when* $\alpha < \alpha^*$ *and over-capacity when* $\alpha > \alpha^*$*.*

$ii)$ *The private firm chooses over-capacity irrespective of whether the products are complements or substitutes.*

Proof. See Appendix.

Let us compare our results with those obtained in the absence of a shadow cost of public funds. Since in the absence of this cost, both the private firm and, if products are complements, the public firm, choose over-capacity, the shadow cost of public firms does not alter the results on capacity choice. But the existence of this cost does alter the results of the public firm's capacity choice if products are substitutes because, in the absence of this cost, the public firm chooses under-capacity, while, in its presence, it continues to choose under-capacity only if this cost is low, while it chooses over-capacity if it is high.

4. Conclusion

We have studied how the existence of a shadow cost of public funds alters the results of capacity choice in a mixed duopoly with product differentiation. We have found that it changes the capacity choice of the public firm if products are substitutes, but that it alters neither the results of the capacity choice of the public firm if products are complements, nor the results of the private firm's capacity choice.

5. Appendix: Proof of Proposition 3.1.

5.1. (i) *Analysis of* $x_2 - q_2$.

Notice, first, that $D < 0$ because $b^6 - 30b^4 + 240b^2 - 288 < 0$ and the coefficients on α^3, α^2 and α are also all negative.

Notice now that since, by assumption, $m < a$, $x_2 - q_2$ and M have different signs when $b < 0$, while they have the same sign when $b > 0$. We will determine the sign of $x_2 - q_2$ by determining the sign of M. Since $(1-b)(b^2-12) < 0$ when α equals zero, M will be negative. On the other hand, since $(b^4 - 4b^3 - 16b^2 + 32b) < 0$ (> 0) when $b < 0$ $(b > 0)$, when α approaches $+\infty$, M will be negative when $b < 0$, and it will be positive when $b > 0$.

The derivative of M with respect to α is equal to:

$$\frac{\partial M}{\partial \alpha} = 3(b^4 - 4b^3 - 16b^2 + 32b)\alpha^2 + 2(2b^4 - 9b^3 - 26b^2 + 72b - 32)\alpha$$

$$+b^4 - 6b^3 - 9b^2 + 52b - 40.$$

This derivative vanishes at the following two roots:

$$\alpha_1 = \frac{-2b^4 + 9b^3 + 26b^2 - 72b + 32 + \sqrt{Q}}{3b^4 - 12b^3 - 48b^2 + 96b}$$

$$\alpha_2 = \frac{-2b^4 + 9b^3 + 26b^2 - 72b + 32 - \sqrt{Q}}{3b^4 - 12b^3 - 48b^2 + 96b}$$

with

$$Q = b^8 - 6b^7 - 20b^6 + 108b^5 + 140b^4 - 288b^3 - 64b^2 - 768b + 1024.$$

Consider first the case, $b \in (0, 1)$. It can be seen that $\alpha_2 < 0 < \alpha_1$. (The denominator of the two roots is positive. The numerator of α_2 is negative while the numerator of α_1 is positive.) Therefore,

α_1 is the only positive value where $\frac{\partial M}{\partial \alpha}$ vanishes. Since $\frac{\partial M}{\partial \alpha} < 0$ when α approaches zero, [it can be seen that $b^4 - 6b^3 - 9b^2 + 52b - 40 < 0$ for $b \in (0, 1)$] while $\frac{\partial M}{\partial \alpha} > 0$ when α approaches $+\infty$, M reaches a local minimum at $\alpha = \alpha_1$. Since M is negative when $\alpha = 0$, and it is positive when α approaches $+\infty$, this implies that there exists $\alpha^*(> \alpha_1)$ such that M is negative for $\alpha < \alpha^*$, and positive for $\alpha > \alpha^*$. Therefore, $x_2 - q_2 < 0 (> 0)$ for $\alpha < \alpha^* (\alpha > \alpha^*)$.

Consider now the case $b \in (-1, 0)$. It can be seen that $\alpha_2 < \alpha_1 < 0$. (The denominator of the roots is negative, while the numerator is positive). Therefore, $\frac{\partial M}{\partial \alpha}$ does not vanish at any $\alpha > 0$. Since $\frac{\partial M}{\partial \alpha} < 0$ when α approaches zero, we have that $\frac{\partial M}{\partial \alpha} < 0$ for all $\alpha > 0$. Since M is strictly decreasing and is negative when α equals zero, it will be negative for all $\alpha > 0$. Therefore, $x_2 - q_2 > 0$.

5.2. (ii) Analysis of $x_1 - q_1$.

Notice first that since both $m-a$ and D are negative, $x_1 - q_1$ has the same sign as L. Notice now that $L > 0$, since $(1-b)(12-b^2) > 0$, and the coefficients of α^2 and α are both positive.

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V. EVOLUTION AND GENERAL EQUILIBRIUM

Elvio Accinelli

Abstract

General Equilibrium plays a central role in the majority of the areas of economics, providing a rigorous analysis of decentralized economies under the assumption that no one in particular sets the prices, but all do it. It is, mainly, a static theory, capable of explaining in a rigorous manner the actual economy, but without a good explanation of the trading process in the long run. It may, perhaps, serve as an exercize to describe an economy out of equilibrium and the process by which this is achieved, which is not an easy task, and may be impossible, at least without strong restrictions in the hypotheses. In this work we attempt to introduce a suitable dynamics along an equilibrium path.

1. Introduction

In this paper, we analyze the theoretical difficulties inherent in the tâtonnement process to consider the global evolution of an economy to an equilibrium system of prices. We will consider the possibility of introducing a complementary dynamics that allows us to explain the evolution of an economy along the equilibrium manifold. This last concept is introduced by Balasko [3].

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By introducing a new dynamics to explain the evolution of economies in their equilibrium variety at the local level, Samuelson's dynamics becomes a good tool to analyze the stability of each equilibrium. In some way both dynamics are complementary.

Following Accinelli and Covarrubias [1] we consider an evolutionary dynamic on a modified Balasko equilibrium manifold and we show that economic crises can be considered as a natural result of the evolution of the economy along an equilibrium path.

The rest of this paper is as follows: in the next section we consider the Samuelson's dyanmics as mathematical formalizations of the tâtonnemet process considered by L. Walras in 1926 (see Walras [9]). In Section 3 we explain our heterodox point of view. The model is introduced in Section 4 and corresponds to the model considered previously in Accinelli and Covarrubias [1]. In Section 5 we give a first step to build a dynamics over the manifold of equilibria. In Section 6 we show a dynamics over a modified version of the Balasko Manifold, and we show that singular economies are the threshold of crises that happen as a result of the choices of owners or managers of firms. Finally, we give some conclusions.

2. Samuelson's dynamics

The main dynamic process in economic theory is given by the tâtonnemet process; however, this is more of a description than an explanation. It works as follows:

Let $z_t(p) = (z_{1t}(p), ..., z_{lt}(p))$ be the aggregate excess demand function of an economy such that in time t there are l different goods, and prices are p. $z_{it}(p) = \sum_{i=1}^{n} x_{it}(p, w_i) - \sum_{i=1}^{n} w_{it}$ where $x_{it}(p, w_i)$ is the demand of the $i-$th agent when prices are p and we represent the initial wealth of this agent by w_i. To simplify the notation, we do not specify the time variable. In addition, we assume a differentiable function of prices $p \in \Re^l_{++}$.

Suppose that in an initial time t_0 prices are $p(t_0)$ and $z(p(t_0)) \neq 0$, then the demand-and-supply principle suggests that prices will adjust upward for goods in excess demand, and downward for those in excess supply. Using elements of differential equations and dynamical systems theory, Samuelson in 1944 and 1947 proposes a specific form for the tâtonement process:

$$\frac{dp_l}{dt} = c_l z_l(p), \text{ for every } l \in \{1, ..., L\}$$

where $c_l > 0$ is a constant affecting the speed of adjustment, and $z_l(p)$ the excess demand function for the $l - th$ commodity. Note that if $z(p^*) = 0$, then p^* is an equilibrium for Samuelson's dynamics and a Walrasian price vector of equilibrium for the economy. Thus, if we can show some kind of global stability of p^*, we can conclude that the economy converges to a steady state. Such was the idea of Walras, but this road is full of stones.

As simple as this equation is, its interpretation is fraught with difficulties.

- Which economic agent is in charge of prices?
- Why must the law of one price hold out of equilibrium?
- What sort of time does t represent? It is not real time because with disequilibrium prices, not all plans can be simultaneously realized.
- Perhaps the most sensible answer to this question is that this axillary variable provides an insight into the equilibrium properties, it helps to distinguish good equilibria from poorly behaving equilibria (restoring an equilibrium after a disturbance).

- It is not necessarily true that the equation $z(p) = 0$ has only one solution. As G. Debreu has shown, a multiplicity of solutions is a generic property (see Balasko [3]).

The most suggestive case is for an economy with two goods.

Example 2.1. In the case of an economy with only two goods, from Walras's law, we can restrict considerations to one given price, for instance, $p_2(t) \equiv 1$, and consider finding the equilibria with only one price out of equilibrium following the law:

$$\dot{p}_1 = kz_1(p_1).$$

- Each equilibrium price $\bar{p}_1$ corresponds to one stationary state $z_1(\bar{p}_1) = 0$ and defines an equilibrium vector price $p = (\bar{p}_1, 1)$.

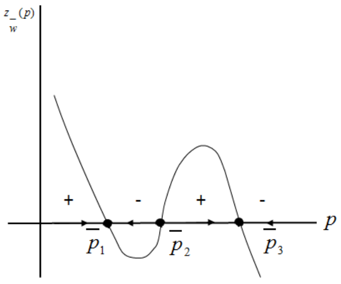
- If the economy is regular, then we have an odd number of equilibria, locally stable, or totally unstable, according to the slope of the excess demand function.

- According to Samuelson's dynamics, every economy (with two goods) out of equilibrium will converge to the equilibrium.

2.1. Some pessimistic considerations

Unfortunately, as soon as $l > 2$, the global consideration corresponds to the case where $L = 2$ cannot be generalized. However, we can consider the local behavior of equilibria after this state is reached using Samuelson's dynamics.

Figure V.1: Stable and unstable equilibria. Corner equilibria are stable.



Example 2.2. Consider an exchange economy with $L = 3$. From the degree zero homogeneity of the excess demand function, we consider the normalized set of prices

$$S = \{p \in R^3_{++} : p_1^2 + p_2^2 + p_3^2 = 1\}$$

and $c_1 = c_2 = c_3 = 1$.

As a consequence of Walras's law, this normalization has the virtue that

$$\frac{d(p_1^2 + p_2^2 + p_3^2)}{dt} = pz(p) = 0.$$

So, if $p(0) \in S$, then $p(t) \in S$ for all $t \geq 0$. The only restrictions to the trajectories imposed by the general theory are those derived from the boundary behavior of the excess demand. In such a case, in the boundary of S given that excess demand becomes positive, the trajectories of prices point inward, near the boundary. However, there are equilibria that are neither locally stable, nor locally unstable, and under some initial conditions, prices may not converge to any equilibrium, depending on the characteristics of the excess demand, and it is possible to obtain cycles. We cannot avoid cycles without strong restrictions in the preferences of the consumers. Thus, we cannot expect to have even local convergence.

But, we can analyze the behavior of equilibria after a small perturbation in the fundamentals of the economy. In conclusion, if $l > 2$, then Samuelson's dynamics does not allow one to make global considerations on convergence to an equilibrium, but it is possible to use this dynamics to analyse the local stability of equilibria.

2.2. *Attempting to do some positive analysis*

In order to be positive, we can say that in the case where we have succeeded proving the uniqueness of an equilibrium, then we could rely on Samuelson's dynamics. Moreover, this equilibrium will be globally stable.

But, unfortunately, even assuming that society maximizes a utility function, in general, uniqueness requires WARP. This is a very restrictive condition. Even if individual utilities verify WARP, it does not necessarily happens in the aggregate (see Debreu [4]).

In contrast to WARP, if the uncompensated law of demand (ULD), i.e., $\forall\ p \neq p'$, $(p'-p)(x_i(p', w)-x(p, w)] \leq 0$ is verified for individual demand functions, then it is verified in the aggregate, and uniqueness of the equilibrium follows. The Mitjushcin and Polterovich [8] condition

$$-\frac{x_i D^2 u_i(x_i) x_i}{x_i \nabla u_i(x_i)} < 4 \text{ for all } x_i$$

implies $x_i(p, w)$ satisfies ULD. This condition doesn't seem to be excessively restrictive.

Assuming that the function of excess demand is differentiable, another possibility for (global) stability is that the Jacobian matrix of the aggregate excess demand (restricted) is negative definite for all p. The Jacobian of the excess demand satisfies $pD_pz(p) = 0$, and so if l is the amount of goods, then $\mathrm{rank}D_pz(p) \leq l - 1$. But if $\mathrm{rank}J_pz(p) = l - 1$, by the homogeneity of degree zero of the excess demand function and from Walras's law, we can restrict the consideration to $l - 1$ markets and $l - 1$ prices for a restricted excess of demand function $\bar{z} : l-1 \to l-1$ and the corresponding restricted Jacobian matrix. In this case ULD follows, and we have only one steady state verifying global stability (see Debreu [4]).

But $D_p(x(p, w_i))$ is negative definite for all p if and only if $x(p, w)$ satisfies the ULD property. In these cases the substitution effect is sufficiently well behaved and overcomes the possible difficulties deriving from the wealth effect. This situation is verified if, for instance, preferences are homothetics.

The weakness of the tâtonnemt's dynamic and its mathematical formulation given by Samuelson lies in the fact that it is determined by the excess demand function, and the main characteristics of the excess demand function are determined by preferences or utilities. Thus, putting restrictions on the excess demand function is equivalent to putting restrictions on preferences.

The Sonnenschein-Mantel-Debreu Theorem (see Mantel [5]) shows that the restrictions imposed by the maximization of consumer preferences on the excess demand function are very weak and the conditions that would validate the tâtonnemt's process require strong conditions put on the characteristics of preferences. Concluding our discussion of this theorem, we can say that claiming the universality of the tâtonnement's dynamics would practically be equivalent to affirming that given any field of vectors in $\Re^l$, their trajectories always converge to the steady state.

However, Samuelson's dynamics can be very useful for the study of local stability, although this certainly was not the intention of Walras. To consider this point of view, let us now introduce the concept of Equilibrum Manifold.

Balasko [3] introduces the concept of Equilibrium Manifold. We will call this manifold the Balasko Manifold. Let $\omega \neq 0 \in \Omega$ be the initial distribution of the wealth of the economy, where Ω characterizes the possible distributions of the initial wealth of the economy among its agents. In our case $\Omega = \Re_+^{lm}$ where l is the amount of different goods in the economy, and m is the number of consumers. The Balasko Manifold is the subset $BM \in \Re^l$ defined by

$$BM = \{(p, w) \in \Delta \times \Omega; z(p, \omega) = 0\}$$

where Δ is the simplex in $\Re_l$, i.e.: $\Delta = \{p \in \Delta; p_1 + ... + p_l = 1\}$. So, by definition, (p, ω) is an equilibrium if $(p, \omega) \in BM$. Note that for an economy defined by $\bar{w} \in R_+^l$, $\bar{p} \neq 0 \in \Delta$ is an equilibrium price if and only if $z(\bar{\omega}, \bar{p}) = 0$. The main problem is the following: suppose that in time t_0, $(\bar{p}, \bar{\omega}) \in BM$ and in some time $t_1 > t_0$, we have that $(\tilde{p}, \tilde{\omega}) \in BM$.

The question still unanswered by economic theory is how the transition occurs from the initial equilibrium existing in t_0 to that existing in t_1. In the equilibrium manifold, there are only economies in equilibrium, and current economic theory can tell us very little about the characteristics of the trajectory along which the economy passes from one equilibrium state to another. Modifications in the initial wealth, or in its distribution among the agents of the economy, can cause the economy to change its equilibrium state, i.e., to pass from the equilibrium $(\bar{p}, \bar{w})$ to another $(\tilde{p}, \tilde{w})$.

Although, for analyzing the overall stability of the equilibrium, Samuelson's dynamics do not seem to be a good tool, it has a very important role to play when we need to analyze the stability of each equilibrium point. That is, locally, Samuelson's dynamics has a very significant role to play. If the initial Samuelson's dynamics is unstable, then the change in the economy and in its fundamentals can be very large. If the equilibrium is stable for this dynamic, not very large (local) changes can be reversed through time. But regardless, we know nothing of what happens outside the Balasko Manifold.

In what follows, we will try to give an explanation of the way this change is processed. For that we will introduce a very heterodox point of view.

3. The heterodox point of view

A respect in which general equilibrium theory remains fundamentally static is that it takes the economy's structure as given and says nothing about how technology, preferences, or products change. We focus our attention on technological change. We introduce a continuous-time replicator dynamics in a production economy with two types of firms.

- We allow the structure of the economy to change. The distribution of the firms in the set of possible technologies available at each moment changes.
- However, we consider the evolution of the economy in equilibrium i.e., in the equilibrium manifold, according to the Balasko Economics-Geometry (see Balasko [3]).
- The profit of each firm is determined by a production plan corresponding to the Walrasian equilibrium.
- We introduce a characterization of the regular and singular economies based on the distribution of the firms over the set of available technologies.

- Up to this point we have been following the classical General Equilibrium Theory (GE).
- From now on we attempt to introduce a dynamic in the framework of GE. But to do this, we need to follow a heterodox approach.

Various traditions and schools of thoughts (e.g., Schumpeterian Economics, evolutionary economics,) may have much to offer to help us understand the dynamics of the economy. Several efforts have been made seeking to integrate these theories in the framework of the Theory of General Equilibrium, for instance, Aghion and Howitt [2] and subsequent work. However, these efforts have halted.

In our model, the evolution of the economy is determined by the owners of firms investing in technologies that offer greater rates of profits.

We assume that managers do not have complete information in the moment of choosing the technology for the next period. Then we introduce a process of imitation of the most successful agent and a dynamics closer to the replicator dynamics. But this dynamics happens along the equilibrium manifold.

4. The model

Consider a production economy such that

- The agents
  - A finite set of agents is $I = \{1, ..., m\}$ distributed in two types:
  - m_1 of them are of type 1 and $I_1 = \{1, ..., m_1\}$ is the set of these agents.

  - The set of agents of type 2 is given by $I_2 = \{m_1 + 1, ..., m\}$.
  - Agents of each type have identical preferences $\succeq_i$ represented by utility functions $u_i : R^2_+ \to R$, and identical endowments $\omega^i = (\omega^i_1, \omega^i_2) \in X \quad i = 1, 2$, and identical consumption space $X \subset R^2$.

- The firms

  - The set of firms is $\mathcal{F} = \{1, ..., n\}$ distributed also between two types according to the technology.
  - By F_k we characterize the set of firms that in a given time t are using the technology $k \in \{1, 2\}$. Let $|F_k| = n_k$ be the amount of firms following the technology $j \in \{1, 2\}$. Thus, $n = n_1 + n_2$.
  - Firms of each type k are characterized by the technological set $Y_k \subset \Re^2, \quad k = 1, 2$ with the habitual properties.

The economy is a private 2-goods ownership economy, i.e., every good is traded in the market at publicly known prices, consumers trade to maximize well-being, and firms produce to maximize profits. The technology is free, so each manager can choose between Y_1 and Y_2. Each consumer $i \in I$ has a claim to a share $\bar{\theta}_{ij}$ of the profit of the firm j such that

$$\sum_{i=1}^{m} \bar{\theta}_{ij} = 1 \text{ for each firm } j \in 1, ..., n.$$

The wealth of consumers is derived from individual endowments of commodities and from ownership claims (shares) of the profits of firms, which are, therefore, thought as being owned by consumers.

$$W_i(n_1, n_2) = p\omega_i + \pi_1(y_1)[\sum_{j \in F_1} \theta_{ij})] + \pi_2(y_2)[\sum_{j \in F_2} \theta_{ij},\ i \in \{1, ..., m\}. \qquad (4.1)$$

For each consumer $h \in \{1, .., m_i\}$ $i \in \{1, 2\}$ the budget set is:

$$B^i = B^i(p, W_h, \theta_h)_h = \left\{x_h^i \in R_+^2 : px_h^i \leq W_i(n_1, n_2)\right\}.$$

- The solution $x_i^* = x_i(p)$ of the maximization problem $\max u_i(x)$ restricted to the budget set, i.e., $x \in B^i$ is called the demand of the $i - th$ agent.

- The solution $y_j^* = y_j(p)$ of the maximization problem $\max p(y)$ in the technological restriction, i.e., $y \in Y_j$ is called the supply of $j - th$ firm.

Note that since we assume that all firms are maximizing their profits, then we have the following system of equalities:

- for the firms' production plans, $y_1^{*j} = ... = y_{n_i}^{*j} = \bar{y}^j$ $j \in \{1, 2\}$ and for analogous reasons

- it follows for consumption plans that $x_1^{*i} = ... = x_{m_i}^{*i} = \bar{x}^i$ $i \in \{1, 2\}$.

Definition 4.1. A Walrasian Equilibrium is a feasible allocation $x^* = (x_1^*, ..., x_m^*, y_1^*, ..., y_n^*) \in X^{m+n}$ and a system of prices $p \in X^*$ such that, at x_i^* the $i - th$ consumer, $i \in I$, is maximizing his preferences given his budget constraint, and at y_j^* the $j - th$ firm, $j \in J$, is maximizing its profits.

- Fixed $\omega = (\omega_1, ..., \omega_m)$, the function $z : \Delta \to R^l$ defined as $z(p) = \sum_{i=1}^{m}(x_i(p) - \omega_i) - \sum_{j=1}^{n} y_j(p)$ is the excess demand function.
- A vector p, is an equilibrium vector price if and only if $z(p) = 0$ and the corresponding allocation $(x(p), y(p)) = (x^*, y^*)$ is an equilibrium allocation.
- So, $\{x(p), y(p), p\}$ is a Walrasian equilbrium.

For our model we have the following definition:

Definition 4.2. Given a private ownership economy with m_1 consumers of type 1 and m_2 consumers of type 2 such that $m_1 + m_2 = m$ and two types of firms, n_1 producing with a technology Y_1 and n_2 firms producing using Y_2 such that $n_1 + n_2 = n$, then: An allocation $(x^{*1}, x^{^*2}, y^{*1}, y^{*2})$ and a system (or vector) of prices $p = (p_1, p_2)$ constitute a Walrasian (or competitive) equilibrium if given the vector prices:

(1) Each coordinate of the vector $y^{*j} = (y_1^{*j}, ..., y_{n_j}^{*j})$, represents an optimal plan of production for each firm in Y_j $j \in \{1, 2\}$, i.e., for each $j \in J$ y_j^*, maximizes the profits of firm j.

(2) Each bundle set in the vector $x^{*i} = (x_1^{*i}, ..., x_{m_i}^{*i}) \in R_+^{2m_i}$, where each $x_h^i = (x_{h1}^i, x_{h2}^i) \in R_+^2$ maximize $u_i(x_{h1}^i, x_{h2}^i)$, $h \in \{1, ..., n_i\}$ $i \in \{1, 2\}$ in the budget set, and

(3) the allocation $(x^{*1}, x^{^*2}, y^{*1}, y^{*2})$ is feasible, i.e.; $\sum_{i=1}^{|m|} x_i^* - \sum_{j=1}^{|n|} y_j^* = \sum_{i=1}^{|m|} w_i$ or equivalently

$$m_1 x_1^* + m_2 x_2^* + n_1 y_1^* + n_2 y_2^* = n_1 w_1 + n_2 w_2.$$

We assume the following:

- In time t_0 the distribution of the firms $n = (n_1, n_2)$ and consumers $m = (m_1, m_2)$ is given.
- The set of equilibria that in time $t = t_0$ are reachable depends precisely on the distribution of firms and consumers over their respective types.

Also, note that:

- Even under the assumption that the economy is in equilibrium in the short run, the profit rates of firms are not necessarily equal.
- *Even under the assumption that firms are maximizing profits, this level depends on the technological characteristics of each firm.*
- Before the time t_0, nobody knows what profits and utilities are possible to reach in equilibrium; this will be known only when the equilibrium is reached.

5. A first step to the dynamics over the equilibrium manifold

At some later time $t_1 > t_0$, when the profits corresponding to the equilibria are publicly well known, managers can then choose to change the technology with which they produce, or continue as before. We assume that it is possible to change the technology without costs because technology is free. Firms can change the technology, but n is constant.

Thus at every time t, the economy will change according to the distribution of the firms over the set of technologies. Let $h_1 = \frac{n_1}{n}$

and $h_2 = \frac{n_2}{n}$ be the distribution of the firms in time $t = t$ so for each distribution h, we have the economy

$$\mathcal{E}_h = \left\{R^2_+, u_i, w_i, I, Y_1, Y_2, h_1, n_2\right\}.$$

Note the wealth of consumers depends on the distribution

$$w_i(h) = pw_i + \pi(y_1^*)\left[\sum_{j\in F_1}\theta_{ij}\right] + \pi(y_1^*)\left[\sum_{j\in F_2}\theta_{ij}\right].$$

Note that $|F_1| = h_1 n$ and $F_2 = h_2 n$. If wealth changes, the demand of consumers change and equilibrium prices change, also.

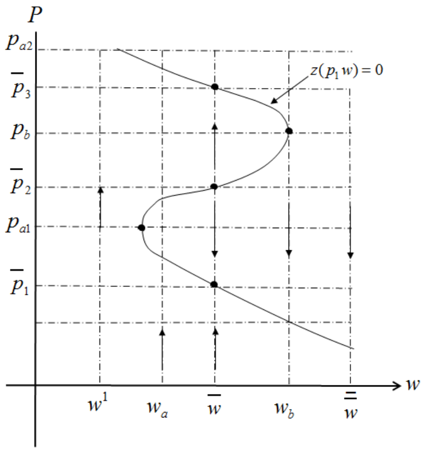
In Accinelli and Covarrubias [1] it is shown that under the habitual hypothesis, $w_i(h)$ is a continuous function of h, i.e., given $\epsilon > 0$ there is $\delta > 0$ such that if $\|\bar{h} - h\| \le \delta$, then $\|w_i(\bar{h}) - w_i(h)\| \le \epsilon$.

6. The modified Balasko Manifold

Economists like to characterize economies by the distribution of wealth among consumers $\omega = (\omega_1, ...\omega_n)$. In this way, if the utilities, technologies and the consumption set is fixed, each economy can be characterized by the initial wealth that each consumer has, i.e., $\omega = (\omega_1, ..., \omega_n)$. The individual wealth is generated by individual shareholdings of firms and by their ownership of stocks of commodities. So, for each w, we have an excess demand function z_ω and, consequently, a set of equilibrium prices $Eq_\omega = \{p; z_\omega(p) = 0\}$.

Definition 6.1. We say that $p \in Eq_\omega$ is regular if $rank z_\omega(p) = l - 1$, where l is the amount of goods in the economy. In another case, we say that $p \in Eq_w\omega$ is critical.

Figure V.2: Equilibrium Manifold for $L = 2$ $Eq = \{(w(h), p) \in \Re \times (0, 1) : z(w(h), p) = 0$



Definition 6.2. We say that the economy ω is regular if for all $p \in Eq_\omega$ $rank z_\omega(p) = l - 1$, where l is the amount of goods in the economy. In another case, we say that the economy is critical or singular.

Note that an economy is regular if and only if 0 is a regular point for the excess demand.

Following Debreu [4], we know that generically in $\Re^l_{++}$ the economies characterized by ω are regular. Precisely, the subset of regular economies is open and dense. Consequently, the subset of singular economies is a very small set, a meager set. In Debreu [4], using index theorems, Mas-Colell [6] shows that the cardinality of Eq_ω for each regular economy ω is odd.

The Balasko manifold is shown in Figure V.2, where the graph represents the set (ω, p); $z(\omega, p) = 0$ in the case in which $l = 2$, and the wealth of the economy can be measured by a positive number w. It is a simplified picture, but takes into account the facts that, generically, economies are regular, and each one has an odd number of equilibrium prices.

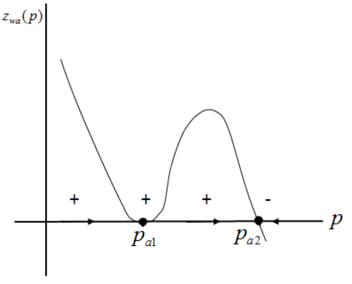
In our model, the wealth of consumers depends on the distribution h of the firms over the set $Y = (Y_1, Y_2)$ of technological possibilities. The wealth of agents is defined as a function of this distribution and denoted by $w(h) = (w_i(h), ..., w_m(h))$. When the distribution of firms changes, the wealth of consumers also changes, and even when all other elements of the economy remain fixed, the demand changes as a result of this change in the wealth of the agents. According to this point of view, we will denote by $z_h(\dot{})$ the excess demand of the economy $\mathcal{E}_h$. Let us denote by Eq_h the set of equilibrium prices of the economy $\mathcal{E}_h$, i.e, $Eq_h = \{p : z_h(p) = 0\}$. Using similar techniques to Debreu [4] and Mas-Colell [6], it is possible to show that generically in h, $\mathcal{E}_h$ is regular, and that the cardinality of Eq_h is odd.

In Accinelli and Covarrubias [1] it is shown that for each regular economy $\mathcal{E}_h$, there exists a continuous selection $\phi(h) \in Eq_h$ such that for all $\epsilon > 0$ there is some δ such that $\phi(\bar{h} - \phi(h)\| \leq \epsilon$ if $\|\bar{h} - h\} \leq \delta$. This continuous selection does not exist if the economy is singular (see Accinelli and Covarrubias [1]). This situation is represented in Figure V.3, where we show the modified Balasko Manifold that is the set

$$MMB = \{(h, p) \in \Delta \times \Delta : z(h, p) = 0\}.$$

We consider that managers choose the technology for the next period according to the expected value of profit rate for the next period. That is, after the prices are revealed, managers of firms k choose to change to technology $l \neq k \in \{1, 2\}$ if $E(\bar{\pi}_l(y_l^*)) > E(\bar{\pi}_k(y_k^*))$ or, in the other case, choose to remain as they were. According to the replicator dynamics (see Maynard

Figure V.3: The critical equilibrium is unstable



Smith and Price [7] and Weibull [10]), we have that

$$\begin{aligned} \dot{h}_1 &= h_i(E(\bar{\pi}_1(y_1^*)) - E(\bar{\pi}_1(y_1^*))) \\ \dot{h}_2 &= -\dot{h}_1. \end{aligned} \tag{6.2}$$

Managers don't have complete information to evaluate these expected values, and need to assign probabilities to the future states of the world, or they can choose to follow an imitation process, (see Accinelli and Covarrubias [1]) and thus, the dynamics of the firms can be written in the form:

$$\begin{aligned} \dot{h}_1 &= h_1\psi((E(\bar{\pi}_1(y_1^*)) - E(\bar{\pi}_1(y_1^*))]) \\ \dot{h}_2 &= -\dot{h}_1 \end{aligned} \tag{6.3}$$

where ψ is an increasing function, such that $\psi(0) = 0$.

Note that a solution of this dynamical system defines a trajectory in the equilibrium manifold, because if $h(t) = h(t, t_0, h_0)$ is a solution with initial condition $h(t_0) = h_0$, then prices change in such way that $z(h(t), p(h(t))) = 0$.

We obtain a steady state for the economy if and only if $E(\bar{\pi}_1(y_1^*)) = E(\bar{\pi}_1(y_1^*)$.

This continuous evolution is not verified if the economy is singular, like w_s in Figure V.2. In this case, a process of abrupt changes in prices occurs. Note that if the fundamentals of a singular economy are perturbed, then, following Samuelson's dynamics, the new economy will be regular, and prices converge to a stable equilibrium. See Figure V.3.

7. Conclusions

Our main objective in this paper was to introduce a dynamics in the framework of General Equilibrium Theory. To do this, we consider firms maximizing profits and managers seeking to maximize profit rates. The choices of managers impact the fundamentals of the economy, in particular, in the wealth of the consumers. So, the demand of commodities changes and, consequently, the equilibrium prices change. These changes are generally smooth, but in a neighborhood of a critical economy, changes can be abrupt and unexpected. In this way, driven by the choices of rational agents, critical economies can be considered the threshold of economic crises.

In this way, we have shown that the choices of managers looking to maximize profits are the engines of economic evolution. Changes in the economic equilibrium can be considered to be the result of rational decisions and, consequently, economic crises are inherent in the base of the theoretical foundations of the economy.

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VI. AN ECONOMY WITH INDIVISIBLE GOODS IN A METRIC SPACE

Saul Mendoza-Palacios and David Cantala

Abstract

This paper presents two economies with indivisible goods where the sets of agents and goods are metric spaces. In addition, each agent has a unique initial indivisible good to be used for trading. We present two models. The first one introduces an economy where the concept of allocation is a measurable function, which assigns to each type of agent one type of good. The second model introduces an economy where the concept of allocation is a probability distribution, which assigns a mass of agents to a mass of goods. We show that the first model is contained in the second model. Finally, we establish general conditions for the no emptiness of the core for both models, and see the relation between them.

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1. Introduction

An economy can be described as a situation where we have a set of agents and a set of commodities. In addition, each agent has a preference relation on the set of all commodities and an initial commodity to be used for trading. The purpose of the model is to explain when the agents have incentives to carry out an exchange, and how this exchange affects their welfare. The manner in which a commodity is assigned to an agent is called an allocation. The core of an economy is a subset of allocations that can be described as follows: if the allocation is in the core, then there is not a set of agents with the possibility of improving their welfare when they carry out an alternative exchange. If an allocation is not in the core (and it is feasible), then there exists a set of agents with the intention of performing an exchange due to the possibility of improving their welfare.

In this paper we are interested in an economy in which each agent has a unique initial indivisible good to be used for trading. One of the first papers that studied these types of economies was Shapley and Scarf [22] in 1974. In our paper we suppose that the set of agents and goods are metric spaces and establish conditions under which the core is not an empty set. The essential idea of this hypothesis is that the economy has a very large number (maybe uncountable) of agents and commodities.

In this paper the agents and commodities are classified in types. The number or mass of agents (and commodities) of a certain type is measured by a probability distribution. We present two models. The first one introduces an economy where the concept of allocation is a function which assigns to each type of agent one type of good. In this first model, we have a very large number of types of agents, and each one has no influence on the economy. If these two conditions are not satisfied, the core may be an empty set. The second model introduces an economy where the concept of allocation is a probability distribution which assigns a mass of agents to a mass of goods. For this second model, we establish general conditions for the existence of the core. Moreover, we show that the first model is contained in the second model. The second model includes another important classic model, the case where the sets of agents and goods are finite.

For models of an economy with indivisible goods where the set of agents and goods are finite, several authors have addressed the problem of no emptiness of the core (see for instance, Shapley and Scarf [22], Roth and Postlewaite [20], Wako [24], Ma [17], Cantala and Pereyra [4]).

Economic models with a set of agents or a set of goods in general measurable spaces are important because the results of these models include other particular cases. Aumann [3] in 1964 proposed a model with a continuum of agents in an economy with divisible goods. There are several models of an economy where the set of agents or goods are continuous spaces, where authors address the problem of no emptiness of the core and characterize a competitive equilibrium: for example, Aumann [3], Mas-Colell and Zame [18], Anderson [2], Accinelli [1], Covarrubias [8]. In these models, goods are divisible, while in our cases, the goods are indivisible. Inoue [14] proposes an economy with indivisible goods (in a finite space) and a continuum of agents, and where, unlike us, each agent can consume more than one good.

In our model, we use concepts and results from the Theory of Mass Transportation Problem (or Optimal Transport Problem) which is a branch of a probability theory with several applications to economic theory (see Carlier [5] and Galichon [10]). The economies with indivisible goods form part of the research on the matching theory. In the matching theory, the theory of mass transportation problem has several contributions for models where the set of agents or goods are continuous spaces (for example see, Gretsky et al. [11], Gretsky et al. [12], Ekeland [9] Chiappori et al. [7]).

2. A first model

In this section, we consider a first approach for an economy in which each agent has a unique indivisible good. Moreover, the economy has a continuum of indivisible goods. In this case, we may assume that we have a population of goods and a population of agents (who act as both sellers and buyers) and the problem is to find an agent-good assignment such that no coalitions exist where agents want to exchange their goods.

2.1. The economy

Consider an economy which has a population of agents and a population of indivisible goods. Each indivisible good is labeled according to its characteristics as a single type of indivisible good g in a set G of types of indivisible goods. Also, each agent is labeled according to his preferences, as a single type of agent a in a set A of types of agents, i.e., a particular type of agent a in A represents a preference relation $\precsim_a$ over G. Assume that A and G are compact metric spaces, and let G, in addition, be separable. For the preference relations $\{\precsim_a\}_{a\in A}$, we assume that

- **H1** *rationality:* for each a in A, $\precsim_a$ is a complete and transitive order relation;

- **H2** *continuity in the goods:* for each a in A and g' in G, the sets $\{g \in G : g' \precsim_a g\}$ and $\{g \in G : g \precsim_a g'\}$ are closed;

- **H3** *continuity in the agents:* for any $g', g \in G$, the set $\{a \in A : g' \precsim_a g\}$ is closed.

Let $\mathcal{B}(A)$ and $\mathcal{B}(G)$ be the Borel σ-algebras of A and G, respectively. Probability measures η and ν assign a population distribution over the sets A and G, severally. Finally, we denote the population of agents and the population of indivisible goods by

$$\mathbf{A} := (A,\ \mathcal{B}(A), \eta) \text{ and} \tag{2.1}$$

$$\mathbf{G} := (G,\ \mathcal{B}(G), \nu), \tag{2.2}$$

respectively.

Theorem 2.1. *Let A be the set of types of agents, and G be the set of types of indivisible goods. Assume as in (2.1) and (2.2) that A and G are compact metric spaces, and let G, in addition, be separable. Suppose that the preference relations $\{\precsim_a\}_{a\in A}$ satisfy* **H1, H2** *and* **H3**. *Then there exists a continuous function $u : A \times G \to [0,1]$ such that*

$$\forall a \in A, \quad g \precsim_a g' \Longleftrightarrow u(a,g) \le u(a,g'). \tag{2.3}$$

Proof. See Rachev and Rüschendorf [19] Theorem 5.5.18 page 337. □

An *economy* is a quadruple $\mathcal{E} := (\mathbf{A}, \mathbf{G}, u, \mu_0,)$, where $\mathbf{A}$ is a population of agents as in (2.1), $\mathbf{G}$ is a population of indivisible goods as in (2.2), u is a continuous function, which satisfies (2.3), and finally, μ_0 is a measurable function $\mu_0 : A \to G$, which assigns for each type of agent a in A, the agent's initial endowment $\mu_0(a)$ in G and satisfies that for all E in $\mathcal{B}(G)$, $\eta(\mu_0^{-1}(E)) = \nu(E)$. The function μ_0 is called the initial endowment.

An *allocation* for the economy $\mathcal{E}$ is a measurable function $\mu : A \to G$. An allocation μ for $\mathcal{E}$ is *feasible* if for each set of types of indivisible goods E in $\mathcal{B}(G)$, the amount $\nu(E)$ of indivisible goods is proportional to the amount $\eta(\mu^{-1}(E))$ of agents. In other words, an allocation μ for $\mathcal{E}$ is feasible if

$$\eta(\mu^{-1}(E)) = \nu(E), \quad \forall E \in \mathcal{B}(G). \tag{2.4}$$

A *coalition* is a set $S \in \mathcal{B}(A)$. An allocation μ_S is *feasible for a coalition* S if $\mu_S(S)$ is in $\mathcal{B}(G)$ and

$$\eta(\mu_S^{-1}(E)) = \nu(E), \quad \forall E \in \mathcal{B}(G) \cap \mu_S(S). \tag{2.5}$$

The following are two examples of economies and allocations:

Example 2.2. Consider an economy $\mathcal{E}$ where A and G are finite sets with the same cardinality n, and the function u in (2.3) is represented as a square matrix $[u(a,g)]_{\{a \in A, g \in G\}}$ of rank n. Let η and ν be uniform probability distributions over the sets A and G, respectively: that is, $\eta(a) = \nu(b) = \frac{1}{n}$ for all $a \in A$ and $g \in G$. In this case, any bijective function $\mu : A \to G$ is a feasible allocation. The economy in this case is equivalent to the one of Shapley and Scarf [22].

Example 2.3. Consider an economy $\mathcal{E}$ where A and G are both the interval $[0,1]$, and the function u in (2.3) is $u(a,g) = a^2 + g^2$. Let η and ν be a uniform probability distribution over the sets A and G, respectively, that is, $d\eta(a) = d\nu(b) = 1$ for all $a \in A$ and $g \in G$. In this case, the functions $\mu_1(a) = a$ and $\mu_2(a) = 1 - a$ are feasible allocations $\mathcal{E}$.

We say that a coalition $S \in \mathcal{B}(A)$ *can improve upon the allocation* μ if $\eta(S) > 0$ and there exists a feasible allocation γ_S for S such that $u(a, \mu(a)) < u(a, \gamma_S(a))$ for all a in S.

Definition 2.4. The core $\mathbf{C}(\mathcal{E})$ of a economy $\mathcal{E}$, is the set of all feasible allocations of $\mathcal{E}$ that no coalition in $\mathcal{B}(A)$ can improve upon.

2.2. *The social planner's problem*

In this section we consider a decision-maker who searches for an assignment that maximizes the social welfare function. This decision-maker is known as the social planner. Let $\mathcal{L}$ be the set of all feasible allocations, i.e.,

$$\mathcal{L} := \left\{\mu : A \to G \ : \ \ \eta(\mu^{-1}(E)) = \nu(E), \forall E \in \mathcal{B}(G)\right\}. \quad (2.6)$$

Consider the social planner's problem

$$\max_{\mu \in \mathcal{L}} \int_A u(a, \mu(a))\eta(da) \quad (2.7)$$

with $\mathcal{L}$ as in 2.6.

The following proposition establishes the relation between the core of the economy $\mathbf{C}(\mathcal{E})$ and the social planner's problem (2.7).

Proposition 2.5. *Suppose that μ^* is a solution of the social planner's problem (2.7), then μ^* is in $\mathbf{C}(\mathcal{E})$.*

Proof. Suppose that μ^* maximizes (2.7) and it is not in $\mathbf{C}(\mathcal{E})$. Then there exists $S \in \mathcal{B}(A)$ with $\eta(S) > 0$ and a feasible allocation γ for S such that $u(a, \gamma(a)) > u(a, \mu(a))$ for all a in S.

Consider the allocation

$$\mu(a) = \begin{cases} \mu^*(a) & \text{if} \quad a \notin S \\ \gamma(a) & \text{if} \quad a \in S \end{cases}.$$

Let $E \in \mathcal{B}(G)$, then

$$\begin{aligned}\eta(\mu^{-1}(E)) =& \eta(\mu^{-1}(E \cap \gamma(S))) + \eta(\mu^{-1}(E \cap (G - \gamma(S))))\\ =& \eta(\gamma^{-1}(E \cap \gamma(S))) + \eta(\mu^{*-1}(E \cap (G - \gamma(S))))\\ =& \eta(\gamma^{-1}(E \cap \gamma(S))) + \eta(\mu^{*-1}(E - E \cap \gamma(S))))\\ =& \eta(\gamma^{-1}(E \cap \gamma(S))) + \eta(\mu^{*-1}(E) - \mu^{*-1}(E \cap \gamma(S)))\\ =& \eta(\gamma^{-1}(E \cap \gamma(S))) + \eta(\mu^{*-1}(E)) - \eta(\mu^{*-1}(E \cap \gamma(S)))\\ =& \nu(E \cap \gamma(S)) + \nu(E) - \nu(E \cap \gamma(S))\\ =& \nu(E).\end{aligned}$$

Then μ is a feasible allocation for $\mathcal{E}$ and satisfies

$$\begin{aligned}\int_A v(a, \mu^*(a))\eta(da) =& \int_{A-S} u(a, \mu^*(a))\eta(da) + \int_S u(a, \mu^*(a))\eta(da)\\ <& \int_{A-S} u(a, \mu^*(a))\eta(da) + \int_S u(a, \gamma(a))\eta(da)\\ =& \int_{A-S} u(a, \mu(a))\eta(da) + \int_S u(a, \mu(a))\eta(da)\\ =& \int_A u(a, \mu(a))\eta(da).\end{aligned}$$

Therefore, μ^* is not optimal for (2.7), which it is a contradiction. □

Example 2.6. Consider an economy $\mathcal{E}$ as in Example 2.2. In this case, $\mathcal{L}$ [as in (2.6)] is the set of all bijective functions $\mu : A \to G$. The social planner's problem is given by the optimization problem:

$$\max_{\mu \in \mathcal{L}} \frac{1}{n} \sum_{a \in A} u(a, \mu(a)).$$

2.3. The core and feasible allocations

Consider an economy $\mathcal{E}$. If μ^* is a solution to the problem (2.7), then by Proposition 2.5, μ^* is in $\mathbf{C}(\mathcal{E})$ and therefore, the core of

$\mathcal{E}$ is not empty. But the set of feasible allocations $\mathcal{L}$ in (2.6) is not necessarily compact or convex; moreover, it may be empty (as in the example 2.7). In any case, (2.7) may have no solution. The following example provides a case where the set of feasible allocations $\mathcal{L}$ is empty.

Example 2.7. Consider a population of agents $\mathbf{A} := (A,\ \mathcal{B}(A), \delta_a)$ and a population of goods $\mathbf{G} := (G,\ \mathcal{B}(G), \nu)$, where δ_a is Dirac probability measure at $a \in A$ and ν is defined by

$$\nu(E) := \frac{1}{2}\delta_{g_1}(E) + \frac{1}{2}\delta_{g_2}(E) \quad \forall E \in \mathcal{B}(G)$$

where δ_{g_1} and δ_{g_2} are Dirac probability measures on G with $g_1 \neq g_2$. This example describes a situation in which we only have two types of goods and one type of agent. In this case $\mathcal{L} = \emptyset$.

3. A continuum economy

Since the set of feasible allocations $\mathcal{L}$ in (2.6) may be empty, we cannot define the concept of an economy for arbitrary populations **A** and **G** (as (2.1) and (2.2), respectively). We work with a different concept of allocation, which ensures that the set of feasible allocations is not empty. Moreover, with this new definition, the social planner's problem always has a solution.

3.1. A continuum economy

As in Section 2.1, we consider a population of agents **A** as in (2.1), and a population of indivisible goods, as in (2.2). Finally, we assume that the preference relations $\{\precsim_a\}_{a\in A}$ satisfy **H1**, **H3** and **H3**.

An economy with a continuum of indivisible goods, or a *continuum economy*, is a quadruple $\mathcal{E}_\Pi := (\mathbf{A}, \mathbf{G}, u, \pi_0)$, where u is a continuous function which satisfies (2.3), and π_0 is a probability measure on $A \times G$, which for each set of types of agents E (for E in $\mathcal{B}(G)$) assigns the proportion $\pi(D \times E)$ of types of agents in D (for D in $\mathcal{B}(A)$). Moreover, π_0 satisfies that for all D in $\mathcal{B}(A)$ and E in $\mathcal{B}(G)$, $\pi_0(D \times G) = \eta(D)$ and $\pi_0(A \times E) = \nu(E)$.

A *continuum allocation* for an economy $\mathcal{E}_\Pi$ is a probability measure π on $A \times G$. A continuum allocation π for $\mathcal{E}_\Pi$ is *feasible* if for all D in $\mathcal{B}(A)$ and E in $\mathcal{B}(G)$,

$$\pi(A \times E) = \nu(E) \ \text{ and } \ \pi(D \times G) = \eta(D). \tag{3.8}$$

Next is an example of a continuum economy.

Example 3.1. Let $\mathcal{E}_\Pi$ be an economy where $\mathbf{A}$, $\mathbf{G}$, and u are as in Example 2.7. The set of continuum allocations Π for $\mathcal{E}_\Pi$ is not empty because it contains at least the product measure $\pi = \delta_a \times \mu$ defined by

$$\pi(E) := \frac{1}{2}\delta_{(a,g_1)}(E) + \frac{1}{2}\delta_{(a,g_2)}(E) \quad \forall E \in \mathcal{B}(A \times G).$$

A *coalition* is a set S in $\mathcal{B}(A)$. A continuum allocation π_S is *feasible for a coalition* S in a set $H \in \mathcal{B}(G)$ if for any E in $\mathcal{B}(G) \cap H$, $\pi_S(A \times E) = \nu(E)$ and for all D in $\mathcal{B}(A) \cap S$, $\pi_S(D \times G) = \eta(D)$.

A coalition S can *improve upon the continuum allocation* π in a set $H \in (G)$ if $\eta(S) > 0$, $\nu(H) > 0$, and there exists a feasible allocation measure π_S for S that satisfies the following conditions:

$i)$ for each E in $\mathcal{B}(G) \cap H$, $\pi_S(S \times E) = \pi(S \times E)$;

$ii)$ for each D in $\mathcal{B}(A) \cap S$, $\pi_S(D \times H) = \pi(D \times H)$;

$iii)$ for each D in $\mathcal{B}(A) \cap S$ and E in $\mathcal{B}(G) \cap H$, with $\eta(D) > 0$ and $\nu(E) > 0$

$$\int_{D \times E} u(a,g)\pi(da,dg) < \int_{D \times E} u(a,g)\pi_S(da,dg).$$

Definition 3.2. The core $\mathbf{C}(\mathcal{E}_\Pi)$ of a continuum economy $\mathcal{E}_\Pi$ is the set of all continuum allocations of $\mathcal{E}_\Pi$ that no coalition in $\mathcal{B}(A)$ can improve upon.

3.2. Allocations in $\mathcal{E}$ and $\mathcal{E}_\Pi$

In this section we compare the definitions of allocations for the economies $\mathcal{E}$ and $\mathcal{E}_\Pi$. Let μ be an allocation of an economy $\mathcal{E}$. We can rewrite μ as a continuum allocation π_μ for an economy $\mathcal{E}_\Pi$ as follows: for any $K \in \mathcal{B}(A \times B)$ let

$$\pi_\mu(K) := \eta(K_a) \text{ with } K_a := \{a \in A \;:\; (a, \mu(a)) \in K\}. \quad (3.9)$$

The following examples illustrate how an assignment μ can be rewritten as a continuum allocation π_μ.

Example 3.3. Let $\mathcal{E}_\Pi$ be an economy where A, G, η, ν and u are as in Example 2.2. Then for any bijective function $\mu : A \to G$, the probability measure π_μ defined by

$$\pi_\mu(a, g) = \begin{cases} \frac{1}{n} & \text{if} \quad g = \mu(a) \\ 0 & \text{otherwise} \end{cases}$$

is a feasible continuum allocation for the economy $\mathcal{E}_\Pi$.

Note that if μ is a feasible allocation for $\mathcal{E}$, then π_μ in (2.9) satisfies (2.8), i.e., π_μ is a feasible continuum allocation of $\mathcal{E}_\Pi$. Furthermore, if μ_S is feasible for a coalition S in economy $\mathcal{E}$, then π_{μ_S} is a feasible continuum allocation for a coalition S in a set $H_S = \mu_S(S) \in \mathcal{B}(G)$, because for any E in $\mathcal{B}(G) \cap H_S$ and D in $\mathcal{B}(A) \cap S$, we have that

$$\pi_{\mu_S}(A \times E) = \eta(\mu_S^{-1}(E)) = \nu(E),$$

$$\pi_{\mu_S}(D \times G) = \eta(D \cap \mu_S^{-1}(\mu_S(D))) = \eta(D).$$

Proposition 3.4. *Suppose that a coalition S can improve upon the allocation μ for an economy $\mathcal{E}$ through the feasible allocation μ_S for S. Then S can improve upon the continuum allocation π_μ for an economy $\mathcal{E}_\Pi$ through the feasible continuum allocation π_{μ_S} for S in a set $H_S = \mu_S(S)$.*

Proof. Note that $H_S = \mu_S(S)$ is in $\mathcal{B}(G)$ and

$$\nu(H_S) = \eta(\mu_S^{-1}(\mu_S(S)) \geq \eta(S) > 0.$$

Moreover, by (2.9), for any feasible allocation φ_S, E in $\mathcal{B}(G) \cap H_S$, and D in $\mathcal{B}(A) \cap S$, we have that

$$\pi_{\varphi_S}(D \times H_S) = \eta(D \cap \varphi_S^{-1}(H_S)) = \eta(D)$$

and

$$\pi_{\varphi_S}(S \times E) = \eta(S \cap \varphi_S^{-1}(E)) = \eta(\varphi_S^{-1}(E)) = \nu(E).$$

Therefore, if E is in $\mathcal{B}(G) \cap H_S$, and D is in $\mathcal{B}(A) \cap S$, then $\pi_{\mu_S}(D \times H_S) = \pi_\mu(D \times H_S)$ and $\pi_{\mu_S}(S \times E) = \pi_\mu(S \times E)$.

On the other hand, note that for any allocation φ of $\mathcal{E}$, E in $\mathcal{B}(G)$, and D in $\mathcal{B}(A)$, we have that

$$\int_{D \times E} u(a,g)\pi_\varphi(da,dg) = \int_{D \cap \varphi^{-1}(E)} u(a,\varphi(a))\eta(da). \quad (3.10)$$

Since $u(a,\mu(a)) < u(a,\mu_S(a))$ for all $a \in S$ and (2.10), if E is in $\mathcal{B}(G) \cap H_S$, and D is in $\mathcal{B}(A) \cap S$, with $\eta(E) > 0$ and $\nu(E) > 0$, then

$$\int_{D \times E} u(a,g)\pi_\mu(da,dg) < \int_{D \times E} u(a,g)\pi_{\mu_S}(da,dg)$$

and we finish the proof. □

3.3. *The social planner's problem*

Consider (2.9), and let $\mathcal{L}$ be as in (2.6). We define the set

$$\Pi_{\mathcal{L}} := \{\pi \in \mathbb{P}(A \times G) : \pi = \pi_{\mu}, \ \ \mu \in \mathcal{L}\}, \tag{3.11}$$

where $\mathbb{P}(A \times G)$ is the set of probability measures on $A \times G$.

In this case, we can rewrite problem (2.7) as

$$\max_{\pi \in \Pi_{\mathcal{L}}} \int_{A \times G} u(a, g)\pi(da, dg). \tag{3.12}$$

The set of feasible allocations (2.11) is not necessarily convex or compact; moreover, it may be empty (as in the example 2.7). In any case, (2.12) may have no solution. For solving this situation, we expand the set of feasible allocations (2.6)-(2.11) by the convex set of Π, which is the set of all feasible continuum allocations, i.e.,

$$\Pi := \{\pi \in \mathbb{P}(A \times G) : \ \ \pi \text{ satisfies } (2.6)\}. \tag{3.13}$$

As in Section 2.2, we consider a social planner who searches an assignment that maximizes the social welfare function. The social planner's problem for a continuum economy $\mathcal{E}_{\Pi}$ is

$$\max_{\pi \in \Pi} \int_{A} u(a, g)\pi(da, dg) \tag{3.14}$$

with Π as in 2.13.

This approach generalizes the following two examples found in the literature.

Example 3.5. Consider an economy $\mathcal{E}_{\Pi}$ as in Example 3.3. Then the set of continuum allocations Π is given by the set of probabilities π that satisfies

$$\sum_{a \in A} \pi(a, g) = 1/n, \quad \sum_{g \in G} \pi(a, g) = 1/n.$$

Hence, the social planner's problem for this economy is

$$\max_{\pi \in \Pi} \sum_{a \in A} \sum_{a \in G} u(a, g)\pi(a, g).$$

The equivalence between the social planner's problem for an economy $\mathcal{E}$ in Examples 2.2 and 2.6, and the social planner's problem for an economy $\mathcal{E}_\pi$ can be seen in Koopmans and Beckmann [16].

The following proposition establishes the relation between the core $\mathbf{C}(\mathcal{E}_\Pi)$ of a continuum economy and the social planner's problem (2.14).

Proposition 3.6. *Suppose that π^* is solution to the social planner's problem (2.14); then π^* is in $\mathbf{C}(\mathcal{E}_\Pi)$.*

Proof. Suppose that π^* maximizes (2.14) and is not in $\mathbf{C}(\mathcal{E}_\Pi)$. Then there exists a coalition S that can *improve upon the continuum allocation* π^* in a set $H \in \mathcal{B}(G)$ (with $\eta(S) > 0$ and $\nu(H) > 0$) through a feasible allocation π_S for S.

Consider the continuum allocation π defined by

$$\pi(H) = \pi^*(H \cap (A \times G)) - \pi^*(H \cap (S \times H)) + \pi_S(H \cap (S \times H))$$

for all H in $\mathcal{B}(A \times G)$. Let D be in $\mathcal{B}(A)$ and E in $\mathcal{B}(G)$, then

$$\begin{aligned}
\pi(D \times G) =& \pi^*((D \times G) \cap (A \times G)) - \pi^*((D \times G) \cap (S \times H)) \\
& + \pi_S((D \times G) \cap (S \times H)) \\
=& \pi^*(D \times G) - \pi^*((D \cap S) \times H) + \pi_S((D \cap S) \times H) \\
=& \pi^*(D \times G) \\
=& \mu(D).
\end{aligned}$$

$$
\begin{aligned}
\pi(A \times E) =& \pi^*((A \times E) \cap (A \times G)) - \pi^*((A \times E) \cap (S \times H)) \\
&+ \pi_S((A \times E) \cap (S \times H)) \\
=& \pi^*(A \times E) - \pi^*(S \times (E \cap H)) + \pi_S(S \times (E \cap H)) \\
=& \pi^*(A \times E) \\
=& \mu(E).
\end{aligned}
$$

Then π is a feasible allocation measure and satisfies

$$
\begin{aligned}
\int_{A \times G} u(a,g)\pi^*(da,dg) <& \int_{A \times G} u(a,g)\pi^*(da,dg) \\
&- \int_{S \times H} u(a,g)\pi^*(da,dg) \\
&+ \int_{S \times H} u(a,g)\pi_S(da,dg) \\
=& \int_{A \times G} u(a,g)\pi(da,dg).
\end{aligned}
$$

Therefore, π^* is not an optimal for problem (2.14), which is a contradiction. □

4. The core of a continuum economy

In this section, we establish conditions that are sufficient for the core of an economy $\mathcal{E}$ not to be empty. Also, we establish important results about the relation between cores of economies $\mathcal{E}$ and $\mathcal{E}_{\Pi}$.

The optimization problem (2.7) is among the oldest and most well known problems in probability theory. It was introduced by Gaspar Monge in 1728, but it was posed as a mathematical linear problem (2.12) by L.V. Kantorovich in 1942. The solvability of (2.14) has been studied under a wide variety of hypotheses on the underlying spaces A and G, and/or the function u. For instance, see Hernández–Lerma and Gabriel [13], Jiménez-Guerra and Rodríguez-Salinas [15]. A classical reference to this topic is Villani [23].

Proposition 4.1. *Consider the economy $\mathcal{E}_\Pi$. Then there exists a solution to (2.14), i.e., there exists a continuum allocation $\pi^* \in \Pi$ (with Π as in (2.12)) such that*

$$\int_A u(a,g)\pi^*(da,dg) = \max_{\pi\in\Pi} \int_A u(a,g)\pi(da,dg). \tag{4.15}$$

Proof. See Santambrogio [21] pages 4-5, Theorem 1.4. □

The next theorem establishes conditions that are sufficient for the core of a continuum economy not to be empty.

Theorem 4.2. *Consider the hypothesis endowed to economy $\mathcal{E}_\Pi$; the core $\mathbf{C}(\mathcal{E}_\Pi)$ is not empty.*

Proof. Consider the hypotheses on the economy $\mathcal{E}_\Pi$. Then by Proposition 4.1, there exists $\pi^* \in \Pi$ that satisfies (4.15), i.e., π^* is a solution to (2.14). By Proposition 3.6, π^* is in $\mathbf{C}(\mathcal{E}_\Pi)$, and so Theorem 4.2 is satisfied. □

5. Comments

In this paper we introduce two economies with indivisible goods where agents and commodities are classified in types, and the sets of types of agents and types of goods are metric spaces. The number or mass of agents (and commodities) of a certain type is measured by a probability distribution. In addition, each agent has no use for more than one indivisible good.

In the first economy, the concept of allocation is a measurable function which assigns to each type of agent one type of good. In the second economy, (which is called continuum economy), the concept of allocation is a probability distribution which assigns a mass of agents to a mass of goods. For each economy, we associated a social planner's problem and established the relation between the core of the economy and the social planner's problem (see Propositions 2.5 and 3.6). We showed that the first model is contained in the second model (see Section 3.2). Finally, we established general conditions for the no emptiness of the core (see Theorem 4.2). In addition, a continuum economy model can be reduced, of course, to the particular case where sets of agents and goods are finite sets.

There are many questions, however, that remain open. For example, for the particular case where the set of agents and goods are finite, Shapley and Scarf [22] prove that the TTC algorithm induces an allocation which is in the core. Is there a "continuum" TTC algorithm for a continuum economy? In that case, what are the relations between the social planner's problem and the "continuum" TTC's algorithm?

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VII. ENTRY MODELS OF THE RADIO AND NEWSPAPER MARKETS IN MEXICO

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Abstract

Theory predicts that larger markets could accommodate more products; however, when consumers have heterogeneous preferences and products are differentiated, the market may remain concentrated even in the presence of numerous consumers. Moreover, when the quality of products comes, mainly, through fixed costs, firms in larger markets would require a higher market share to receive positive profits. In this chapter, we explore these hypotheses in the context of Mexico. We find that larger municipalities have more radio stations and daily newspapers. While both relationships are positive, the elasticity of the number of stations with respect to population is 0.8, and the elasticity of the number of newspapers is 0.2.

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1. Introduction

Theory predicts that as market size increases, the number of firms increases as well. As demand expands, the market could accommodate more firms; however, the intensity of this positive relationship depends on the magnitude of sunk costs and the competition between the firms operating in that market (Sutton [10]; Asplund and Sandin [2]).

Asplund and Sandin [2] explain that when new firms enter as a market expands, assuming low or null barriers to entry, variable profits per firm diminish, and the market share of firms should be higher to cover fixed costs. Moreover, if firms lower prices in the presence of more competitors due to broader market size, net profit per firm falls, as well.

Shaked and Sutton [8] state that under certain assumptions, as the size of the market increases,[1] the level of concentration may remain unaffected. In this way, even when the number of firms in the industry becomes arbitrarily large, at least one firm may keep a positive market share and many products that otherwise would disappear, remain in the market.[2]

In this chapter, we examine descriptive data on the relationship between market size and entry of media products, where the observations are cross-sections of Mexico's market areas (mostly municipalities).

In particular, we study the association between the number of radio stations/newspapers and the size of the population within a municipality in Mexico, excluding Mexico City.[3] We would expect

[1]Total number of consumers.

[2]For instance, multi-product firms which are common in differentiated markets.

[3]We exclude Mexico City because we do not have disaggregated informa-

to see a positive relationship where large populations accommodate more radio stations and newspapers.[4]

It is worth noting that, although the market size in media outlets can be defined from the advertisers' point of view or population served, we concentrate on the population that could potentially listen, or read papers, in a specific geographical area. However, we are aware that the total population in an area may not be all media outlet consumers. This analysis is a first approach to study some stylized facts related to the entry of radio stations and newspapers in Mexico.

Our chapter contributes to the literature that documents the relationship between market size and entry (Berry and Waldfogel [4]; Berry and Waldfogel [5]). To our best knowledge, this is the first attempt to provide evidence in this area for Mexico.

2. Industry background

Media outlets share the characteristic that their cost structure is mainly fixed costs, which are usually high,[5] and the availability of media products relies on many consumers wanting them (Anderson and Waldfogel [1]).

Media products cater to different consumers' preferences. For instance, the US radio broadcasting industry divides its formats into around 30 station types (Anderson and Waldfogel [1]). These

tion at the municipality-level for radio stations, and since we want to compare the radio and newspaper industries, this allows us to make the samples as comparable as possible.

[4]We would assume that a large population implicitly means a large market with a large number of potential radio listeners and newspapers readers.

[5]Fixed costs in daily newspapers are large relative to the market size (George and Waldfogel [6].

formats cater to different demographic, religious, or political backgrounds.[6]

Henceforth, in markets with substantial fixed costs and heterogeneous preferences, such as media outlets, as market size increases, the number of products available increases, as well (Waldfogel [11]). Also, the set of available products is determined, at least in part, by the size of populations wanting them (Berry et al. [3]).

Large markets do not mean "fragmentation" when products are differentiated and costs to achieve better quality rely on the fixed costs (Shaked and Sutton [9]). Radio stations and newspapers are both horizontally and vertically differentiated (Waldfogel [11]): horizontally because they differ in their content, and vertically, because they differ in quality.

Moreover, there exists an upper bound to the number of firms that could make positive profits when consumers always prefer a better quality product, and in equilibrium, even products with low quality may not survive at a price equal to zero (Shaked and Sutton [8]).

Waldfogel [11] coins the term "preference externalities" to explain the phenomenon where agents consume goods that they find attractive as long as others share their preferences. To exemplify the latter, consider the following: consider a two-type population with non-overlapping preferences, where each listens only to its own-type of radio format and each radio station chooses only one format. Without loss of generality, assume that Type 1 only listens to *News*, and Type 2, only to *Pop music*. In that case, cross-group preferences are zero, and only the growth of each type gives rise to more stations with *News* or *Pop music* formats,

[6]Waldfogel [11] finds that across 247 US radio markets, there exist sharp radio formatting preference differences between Hispanics and non-Hispanics.

respectively.

Now, assume that Type 2 are the minority and that some proportion listens to *News* stations. The fact that Type 2 has a second-choice format initially attracts more *News* (Type 1 first-choice) radio stations to enter; however, as the size of Type 2 increases, firms find it profitable to enter as *Pop music* stations (Type 2 first-choice) and, at some point, the number of *News* stations may fall.

George and Waldfogel [6] present evidence about the US market for printed newspaper consumption. They document the effect of the size of a population, race and Hispanic status on the product variety within MSA. The authors find positive own-race (or Hispanic) externalities, where, for instance, additional white consumers only bring more white-targeted newspapers to the MSA.

In this chapter, we consider both radio stations and newspapers that are physically present in a constrained geographical area (municipalities); nevertheless, when technology decreases the cost of distribution in media outlets, the link between local preferences and local products may fade, and local consumers may favor outside options that better cater to their preferences (George and Waldfogel [7]).

Although we concentrate on the relationship between the size of the market and the number of products, it is important to discuss that quality in media outlets and fixed costs also play an essential role in the number and type of products.

Since consumers have heterogeneous preferences and decide which product to choose according to their tastes and/or their willingness to pay for quality, one fundamental relationship that defines the number of firms present in the market is the consumer's willingness to pay for quality, and whether this improvement comes through fixed or variable costs (Shaked and Sutton [9]).

It is worth noting that when variable profit covers fixed costs, product proliferation arises, and, in equilibrium, every segment of the quality line eventually contains at least some product (Berry and Waldfogel [5]). Thus, in a large market, we would expect to see, on average, more products of all quality types.

However, this positive relationship may be weakened, and the entrance of new firms/products may be slower in much larger markets. For instance, in large cities: i) the cost of living is more expensive and, ii) radio stations may have the incentive to spend more than is proportional to increase their quality (Anderson and Waldfogel [1]).

To illustrate an enhancement in a media outlet, consider that the owner of a radio station or newspaper wishes to improve the quality of a format, or at least to be perceived as enhancing quality. To do that, it would require a better-paid staff of reporters, producers and well-known personalities, regardless of the number of readers or listeners.

Thus, the enrichment of quality in media outlets comes, mainly, through fixed costs and an almost negligible increase in variable costs. Moreover, an entrant may be required to pay an entry cost such as a radio concession or certification to legally print and circulate a written issue, as is the case in Mexico.

3. Market size and number of firms

In this section, we present a simple model to illustrate why we would expect that when the market size is large, we should see more media outlets present in it. Following Anderson and Waldfogel [1], consider a logit model and the following assumptions:

- One product is targeting one single type of consumers.
- Products are symmetrically differentiated.

- All products have equal market shares.
- Each product has an entry cost, F.
- The economic mass, M, is defined as *Number of consumers* x *Economic value to advertisers*. Consider M as the market size.

The share of population consuming product i is given by:

$$P_i = \frac{e^s}{1 + ne^s} \tag{3.1}$$

where n is the number of products entering the market, and s is the attractiveness of listening/reading.

Note from Eq. 3.1 that under free entry:[7]

$$F = M * P_i = \frac{M}{n + e^{-s}}. \tag{3.2}$$

Thus, using Eq. 3.1 and Eq. 3.2, the number of products [8] in equilibrium is:

$$n = \frac{M}{F} - e^{-s}. \tag{3.3}$$

The number of products increases in the following cases:

1. When the economic mass (M) increases. It may be because the number of consumers increases, or because their economic value increases, or both.
2. When the attractiveness to listen/read (s) increases.

[7]Firms will enter until profits equal zero.
[8]The number of products, n, is an integer.

3. When the entry cost (F) decreases.

Thus, the direct empirical implication of this model is that we might expect to see more radio stations and newspapers in larger markets (with a higher M), where the size of the market is proxied by the municipality's population in our empirical setting, keeping constant the entry cost and attractiveness to consume. Ideally, in our analysis, we would want to control for F and s. However, due to data limitations in the empirical section, we have not been able to do this.

4. Data sources and legal framework in Mexico

The dataset for this chapter is a set of cross sections of product availability at the municipal level[9] in the daily newspaper and radio broadcasting industries for municipalities with populations equal to or greater than 50,000. Demographic characteristics of the municipalities, such as population, median income, the fraction of people with college educations, master's degrees or PhDs, the fraction of people younger than 25 years old, or 65 years old or older, etc. come from the 2015 Intercensal Survey published by the National Institute of Statistics and Geography (INEGI).

[9] There exist 2,457 municipalities in Mexico.

4.1. Radio stations

On July 14, 2014, the New Federal Telecommunications and Broadcasting Law (“Law") was published in the Official Federal Gazette of Mexico. According to the Law, the Federal Telecommunications Institute (“IFT" for its acronym in Spanish) is an autonomous and independent public agency responsible for regulating and promoting competition in the telecommunications markets within Mexico.

In Mexico, the law establishes that the radio spectrum is the sole property of the Nation, and the State is responsible for holding and managing it. Also, it establishes four types of radio broadcasting concessions according to its purposes: i) Commercial, ii) Public, iii) Private and iv) Social.

The IFT is responsible for implementing the bidding scheme (public auctions for commercial use) to allocate the radio spectrum frequency bands. Also, the IFT provides data on the total number of commercial radio stations at the local level updated in February of 2019.

Since a radio station operates in an area around the size of a municipality,[10] we assign a radio station operating in locality l in municipality m to municipality m, no matter where it is located.

Table VII.1 reports means, standard deviations, minima, and maxima of the municipality-level variables for the 258 municipalities included in the estimates. The population of the municipalities in the sample ranges from 50,377, to over 1.6 million. The number of commercial radio stations varies from 1 to 27, with an average of 5.3. The average of the median head of household income in the municipalities of the sample is \$4,291 (219 USD) and the average fraction having college or higher than college is 12%.

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[10] According to our talks with people at the IFT.

Table VII.1: Sample Characteristics: Radio Industry

Variable	Mean	Std. Dev.	Min	Max
		Radio (Obs. 258)		
Number of stations	5.3	5.5	1.0	27.0
Population (thousands)	253.63	303.38	50.38	1641.57
Median Income	4,290.53	917.22	2,143.00	6,800.00
Fraction with college or higher	0.12	0.06	0.03	0.29
% younger than 25 years old	0.46	0.03	0.36	0.56
% 65 years or older	0.07	0.02	0.02	0.12
% of women	0.51	0.01	0.48	0.55

4.2. Newspapers

The Ministry of the Interior (“SEGOB" for its acronym in Spanish) is responsible for certifying the legal circulation of print media in Mexico, including newspapers. Our data on daily newspapers comes from the National Registry of Print Media, subordinated to SEGOB, which provides an exhaustive list of print media throughout the Mexican territory.

The Registry contains detailed certified information about printed media. Specifically:

- Circulation: average number of copies according to publication frequency.
- Geographical coverage: states and municipalities where it is distributed.

For newspapers, we consider two datasets: full and restricted. Our full dataset includes all national and state level newspapers that are recorded as circulating in a given state(s) without detailing the circulation per municipality; when this happens, we impute that newspaper to all the municipalities of that state(s). Our restricted dataset includes only those newspapers that have information on circulation per municipality; in this case, we are sure that a newspaper is, indeed, present.

From the circulation figures, we compute the market share of the largest firm.[11] Our data are at the level of the municipality for municipalities with a population greater than 50,000.

In Table $VII.2$ we present summary statistics of the newspaper industry where, as described above, the mean of the number of products in a municipality is much higher in the full sample (34.2) than in the restricted one (5.7). As expected, we observe that other characteristics of the municipalities in both datasets are similar (population and median income) or equal.

5. Empirical Results on Market Size

As discussed in previous sections, we expect to see: i) more radio stations and newspapers in large markets, and ii) there is fragmentation in the newspaper industry; thus, there is no positive correlation between population and largest market share.

We show in Figure VII.2 the municipal population and number of radio stations (top) and the number of newspapers (bottom). Both show a positive relationship, but the linearity is more evident for stations than for newspapers.

There are several small municipalities with numerous newspapers, in number, much higher than their corresponding number of radio stations in the full sample. One possible explanation for this is that we impute a national or state level newspaper[12] to all municipalities within the state, regardless of its circulation.

[11] When calculating this measure, we exclude newspapers whose circulation is not disaggregated at the municipal level.

[12] There are no state or national radio stations because each one has a limited radio frequency.

Table VII.2: Sample Characteristics: Newspaper Industry

Variable	Mean	Std. Dev.	Min.	Max.
Panel A	Newspapers - Full sample (Obs. 409)			
Number of newspapers	34.2	11.4	2.0	58.0
Population (thousands)	203.02	262.69	50.38	1677.68
Median Income	4,102.82	1,110.53	0.00	8,571.00
Fraction with college or higher	0.10	0.06	0.00	0.41
% younger than 25 years old	0.47	0.04	0.36	0.63
% 65 years or older	0.07	0.02	0.02	0.12
% of women	0.51	0.01	0.48	0.55
Share of Largest Newspaper	0.5	0.2	0.2	1.0
Panel B	Newspapers - Restricted sample (Obs. 377)			
Number of newspapers	5.7	3.1	1.0	24.0
Population (thousands)	202.17	251.81	50.38	1641.57
Median Income	4,118.49	1,092.46	0.00	8,571.00
Fraction with college or higher	0.10	0.06	0.01	0.41
% younger than 25 years old	0.47	0.04	0.36	0.62
% 65 years or older	0.07	0.02	0.02	0.12
% of women	0.51	0.01	0.48	0.55
Share of Largest Newspaper	0.5	0.2	0.2	1.0

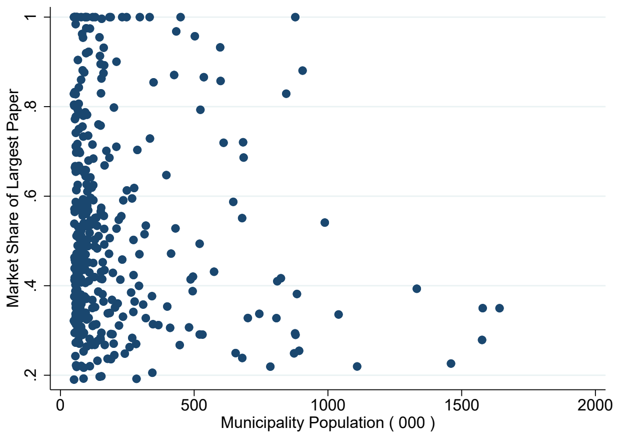
Figure VII.2 shows that the number of radio stations and the number of newspapers increase with market size, but the increase is faster for radio stations compared to both the full and restricted sample of newspapers.

Moreover, there is a positive association between market size and the number of newspapers, independent of the sample used in the analysis.

Our findings are in line with previous studies. For instance, Berry and Waldfogel [5] study the number of newspapers present in a Metropolitan Statistical Area in the US and find a positive relationship between the log number of outlets and market size proxied by log population. But while in the restaurant industry, this relationship is proportional (close to 1), in the newspaper industry it is 0.5. Their explanation for this difference relies on the fixed cost structures of media outlets.

Figure VII.1 shows the market share of the largest newspaper per municipal population, where the lower bound is at around 0.2.[13]

Figure VII.1: Market Size and Share of Largest Paper



Notes. The figure shows the largest newspaper's market share among readers across municipalities.

We can think of the "maximum share of the largest newspaper" as a proxy of the concentration of the industry. As the market grows larger, we would expect to see a more fragmented market. However, in markets with high fixed costs, differentiated products and consumers with heterogeneous preferences, the market may remain concentrated even in the presence of many consumers (Sutton [10]). Moreover, as discussed before, in media outlets we

[13]Berry and Waldfogel [5] find a lower bound of 0.2, as well.

expect to see a lower bound of market share for all firms to operate with positive profits in large markets.

We do not observe a systematic decline of the maximum market share as the population grows. Moreover, there are small municipalities with highly concentrated markets or highly fragmented ones (Figure VII.1).

Table VII.3 shows regressions of the numbers of newspapers or radio stations on market size, both in the log form of the raw numbers. The coefficients on log population (market size) are much smaller in newspapers than in radio stations, with and without controls, but are positive and significant.

The numbers of both newspapers and radio stations increase in market size, but the increase is much slower for newspapers (for both the full and restricted datasets). Without controls, the elasticity of population and market size is 0.84 for radio stations and 0.1 for newspapers in the full data, and 0.2 for the restricted.

The coefficients do no change drastically when we add controls on population characteristics: for radio stations and the restricted data of newspapers, these decrease to 0.75 and 0.15, respectively, and for newspapers in the full data, this remains almost the same.

6. Concluding Discussion

In this chapter, we analyze the relationship between the number of radio stations and newspapers and the size of the market in Mexico. As discussed, in markets in the presence of consumers with heterogeneous preferences (e.g., religion, political views or gender), or with differentiated goods and high fixed costs, such as media outlets, we would expect to see that fragmentation of the market does not increase as the size of the market increases.

We found a positive relationship in both markets between market size, and the number of products, with a higher elasticity existing in the number of radio stations with respect to population, than in the elasticity of newspapers. Moreover, when considering market concentration, we find that the market share of the largest newspaper in each municipality does not systematically decrease as the population grows.

As explored by Berry and Waldfogel [4], the free entry condition could lead to excessive entry in media markets compared to what is socially optimal. Further research could investigate, in the context of Mexico, what the socially optimal number of radio stations and newspapers should be. In this way, policymakers could potentially find another tool to set the rules of bidding to allocate the radio spectrum, or to promote more/fewer newspapers in some areas of the country.

Moreover, the duplicity problem in media outlets is also a well-known phenomenon which could be investigated for the case of Mexico to understand whether this represents a loss in welfare.

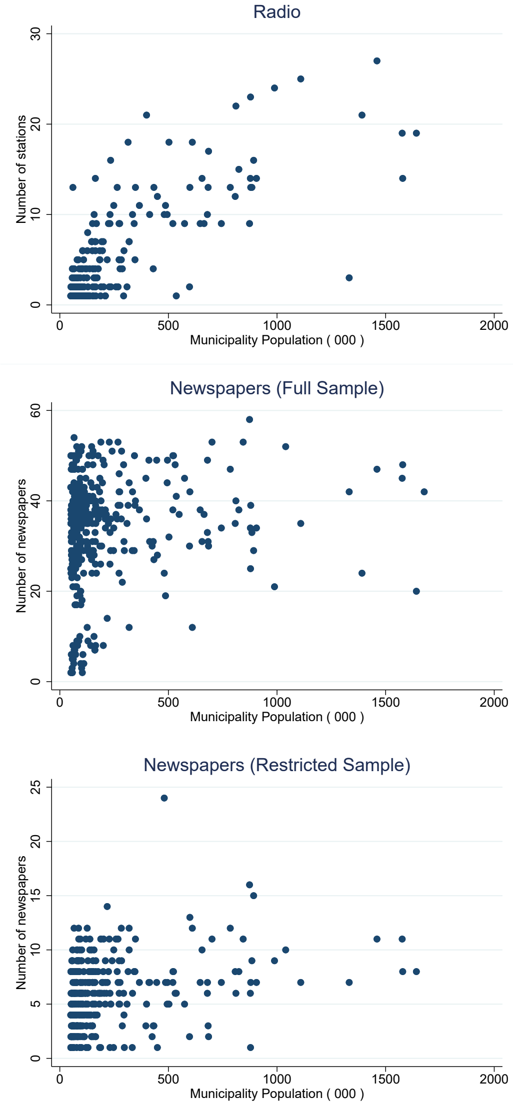
This chapter is a first attempt to provide stylized facts related to the current state of the radio and newspapers markets in Mexico.

Table VII.3: Market Size: Radio and Newspapers

	Radio		Newspapers (Full sample)		Newspapers (Restricted sample)	
Variables	Log N	Log N	Log N	Log N	Log N	Log N
Log Population	0.835***	0.750***	0.114***	0.108***	0.232***	0.153***
	(0.0508)	(0.0671)	(0.0342)	(0.0413)	(0.0384)	(0.0456)
Median Income		7.48e-05		0.000160***		-0.000226***
		(8.02e-05)		(4.02e-05)		(4.46e-05)
Fraction with College or higher		1.178		-1.033		1.065
		(1.390)		(0.807)		(0.876)
Fraction younger than 25 years old		-2.622		3.837***		-7.190***
		(2.588)		(1.244)		(1.352)
Fraction 65 years or older		3.883		-2.367		-13.09***
		(4.171)		(2.198)		(2.385)
Fraction of women		-5.954		11.44***		11.60***
		(5.200)		(3.127)		(3.452)
Constant	-3.074***	0.879	2.868***	-5.180***	0.434**	-0.0705
	(0.262)	(3.064)	(0.169)	(1.788)	(0.190)	(1.967)
Observations	226	226	409	409	377	377
R-squared	0.546	0.585	0.027	0.136	0.089	0.230

Notes. Standard errors in parentheses. *** Significant at 1%. **Significant at 5%. *Significant at 10%. The unit of observation for the newspaper and radio data is the municipality.

Figure VII.2: Number of Products and Market Size



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VIII. A NOTE ON THE BIG PUSH AS INDUSTRIALIZATION PROCESS

Saul Mendoza-Palacios and Alfonso Mercado

Abstract

This paper relates to the debate and formalization of some essential aspects of the "Big Push" (Rosenstein-Rodan [8]). We propose two economic models: static and dynamic. The static model is a simplified version of the Murphy et al. [6] model. In the dynamic model, we establish three possible steady states (SSs). In the first SS, all firms produce based on a traditional technology and both wages and aggregate income (AI) are low. In the second SS, all firms produce using a modern technology and both wages and AI are high. These two SSs are stable. The third SS is unstable and has characteristics of the first and second SSs. Firms choose the SS through an evolutionary dynamic, which explains the interrelation with the economic system. Finally, we establish general conditions under which firms select the best SS.

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1. Introduction

This paper relates to the debate and formalization of some essential aspects of the “Big Push” (Rosenstein-Rodan [8]),[1] an idea revised by Murphy et al. [6], Krugman [5] and others. In particular, our concern is which economic dynamics allow some countries, but not others, to successfully and rapidly increase their productivity and living standards, and how government intervention can accelerate this process. This paper proposes a simplified version of the Murphy et al. [6] and Basu [1] models, with two formulations; static and dynamic. The static model is presented in Section 2, and the dynamic model is explained in Section 3. From here, critical points of economic evolution and coordination are incorporated in Section 4, and Section 5 presents our final remarks.

This first static model is based on the Murphy et al. [6] model and is similar to the version of Basu [1]. In Section 2.3 we aggregate a dynamics to explain the evolution of the industrialization process, which moves an economic system (see Equations (3.21)-(3.25)).

[1]As Krugman [5] considered, the Big Push, proposed by Rosenstein-Rodan [8], has inspired many interesting interpretations and formalizations. Two of the closest formalizations to this proposal are the models of Murphy et al. [6], and Krugman [5], himself. The central idea is that the simultaneous industrialization of many interlinked sectors of the economy can be profitable even if none of them can be industrialized effectively by themselves. This idea is analyzed in the context of positive external economies.

2. A static economy model

This is a simplified version of the Murphy et al. [6] model and is similar to Basu [1]. Like these two models, here we assume a closed economy, producing N *types of commodities*, where there exist N *types of economic sectors*. More details about the aggregate demand, technology and aggregate income are given as follows:

2.1. The aggregate demand

Consider an inelastic supply of L *units of labor*. The consumers have the same preferences. The utility consumer's function is given by

$$u = x_1 x_2 \cdots x_N. \tag{2.1}$$

Let y be the *aggregate income* which will be defined in equation (2.11), and for $i = 1, \ldots, N$, let p_i be the *price of the good* i. Then the consumer problem is to maximize utility function (2.1) subject to the budget constraint

$$p_1 x_1 + \ldots + p_N x_N = y.$$

Hence, the *demand function* for the good i is

$$x_i := \frac{y}{N p_i}. \tag{2.2}$$

2.2. Market structure and technology

Firms select their technology to produce a commodity, either a traditional, or a modern technology. This defines two broad types of industry (traditional and modern). Firms also decide their commodity price. We assume that each sector i is represented by a decision making firm that decides the type of technology used. That is, each sector produces using only one type of technology.

Traditional firms

If the firm produces its commodity using a *traditional technology*, it is called a *traditional firm*. Like Murphy et al. [6] and Basu [1], we assume a perfect competition market for traditional firms. They transform one unit of labor l into one unit of output x_i. If a traditional firm in sector i pays a wage w_T, then its profit is given by

$$\pi_i^T = p_i x_i - w_T l. \tag{2.3}$$

If the economic sectors have a competitive market, then given a demand x_i, we have two equilibrium market conditions for a traditional firm in sector i

$$x_i = l, \tag{2.4}$$

$$p_i = w_T. \tag{2.5}$$

Modern firms

Modern firms are those that choose a *modern technology* for their production. In this case, we assume economies of scale at the firm level (as Murphy et al. [6] and Basu [1]), but no economies of scope nor economies of multi-sectoral operations. The technology is the same for all modern firms in the market for each type of commodity produced, involving a fixed input of F and marginal input requirement $c \in (0,1)$. Thus, assuming for the moment that the only input is labor, the production of a quantity x_i of a modern firm in sector i requires labor input given by

$$l = F + c x_i. \tag{2.6}$$

A modern firm does not have complete control over the price of its product, and it produces with economies of scale, using a fixed input of F of labor. That is, the modern firm is in a monopolistic competition market, so it may incur positive or negative profits. We assume that a modern firm in sector i pays a wage $w_M > w_T$. Then, with a mill price p_i and a demand x_i, its profit is given by

$$\pi_M^i = p_i x_i - w_M(F + cx_i). \tag{2.7}$$

The process of industrialization is a transition from a traditional to a modern industry. If the representative firm of the sector i is a modern firm, we say that sector i has been industrialized. From Equations (2.2), (2.5) if a sector i is industrialized, then the profit of a modern firm is

$$\pi_M(y) = \pi_i^M = \left(1 - \frac{w_M}{w_T}c\right)\frac{y}{N} - w_M F. \tag{2.8}$$

Note that we need the following condition:

$$\frac{w_M}{w_T} < \frac{1}{c}, \tag{2.9}$$

for a possible $\pi_M(y) \geq 0$. The expression, $\frac{1}{c} > 1$ represents the number of units of output by a unit of labor after incurring a fixed cost of F units of labor.

Let $\pi_M(n, y)$ be the sum of profits of n representative firms that choose a modern technology. Using Equation (2.8), we have that

$$\begin{aligned} \pi_M(y, n) &= n\pi_M(y) \\ &= n\left[\left(1 - \frac{w_M}{w_T}c\right)\frac{y}{N} - w_M F\right]. \end{aligned} \tag{2.10}$$

2.3. The aggregate income

All traditional firms in a given market use $\frac{y}{NW_T}$ units of labor. Hence, if n of these representative firms are industrialized, then by Equation (2.2), and Conditions (2.4) and (2.5), the economy uses $(N-n)\frac{y}{Nw_T}$ units of labor in the traditional productive sector, and $L-(N-n)\frac{y}{Nw_T}$ units of labor in the modern sector (already expanded by the new entrants). Moreover, the aggregate income when n sectors are industrialized is

$$\begin{aligned} y(n) &= \pi_M(y,n)+\pi_T(y,n)+w_T\left[(N-n)\frac{y}{Nw_T}\right] \\ &\quad +w_M\left[L-(N-n)\frac{y}{Nw_T}\right] \\ &= \left(\frac{w_T-w_M c}{w_T}\right)\frac{yn}{N}-\left(\frac{w_M-w_T}{w_T}\right)(N-n)\frac{y}{N} \\ &\quad +[L-nF]w_M \end{aligned} \tag{2.11}$$

where $\pi_T(n,y)$ is the sum of profits of $N-n$ industries (or representative firms) that are not industrialized which satisfies $\pi_T(n,y)=0$ by Equation (2.3), and Conditions (2.4) and (2.5). If each y in Equation (2.11) depended implicitly on n, that is $y=y(n)$, then we would have that

$$y(n)=\frac{[L-nF]Nw_T}{N-n(1-c)}. \tag{2.12}$$

Note that if no productive sector is industrialized, then

$$y(0)=\pi_T(0,y)+w_TL,$$

with $\pi_T(0,y)=0$.

2.4. *The profit of a modern firm*

If n productive sectors are industrialized, then by Equation (2.12) the profit of a representative modern firm π_M in Equation (2.8) is

$$\pi_M(n) = \frac{F[(w_T - w_M c)((L/F) - n) - w_M(N - n(1-c))]}{[N - n(1-c)]}. \tag{2.13}$$

Let $\lambda_M := \frac{n}{N}$ be the proportion of representative firms that choose a modern technology; then Equation (2.13) can be rewritten in terms of λ_T as

$$\pi_M(\lambda_M) = \frac{F[(w_T - w_M c)((L/NF) - \lambda_M) - w_M(1 - \lambda_M(1-c))]}{[1 - \lambda_M(1-c)]}. \tag{2.14}$$

If all the representative firms are industrialized, then the economy must supply at least NF labor units, and this condition is

$$\frac{L}{NF} > 1. \tag{2.15}$$

Since $0 < c < 1$, if condition (2.5) is satisfied, then the sign of $\pi_M(\lambda_T)$ is the same as

$$(w_T - w_M c)((L/NF) - \lambda_M) - w_M(1 - \lambda_M(1-c)), \tag{2.16}$$

and the sign of $\pi_M(\lambda_T)$ depends directly on the magnitude of λ_M.

3. A dynamic economy model

Let us make this model more dynamic to study the evolution of an industrialization process. First, we introduce an imitative-evolutionary dynamics to explain how the proportions of both modern and traditional firms are changing over time. Then we explain how this imitative-dynamics moves key variables of the economic system in Equations (3.21) to (3.25).

3.1. The industrialization process

We assume that firms choose a technology every time t, either a traditional or modern technology. The proportion of firms selecting a modern technology in time t is described by $\lambda_M(t) := \frac{n(t)}{N}$, and the proportion of firms choosing a traditional technology is described by $\lambda_T(t) := 1 - \lambda_M(t)$.

An imitative-evolutionary dynamics explains how the proportions of both groups of firms (those using modern technology and those with traditional technology) are changing over the time. Then, with this dynamic view, firms choose a technology through imitation, i.e., each firm selects a technology if its use is profitable. So, in this way, firms follow an imitative evolutionary dynamics known as replicator dynamics, which has interesting properties such as a simple mathematical form. It has a natural interpretation and describes imitation behaviors (see Hofbauer and Sigmund [3], Hofbauer and Sigmund [4], and Webb [9]). Moreover, it can be derived from models of interactive learning processes, (as Gale et al. [2]). The evolution of $\lambda_M(t)$ and $\lambda_T(t)$ is described by replicator dynamics as follows:

$$\dot{\lambda}_T(t) = \mu[\pi_T(\lambda(t)) - \bar{\pi}(\lambda(t))]\lambda_T(t) \tag{3.17}$$

$$\dot{\lambda}_M(t) = \mu[\pi_M(\lambda(t)) - \bar{\pi}(\lambda(t))]\lambda_M(t) \tag{3.18}$$

$$1 = \lambda_M(t) + \lambda_T(t), \tag{3.19}$$

where $\gamma > 0$ is a real number that explains the speed of imitation; $\pi_T(t)$ is the profit of a traditional firm and satisfies $\pi_T(t) = 0$ in any time t; $\pi_M(t)$ is the profit of a modern firm described by Equation (2.14); and finally,

$$\begin{aligned} \bar{\pi}(\lambda(t)) &:= \lambda_M(t)\pi_M(\lambda(t)) + \lambda_T(t)\pi_M(\lambda(t)) \\ &= \lambda_M(t)\pi_M(\lambda(t)). \end{aligned} \tag{3.20}$$

3.2. *The economy and technological change*

By Equation (3.19), $\dot{\lambda}_T(t) = -\dot{\lambda}_M(t)$ for all t. Hence, we can simplify the system of Equations (3.17) to (3.19), substituting Equation (3.20) in Equation (4.18), and using $\lambda(t) = \lambda_M(t), 1-\lambda(t) = \lambda_T(t)$, to obtain the *technological change equation* (3.21), which explains the technological change process. The evolution of technological change directly affects the economy as follows:

Technological change equation

$$\dot{\lambda}(t) = \mu\pi_M(\lambda(t))\lambda(t)[1-\lambda(t)], \tag{3.21}$$

Aggregate income

$$y(\lambda(t)) = \frac{[L/N - \lambda(t)F]Nw_T}{1-\lambda(t)(1-c)}. \tag{3.22}$$

Demand of good i

$$x_i(\lambda(t)) = \frac{y(\lambda(t))}{Nw_M}. \tag{3.23}$$

Aggregate welfare function

$$u(\lambda(t)) = x_1(\lambda(t))x_2(\lambda(t))\cdots x_N(\lambda(t)), \tag{3.24}$$

Benefit of a modern firm

$$\pi_M(\lambda(t)) = \frac{F\left[[w_T - w_M c][(L/NF) - \lambda(t)] - w_M[1-\lambda(t)[1-c]]\right]}{1-\lambda(t)[1-c]}. \tag{3.25}$$

4. Economic evolution, industrialization process and coordination

4.1. The steady states

Our Technological change equation (3.21) is a source of the dynamic process driving the economic system, as formulated by Equations (3.22) to (3.25). This equation has three critical points or steady states, i.e., values of λ where

$$\mu\pi_M(\lambda)\lambda[1-\lambda] = 0.$$

The three critical points of Equation (3.21) are:

$$\lambda^* = 0, \quad (4.26)$$

$$\lambda^* = 1, \quad (4.27)$$

$$\lambda^* = \frac{w_M}{w_M - w_T} - \frac{w_T - w_M c}{w_M - w_T}\frac{L}{NF}, \quad (4.28)$$

where Equation (4.26) refers to the case when all the firms use traditional technology, Equation (4.27) refers to the case when all the firms use modern technology, and Equation (4.28) refers to the case when $\pi(\lambda^*) = 0$ (for $\pi_M(\cdot)$, as in Equation (2.14)).

A steady state λ^* of Equation (3.21), is called

- *asymptotically stable state* (ASS) if for any small $\epsilon > 0$, there exists $\delta > 0$ such that if $|\lambda(0)-\lambda^*| < \delta$, then $|\lambda(t)-\lambda^*| < \epsilon$ for all $t > 0$, and $\lim_{t\to 0}\lambda(t) = \lambda^*$.

- *repulsive state* (RS) if for any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that $|\lambda(t) - \lambda^*| > \epsilon$ for all $t > t_\epsilon > 0$.

Theorem 4.1. *Consider Conditions (2.9), (3.15) and*

$$\frac{w_M}{w_T} > \left(1 - \frac{w_M c}{w_T}\right)\left(\frac{L}{NF}\right) > 1. \tag{4.29}$$

Then for the Technological change equation (3.21) we have that

$i)$ *State (4.26) is an ASS, and in this state, all firms use traditional technology;*

$ii)$ *State (4.27) is an ASS, and in this state, all firms use modern technology; and*

$iii)$ *State (4.28) it is a RS, and in this state, all firms decide to use traditional or modern technology (that is $\pi_T = \pi_M(\lambda^*) = 0$).*

Proof. Let $G(\lambda) := \mu\pi_M(\lambda)\lambda[1-\lambda]$, then System (3.21) can be rewritten as $\dot{\lambda} = G(\lambda)$. Hence,

- if $\frac{dG}{d\lambda}(\lambda^*) < 0$, λ^* is an asymptotically stable state;
- if $\frac{dG}{d\lambda}(\lambda^*) > 0$, λ^* is a repulsive state.

See, for example, Perko [7]. Now,

$$\frac{dG}{d\lambda} = \mu[\pi_M(\lambda)[1-2\lambda] + \lambda(1-\lambda)\pi_M'(\lambda)]$$

where

$$\pi_M'(\lambda) = \frac{F[(w_M - w_T) - \pi_M(\lambda)]}{[1-\lambda(1-c)]}.$$

Under Conditions (2.9) and (3.15), we have that $\pi_M(0) < 0$, see Appendix 6.1. Then, for the steady state $\lambda^* = 0$, we have that

$$\frac{dG}{d\lambda}(0) = \mu\pi_M(0) < 0,$$

which proves $i)$.

Under Conditions (2.9), (3.15) and (3.25), we have that $\pi_M(1) > 0$, see Appendix 6.2. Then for the steady state $\lambda^* = 1$, we have that

$$\frac{dG}{d\lambda}(1) = -\mu\pi_M(1) < 0,$$

which proves $ii)$. Consider the steady state λ^* defined in Equation (3.24). For this steady state $\pi_M(\lambda^*) = 0$, and $\pi'_M(\lambda^*) = \frac{F[w_M - w_T]}{1-\lambda^*(1-c)} > 0$. Then,

$$\frac{dG}{d\lambda}(\lambda^*) = \mu\lambda^*[1-\lambda^*]\pi'_M(\lambda^*) > 0,$$

which proves $iii)$. □

4.2. *Coordination effort*

Consider the steady state λ^* as in (4.28) and let $\lambda_0 = \lambda(0)$ be the initial state of industrialization in the economy. Then, by Theorem 4.1 we have that

- $i)$ if $\lambda_0 < \lambda^*$, then $\lambda(t)$ is decreasing and $\lim_{t\to\infty} \lambda(t) = 0$, i.e., the evolution of industrialization process is decreasing;
- $ii)$ if $\lambda_0 > \lambda^*$, then $\lambda(t)$ is increasing and $\lim_{t\to\infty} \lambda(t) = 1$, i.e., the evolution of industrialization process is increasing.

For all of the above arguments, the steady state λ^*, as in Equation (4.28), is called *the state of critical coordination effort.*

The value of the state of critical coordination effort λ^* is important because λ^* [as in Equation (4.28)] is the minimum proportion of firms that must be coordinated in an industrial policy to obtain an effective process of industrialization. For this, we are interested in variables that affect this value of λ^*.

Fixed and marginal costs

Consider Equation (4.28), then by Conditions (2.9) and (3.15), we have that

$$\frac{d\lambda^*}{dF} = \frac{w_T - w_M c}{w_M - w_T}\frac{L}{NF^2} > 0, \tag{4.30}$$
$$\frac{d\lambda^*}{dc} = \frac{w_M}{w_M - w_T}\frac{L}{NF} > 0, \tag{4.31}$$

which means that if there is a high fixed input cost F or a high marginal input cost c (for example, a value of c close to 1), then the state of critical coordination effort λ^* is also high.

The number of firms in the market and units of labor

Consider Equation (4.28), then, by Conditions (2.9) and (3.15) we have that

$$\frac{d\lambda^*}{dN} = \frac{w_T - w_M c}{w_M - w_T}\frac{L}{N^2 F} > 0, \tag{4.32}$$
$$\frac{d\lambda^*}{dL} = -\frac{w_T - w_M c}{w_M - w_T}\frac{1}{NF} < 0. \tag{4.33}$$

By the sign of Equation (4.32), we can affirm that the participation of new firms in the market implies a higher value in the critical minimum effort of coordination λ^*. Contrary to the effect of increments in N, the sign of Equation (4.33) means that if the quantity of labor L increases, then the state of critical coordination effort λ^* decreases.

The wages of modern firms w_M

Consider Equation (4.28), then, by Conditions (2.9) and (3.15), we have that

$$\begin{aligned} \frac{d\lambda^*}{dw_M} &= \frac{(w_T - w_M c)L}{(w_M - w_T)^2 NF} + \frac{NF + cL}{(w_M - w_T)NF} - \frac{w_M}{w_M - w_T} \\ &= \frac{w_T(L(1-c) - NF)}{(w_M - w_T)^2 NF}. \end{aligned} \tag{4.34}$$

An increment in wages of modern firms w_M has two opposing effects: the first effect is that the aggregated demand can increase, which implies a negative effect in λ^*; the second effect is that the cost of the firm can increase, which implies a positive effect on λ^*.

If all sectors are industrialized, then we need at least

$$L \geq \frac{NF}{1-c} \tag{4.35}$$

units of labor. If Condition (4.35) is satisfied, then $\frac{d\lambda^*}{dw_M} \geq 0$. This means that if we have increments in W_M, then we have a higher value in the state of critical coordination effort λ^*.

5. Final Remarks

Concerning the issue of Big Push, this paper formulates a simplified version of the Murphy et al. [6] and Basu [1] models, and incorporates replicator dynamics to consider imitative behavior of producers choosing their technology. There are three outcomes given by an evolutionary dynamic interrelation between the traditional and the modern sectors. First, traditional firms can continue choosing traditional technology, with low wages and low output.

Second, they can choose modern technology, with high wages and high output. Third, some of them choose modern technology and others choose traditional technology. These outcomes bring three possible equilibria, like the Murphy et al. [6] model: a) The economy tends to high industrialization; b) it tends to never industrialize, being mostly traditional; c) it remains in an intermediate situation, partially traditional (this third equilibrium is unstable). In the first of these three equilibria, the economy as a whole tends towards the best equilibrium (the most productive one) under some general conditions, in particular, a critical coordination effort by the government through an industrial policy. Just as Murphy et al. [6] and Krugman [5] conclude, we have a multi-equilibrium outcome, with three possible equilibria (one of them is unstable), but with the difference that our model incorporates both dynamism and government coordination to accelerate industrialization.

6. Appendix

6.1. *The profit* $\pi_M(0) < 0$

Using Equation (2.14), we have that

$$\pi_M(0) = F[(w_T - w_M c)(L/NF) - w_M].$$

By Condition (4.29)

$$\begin{aligned}(4.29) \quad &\Rightarrow \quad \frac{(w_T - w_M c)L}{w_T NF} < \frac{w_M}{w_T} \\ &\Rightarrow \quad (w_T - w_M c)(L/(NF)) - w_M < 0 \\ &\Rightarrow \quad \pi_M(0) < 0.\end{aligned}$$

6.2. The profit $\pi_M(1) > 0$

Using Equation (2.14), we have that

$$\pi_M(1) = \frac{F[(w_T - w_M c)((L/(NF)) - 1) - w_M c]}{c}.$$

By Condition (4.29)

$$\begin{aligned}
(4.29) \quad &\Rightarrow \quad \frac{(w_T - w_M c)L}{w_T NF} > 1 \\
&\Rightarrow \quad (w_T - w_M c)(L/(NF)) - w_T > 0 \\
&\Rightarrow \quad (w_T - w_M c)(L/(NF)) - (w_T - w_M c) - w_M c > 0 \\
&\Rightarrow \quad (w_T - w_M c)((L/(NF)) - 1) - w_M c > 0 \\
&\Rightarrow \quad \pi_M(1) > 0.
\end{aligned}$$

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IX. COLLECTIVE AGREEMENT ON FOREST RESOURCES POLICY: AN EVOLUTIONARY DYNAMIC APPROACH

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Abstract

Collective agreement on forestry resources is the activity performed by a rural forest community considering a cooperative agreement, without government intervention. In this paper, we study the collective agreement in a population that is divided in three types of mutually exclusive sub-populations: cooperators, defectors and enforcers. In this model, we incorporate a forest resource dynamics similar to the golden rule in economic growth. By means of the replicator dynamics, we analyze the interaction

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between sub-populations and forest resources. Finally, the stability of the dynamic system is studied in order to determine the public policies to be followed by this community to achieve sustainability.

1. Introduction

According to the 2015 Global Forest Resources Assessments FAO [3] elaborated by the Food and Agriculture Organization (FAO) of the United Nations during the period of 1990 to 2015, the world's forest area has decreased by approximately 3.1%, making safeguarding of natural resources one of the main global goals all over the world. However, there exists evidence that some countries have not always taken the right decisions in this matter. Sometimes, economical reasons underlie these decisions, and in other cases, it is just because of the impossibility of a total supervision by governmental authorities Ostrom [9]. For a description of conflicts around the world see Temper et al. [14].

In this paper our goal is to study the evolution of a forest community and the forest resource. We define *collective agreement on forestry resources* as the activity performed by a rural forest community considering a cooperative agreement, without government intervention. An evolutionary dynamics explains how the structure of the population in the community is modified in terms of strategies, which are determined by their sustainable behavior with respect to the forest resource by means of changes in the stock through time. As a result, some conditions are given in order to determine the stability of each critical point.

There exists evidence of a successful process of sustainable exploitation applying collective agreements in some forest communities: for instance, Cherán's community in Mexico. Unless some conditions are met, there is no way to assure that the entire population will take care of the environment. In this scenario, we may expect a framework similar to Shahi and Kant [11], where members of the community could play different roles in this society, namely, cooperators, defectors and enforcers, all of them with a rate of exploitation of the resource assigned according to their behavioral strategy. In this setting the model describes the process of exploitation of the forest by the population of a given forest community, considering that only they have access to the forest resources in order to exploit them.

In Section 2, we provide a brief overview of the related literature about renewable resources, collective management and sustainability. In Section 3, we introduce our model. In Section 4, we highlight some results derived from the model in terms of sustainability. In Section 5, we explain the main critical points of the system. In Section 6, we discuss the stability of the system. Finally, we present our conclusions.

2. The model

In order to protect natural resources, many management programs have been proposed. For instance, the Joint Forest Management Regime (JFM), a program developed in India, consists of an agreement among forest communities, represented by a Forest Protection Committee and government to safeguard the resource in exchange for a share of the revenue from the sale of timber products and other non timber products. Moreover, in Europe the Sustainable Forest Management program was launched, which tries to balance human needs and the survival of the forests.

In this paper, we propose a *Collective agreement* describing a forest community which survives by the exploitation of available natural resources, like a subsistence economy. This does not mean that individuals cannot access other jobs in order to get a wage, but if they do, this wage represents a small proportion of their total income. So, we focus on the part of the income explained by the exploitation of the forest resource. We assume that there exists a community agreement to exploit the resource at a given rate, without the intervention of government or any other kind of authority. This agreement is a product of a collective consensus that looks for a sustainable use of the forest. It is important to mention that although some authors have previously worked on the sustainable management problem in Mexico, we do not try to explain the community as a forestry enterprise (see Bray et al. [1]) nor the production of timber, nor any other product (see Torres-Rojo et al. [15]). We describe the possible paths that a rural forest community might reach when a self-management program is applied to preclude overusing the forest resources.

2.1. Population

We are interested in the dynamics of the renewable resource and the dynamics of the population. Based on their behavioral strategies, the community is divided in three types of sub-populations: *cooperators (c), defectors (d) and enforcers (e).* Each individual belongs only to one type of sub-population.

The total number of members in the community is n, which is fixed at every moment in time. Each sub-population consists of n_k individuals, for $k = c, d, e$, and

$$n = n_c + n_d + n_e.$$

In the community, there exists a stock of renewable resource, r. This resource is consumed with different rates of exploitation.

Cooperators

Cooperators are those who are dedicated to caring for the forest. The entire community needs to consume a part of the resource. Both sub-populations, cooperators and enforcers consume a proportion λ of the resource, which is called the *collective agreement rate of exploitation of the resource*, and $\lambda \in (0, 1)$. The total of cooperators' payoffs, π_c, is

$$\pi_c = \left[\frac{\lambda r}{n_c + n_e}\right] n_c. \tag{2.1}$$

In words, Equation (2.1) states that the payoff for cooperators is the fraction of the resource exploited under the collective agreement, over the total number of individuals that respect it, times the number of cooperators.

Defectors

Defectors are those who are not willing to maintain the agreement. They take advantage and make some extra profit via a higher rate of exploitation, δ in $(\lambda, 1)$. Defectors have been assigned a payoff that depends on the probability $b \in [0, 1]$ of being caught for exploiting the resource with a higher rate of exploitation δ.

The total defectors' payoff, π_d, is

$$\pi_d = [(1 - b)\delta r] n_d. \tag{2.2}$$

That is, the payoff for the defectors is equal to the probability of not being caught, times the fraction of the resource overexploited, times the number of defectors.

Enforcers

Enforcers are the ones with the authority to sanction any excessive use of the resource and, in addition, to collect the money from the fees paid by visitors. The fraction of the quantity recovered of the forest resources from the captured defectors is represented by μ; enforcers take into account a fixed cost for their activities, F. As in Shahi and Kant [11], we consider a variable cost faced by the enforcers in order to catch defectors, β, and S is the fixed salary of the enforcer. By hypothesis, $S > F > 0$ and $1 \geq \mu \geq \beta \geq 0$.

The sum of the enforcer's payoff, π_e, is

$$\pi_e = \left[\frac{\pi_c}{n_c}\right] n_e + (S - F)n_e + \delta b(\mu - \beta) r n_d. \tag{2.3}$$

Hence, the payoff of enforcers is the payoff of the cooperators over the number of cooperators, times the number of enforcers, plus a quantity that depends on the number of defectors who have to pay a fine, plus a fixed income which we separate as a difference between a fixed salary and a fixed cost, to encourage a possible policy to enhance the performance of enforcers.

2.2. Payoffs

In the next section, we introduce the replicator dynamics, so it is necessary to replace the number of individuals belonging to each group of strategies by the fraction of them; that is, $s_i = \frac{n_i}{n}$ for $i = c, d, e$. As a result, our payoff functions in (2.1) - (2.3) are now given by:

$i)$ cooperators

$$\pi_c = \left[\frac{\lambda r}{s_c + s_e}\right] s_c, \tag{2.4}$$

$ii)$ defectors

$$\pi_d = [(1-b)\delta r]ns_d, \tag{2.5}$$

$iii)$ enforcers

$$\pi_e = \left[\frac{\pi_c}{s_c}\right] s_e + (S-F)ns_e + \delta b(\mu - \beta)rns_d. \tag{2.6}$$

2.3. Evolutionary dynamics

An evolutionary dynamics describes the behavior and interaction of strategies in a population. In our model, at each time $t \geq 0$, every member of the population selects a sub-population from the options previously mentioned. Then, at each $t \geq 0$, the proportion of individuals in the population with a certain activity is changing. Let s_c, s_e and s_d be the proportions of individuals belonging to the sub-populations of cooperators, enforcers and defectors, respectively. An evolutionary dynamics explains how the proportions of sub-populations are changing in time; in other words, it explains how the individuals select their activities as a function of time. In an imitative evolutionary-dynamics, the individuals select a sub-population through imitation. Each individual selects a subpopulation if he sees that the payoff in that sub-population is better than the expected payoff of the population and eliminates it (see Hofbauer and Sigmund [6]). Here, we select an imitative evolutionary-dynamics known as the replicator dynamics. The replicator dynamics has many interesting properties; in particular, it has a simple mathematical form, has a natural interpretation, describes imitation behaviors (see Hofbauer and Sigmund [6, 5] and Webb [16]), and its local stability is similar to the local stability of a family of evolutionary dynamics (Cressman [2]). Moreover, it can be derived from models of interactive learning processes, (as in Gale et al. [4]). In our case, the evolution of the population is described by the replicator dynamics as follows:

$$\dot{s_c} = s_c(\pi_c - \bar{\pi}), \tag{2.7}$$

$$\dot{s_d} = s_d(\pi_d - \bar{\pi}), \tag{2.8}$$

$$\dot{s_e} = s_e(\pi_e - \bar{\pi}), \tag{2.9}$$

where

$$s_c + s_d + s_e = 1. \tag{2.10}$$

Equation (2.10) indicates that the sum of the sub-populations adds up to the entire population, and the number

$$\bar{\pi} := s_c\pi_c + s_d\pi_d + s_e\pi_e \tag{2.11}$$

in (2.7) - (2.9) is the average payoff function.

Note that $\dot{s}_k \geq 0$ if and only if $\pi_k - \bar{\pi} \geq 0$, for $k = c, e, d$. Then, each sub-population grows if and only if the sub-population's payoff is better than the expected payoff of the population.

2.4. Resource dynamics

To complete the model, we introduce a dynamic model of the biomass, that is, renewable forestry resource. Therefore, we are no longer in an evolutionary game, but in a dynamic system. The decisions of any person in the community are influenced by the level of the renewable resource at any given time. Let's assume that the renewable resource has a natural growth function of the form

$$f(r) = Ar^{\alpha}, \tag{2.12}$$

which, in economic terms, represents a production function. In (2.12) A represents a constant friendly technology for the environment, and $\alpha \in (0, 1)$ is the weight of the production factor r. This growth function behaves according to the first six assumptions of the model for conservation of renewable resources in Olson and Roy [8], and so it has important properties such as monotonicity, continuity, differentiability, bounded growth and others.

Moreover, A is the reforestation technology used by the community. In addition, the resource biomass evolves as

$$\dot{r} = Ar^{\alpha} - \delta s_d r - \lambda(s_c + s_e)r. \tag{2.13}$$

This expression is similar to the fundamental equation of Solow-Swan (see Solow [12] and Swan [13]). However, in (2.13) the depreciation comes from the exploitation, sustainable or not, of the resource.

It is important to mention that the parameter λ does not guarantee sustainability, by itself, nor does δ mean non-sustainability. These parameters are just rates of exploitation such that $\lambda < \delta$. Sustainability is expressed in terms of the stock of the resource. Specifically, we assume that there exists $r_0 > 0$, which is the minimum level possible to reforest, and if

$$r_0 \leq r_t \text{ for all } t \geq 0, \tag{2.14}$$

then if (2.14) holds, the renewable resource is conserved and, consequently, there is no loss in biodiversity.

3. Critical points

In view of (2.10), we may rewrite (2.7) - (2.9) and (2.13) as

$$\dot{s_c} = s_c(\pi_c - \bar{\pi}), \tag{3.15}$$

$$\dot{s_d} = s_d(\pi_d - \bar{\pi}), \tag{3.16}$$

$$\dot{r} = Ar^{\alpha} - \delta s_d r - \lambda(1 - s_d)r. \tag{3.17}$$

We will now determine the critical points of this system.

Cooperators. In the context of an economy of subsistence in a forestry community like Cherán, the concept of cooperation is highly developed; they share history and social values, which makes it possible that the entire population behaves like a cooperator, that is, $s_c = 1$. Hence, (3.15) - (3.17) yield

$$\pi_c = \bar{\pi}, r = \left[\frac{A}{\lambda}\right]^{\frac{1}{1-\alpha}}.$$

Defectors. Culture and social customs play an important role in the actions of the community members, so if they were caught overexploiting the resource, they could be expelled or socially rejected. In this case, we study the situation in which everybody is a defector, that is, $s_d = 1$. Therefore, from (3.15) - (3.17),

$$\pi_d = \bar{\pi}, r = \left[\frac{A}{\delta}\right]^{\frac{1}{1-\alpha}}.$$

Enforcers. An ideal model is when the whole community acts as an enforcer, so $s_c + s_d = 0$. In this case, we obtain

$$\pi_e = \bar{\pi}, r = \left[\frac{A}{\lambda}\right]^{\frac{1}{1-\alpha}}.$$

No defectors. Cherán's community provides a remarkable example of an economy of subsistence in which the social rules are established by means of a self-management regime based on collective action, and the population is divided in two subpopulations: cooperators and enforcers. Therefore, $s_d = 0, s_c \neq 0, 1 - s_c \neq 0$. Hence $\pi_c = \bar{\pi} = \pi_e, r = \left[\frac{A}{\lambda}\right]^{\frac{1}{1-\alpha}}$,

$$s_c = 1 - \left[\frac{\lambda r}{2\lambda r + (S - F)n}\right].$$

4. Stability

The criterion for stability is given in terms of the determinant which is computed in Section 9 below.

$$0 = |J - \xi I| = \begin{vmatrix} \frac{\partial \dot{s_c}}{\partial s_c} - \xi_1 & \frac{\partial \dot{s_c}}{\partial s_d} & \frac{\partial \dot{s_c}}{\partial r} \\ \frac{\partial \dot{s_d}}{\partial s_c} & \frac{\partial \dot{s_d}}{\partial s_d} - \xi_2 & \frac{\partial \dot{s_d}}{\partial r} \\ \frac{\partial \dot{r}}{\partial s_c} & \frac{\partial \dot{r}}{\partial s_d} & \frac{\partial \dot{r}}{\partial r} - \xi_3 \end{vmatrix}. \tag{4.18}$$

Cooperators. To determine the stability of this system, which consists only of cooperators, we have to analyze the eigenvalues of the determinant (4.18). From Section 9, these eigenvalues are given by

$$\xi_1 = \xi_2 = -\lambda \left[\frac{A}{\lambda}\right]^{\frac{1}{1-\alpha}} < 0,$$
$$\xi_3 = -\lambda(1-\alpha) < 0.$$

Since one value is repeated, then this point is degenerate. However, all the eigenvalues are negative, so it is also an attractor. This result is very interesting, since we find the possibility to describe an unconditioned public policy, that is, we do not need to add any other conditions on the parameters.

Defectors. To determine the conditions to create a public policy avoiding the possibility of defector behavior, we study the case where

$$\xi_1 = -(1-b)\delta n \left[\frac{A}{\delta}\right]^{\frac{1}{1-\alpha}} < 0,$$
$$\xi_2 = -\delta(1-\alpha) < 0,$$
$$\xi_3 = \delta n[b(\mu-\beta) - (1-b)] \left[\frac{A}{\delta}\right]^{\frac{1}{1-\alpha}}.$$

The first two eigenvalues are always negative. If the third eigenvalue is positive, which is when $1 + \mu - \beta > \frac{1}{b}$, then the critical point is nondegenerate and repulsive. The latter condition highlights the importance of self-management in forestry communities populated by natives. The parameter b when the government is in charge is too low, almost zero. However, when the forestry management is developed by the community following collective action, then this parameter is high because of the social relevance of being excluded by the entire community. The social impact in the day-to-day life of a defector is not as simple as a fine; there is an implicit social stigma for him and his family.

Enforcers. Once again, the eigenvalues are obtained throughout the product of the diagonal:

$$\xi_1 = \xi_2 = -\left([\lambda \left[\frac{A}{\lambda}\right]^{\frac{1}{1-\alpha}}] + (S - F)n \right) < 0,$$
$$\xi_3 = -\lambda(1 - \alpha) < 0.$$

As in the cooperators' case, one value is repeated, and so this critical point is degenerate. Also, all the eigenvalues are negative, so it is an attractor. The difference between the fixed salary S and the fixed cost F must be positive; on the other hand, there is no incentive for a community member to act as an enforcer.

No defectors. In this case,

$$\xi_1 = -[\lambda r(1 - 6s_c(1 - s_c) + (S - F)n(1 - s_c)(1 - 3s_c)],$$
$$\xi_2 = -[\lambda r s_c^2 + (1 - s_c)^2(\lambda r + (S - F)n],$$
$$\xi_3 = -[\lambda(1 - \alpha)].$$

The eigenvalues are different from zero and among them, so it is a non degenerate critical point. Besides, the last two eigenvalues are always negatives such that the repulsion or attraction depends on the first eigenvalue. If the sum inside the brackets is positive, then it is an attractor, which is a good public policy to follow.

5. Collective agreement regime in forestry communities

Native communities live mainly in forestry areas. Historically, these communities have defended their resources. For instance, in 1543 in Taxco, now in the state of Guerrero, natives complained of mining activities because forests were running out. As far as we know, the number of incidents of this kind increased through time Madrigal González [7]. This increment was possible, and, in fact, it is also today because of overexploitation practices of firms and the lack of enforcement of laws. This contradicts some of Hardin's postulates.

Under these circumstances, forestry communities face extortion, kidnapping and the murder of their leaders. Eventually, many communities have decided to take actions such as the substitution of their police corps (usually associated with organized crime), their mayors (who cooperate with the criminal cartels) and the laws, which did not provide enough protection to the communities, themselves, and the natural resources. These factors create conditions in the population to behave as conditional cooperators and willing punishers as in Ostrom [10].

6. Discussion

In this paper, we analyze the interaction between a population which is divided into three sub-populations according to the behavioral strategies of the community members, and a renewable resource, in general terms, a forest. In our model, the native rural community takes care of the renewable resource by means of a collective agreement regime. This proposal has the advantage that it minimizes the desire of any individual for being a defector since the entire population might discard him or her from the decision making process. Taking this into account, our stability analysis shows the conditions to attain trajectories in order to achieve sustainability as long as $r \geq r_0$. This is a remarkable fact because it shows how it is possible to solve the overexploitation problem without the interference of governmental authorities, which usually exhibit a high propensity for acting corruptly.

A collective agreement regime is not the solution to all the problems in forest communities; however, it sheds light on considering other solutions. So far, we only are worried about the dynamics of the renewable resource and the dynamics of the population, but we are aware that in many communities, there is no total participation of the population, since by "uses and costumes," women and the young do not participate in the process of decision making about how to organize the community. This is a problem with which we will work on later.

7. Appendix

The criterion for stability is given by

$$0 = |J - \xi I| = \begin{vmatrix} \frac{\partial \dot{s_c}}{\partial s_c} - \xi_1 & \frac{\partial \dot{s_c}}{\partial s_d} & \frac{\partial \dot{s_c}}{\partial r} \\ \frac{\partial \dot{s_d}}{\partial s_c} & \frac{\partial \dot{s_d}}{\partial s_d} - \xi_2 & \frac{\partial \dot{s_d}}{\partial r} \\ \frac{\partial \dot{r}}{\partial s_c} & \frac{\partial \dot{r}}{\partial s_d} & \frac{\partial \dot{r}}{\partial r} - \xi_3 \end{vmatrix}.$$

If $s_c = 1$, then

$$|J - \xi I| = \begin{vmatrix} -\lambda r - \xi_1 & 0 & 0 \\ 0 & -\lambda r - \xi_2 & 0 \\ 0 & -(\delta - \lambda) r & \alpha A r^{-(1-\alpha)} - \lambda - \xi_3 \end{vmatrix}$$

$$= (\lambda r - \xi_1)(\lambda r - \xi_2)(\alpha A r^{-(1-\alpha)} - \lambda - \xi_3) = 0.$$

If $s_d = 1$, then

$$|J - \xi I| = \begin{vmatrix} -(1-b)\delta rn - \xi_1 & 0 & 0 \\ -\delta b(\mu - \beta) rn & -(1-b)\delta rn + \delta b(\mu - \beta) rn - \xi_2 & 0 \\ 0 & -(\delta - \lambda) r & \alpha A r^{-(1-\alpha)} - \delta - \xi_3 \end{vmatrix}$$

$$= (-(1-b)\delta rn - \xi_1)(-(1-b)\delta rn + \delta b(\mu - \beta) rn - \xi_2)(\alpha A r^{-(1-\alpha)} - \delta - \xi_3) = 0.$$

If $s_c + s_d = 0$, then

$$|J - \xi I| = \begin{vmatrix} -\lambda r - (S-F)n - \xi_1 & 0 & 0 \\ 0 & -\lambda r - (S-F)n - \xi_2 & 0 \\ 0 & -(\delta - \lambda) r & \alpha A r^{-(1-\alpha)} - \lambda - \xi_3 \end{vmatrix}$$

$$= (\lambda r - (S-F)n\xi_1)(\lambda r - (S-F)n - \xi_2)(\alpha A r^{-(1-\alpha)} - \lambda - \xi_3) = 0.$$

If $s_d = 0$, then let

$$a_{11} = -\lambda r(1 - 6s_c(1 - s_c)) - (S - F)n(1 - s_c)(1 - 3s_c) - \xi_1,$$
$$a_{12} = \lambda r s_c^2 - s_c(2\lambda r s_c^2 - \lambda r - 2(S - F)n(1 - s_c) + \delta b(\mu - \beta)rn(1 - s_c)),$$
$$a_{13} = \lambda s_c^2 - \lambda s_c^3 - \lambda s_c(1 - s_c)^2,$$
$$a_{21} = 0,$$
$$a_{22} = -\lambda r s_c^2 - \lambda r(1 - s_c)^2 - (S - F)n(1 - s_c)^2 - \xi_2,$$
$$a_{23} = 0,$$
$$a_{31} = 0,$$
$$a_{32} = -(\delta - \lambda)r,$$
$$a_{33} = \alpha A r^{-(1-\alpha)} - \lambda - \xi_3,$$

then

$$|J - \xi I| = \begin{vmatrix} a_{11} - \xi_1 & a_{12} & a_{13} \\ a_{21} & a_{22} - \xi_2 & a_{23} \\ a_{31} & a_{32} & a_{33} - \xi_3 \end{vmatrix}$$
$$= (a_{11} - \xi_1)(a_{22} - \xi_2)(a_{33} - \xi_3) = 0.$$

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X. AN OPTIMAL CONTROL PROBLEM IN FOREST MANAGEMENT

Leonardo R. Laura-Guarachi

Abstract

In this work the Mitra-Wan forestry model is studied as a discrete-time optimal control problem. We consider a profit function depending on timber production and the forest maintenance cost. The *optimal rotational age* is determined in terms of the average timber production and the average cost of forest maintenance. On the other hand, to study the non stationary optimal control policies, we consider the *average optimality*, *good policy* and *bias optimality*. The set of average optimality policies contains strictly the good policies, and in turn, the set of good policies contains the bias optimal policies. Moreover, the good policies converge to the optimal stationary policy (*turnpike property*).

1. Introduction

A forest manager has a unit of land that is completely covered by trees of the same species but, possibly, of different ages. The forest configuration evolves according to the time. Then, one of the main forest management problems is to establish a harvest plan that maximizes the accumulated revenue along some determined time horizon.

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The literature on forest economics can be traced back at least to Faustman's (in 1849) and Pressler's (in 1860) studies on the existence of optimal rotation age, that is, the "right" age at which the trees should be harvested. In more recent years, among other efforts to solve the problem, Mitra and Wan [12, 13] have introduced a discrete time dynamical model to study the forest structure and behavior. They have found that the solutions evolve according to the so called "*turnpike property*", that is, when the time horizon is finite, the solution approaches to some *stationary state*, stays close to it, and at the end, exists in order to reach the final state; when the horizon is infinite, the solutions converge to the stationary state. This analysis was extended by Salo and Tahvonen [14]. Further advances were made recently by Khan and Piazza [9, 8] establishing turnpike properties for a broader class of utility functions.

On the other hand, the turnpike properties have been studied since Dorfman et al. [5], if not earlier. Recent works – for example, Damm et al. [3], Zaslavski [19], Trélat and Zuazua [16] – show that the turnpike properties are part of a wide class of optimal control problems in discrete time horizon, as well as in continuous time.

In this work, we will study the Mitra-Wan forestry model as an infinite horizon discrete time optimal control problem Laura-Guarachi and Hernández-Lerma [10]. Regardless, in addition to previous works on this topic, where the utility function depends only on the amount of timber harvested and no plantation costs are considered, we assume that the forest plantation (or maintenance) has some cost, for example, the limited disposal of water could have some price; or, for optimal growth of the trees, the land could need some amount of nutrients (fertilizers). This consideration could refine the determination of optimal stationary forest configuration, as it is shown in Lemma 3.9, Corollary 3.10, Theorem 4.2, and Example 5.1. Based on the optimal stationary state, the following ergodic optimality criteria are defined: *average optimality*, *good policy* and *bias optimality*. We review the relations among these criteria and their properties.

The next sections are organized as follows: The Mitra-Wan forestry model is presented in Section 1.2; Optimal stationary states are studied in Section 1.3; Section 1.4 is about the average optimality, good policy and bias optimality. Finally, Section 1.5 treats some numerical examples that illustrate the theoretical results.

2. The Mitra-Wan Forestry Model

We consider the so called, Mitra-Wan Forestry Model, introduced originally by Mitra and Wan [12, 13]. This model considers a unit of land which is completely occupied by trees of a certain species, with ages ranging from 1 to n. The age n is supposed to be the age after which a tree dies or loses its economic value.

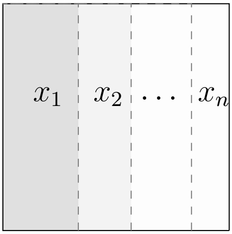
Forest state. Let $t = 0, 1, \cdots, T-1$ be the period of time. Suppose that the surface proportion occupied by trees of age $i = 1, 2, \cdots, n$ at period t is given by $x_i(t) \geq 0$. Then, the state of the forest at period t can be represented by a vector $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ that satisfies

$$x_1(t)+x_2(t)+\cdots+x_n(t) = 1, \text{ for all } t = 0, 1, \cdots, T-1. \quad (2.1)$$

So, a forest configuration can be represented as some point of the simplex space

$$\Delta := \{x \in \mathbb{R}^n : x_1 + x_2 + \cdots + x_n = 1, x_i \geq 1, i = 1, ..., n\}.$$

Figure X.1: Forest configuration.



Harvest plan. Given a forest configuration $x(t)$ at the end of period t, we must decide how much land to harvest of every age class. Say

$$u(t) = (u_1(t), u_2(t), \cdots, u_n(t)) \text{ with } 0 \le u_i(t) \le x_i(t). \quad (2.2)$$

Because a tree has no value after age n, the proportion occupied by trees of age n will be completely harvested, that is, $u_n(t) = x_n(t)$ for all $t = 0, 1, \cdots, T-1$.

The total harvested area at the end of period t will be $u_1(t) + \cdots + u_n(t)$ and, at the beginning of the following period, it will be covered by trees of age 1, that is

$$x_1(t+1) = u_1(t) + \cdots + u_n(t). \quad (2.3)$$

The proportion of area occupied by $(i+1)$-aged trees in the period $t+1$ will be $x_i(t) - u_i(t)$, which is

$$x_{i+1}(t+1) = x_i(t) - u_i(t) \text{ for } i = 1, \cdots, n-1. \quad (2.4)$$

Thus, given a forest configuration $x \in \Delta$, the feasible control (harvest plan) set at the state x is $U(x) := \{u \in \mathbb{R}^n : u_i \le x_i, u_n = x_n, i = 1, ..., n\}$, and the set of *feasible state-control* pairs is defined by $\mathbb{K} := \{(x, u) : x \in \Delta, u \in U(x)\}$.

Control system. From (2.3) and (2.4), at the beginning of the period $t+1$, the state of the forest is given by

$$x(t+1) = \left(\sum_{i=1}^{n} u_i(t), x_1(t) - u_1(t), ..., x_{n-1}(t) - u_{n-1}(t) \right),$$

which can be written in compact form as a discrete-time linear control system

$$x(t+1) = f(x,u) := Ax(t) + Bu(t), \quad \text{for} \quad t = 0, 1, \cdots, T-1, \tag{2.5}$$

where

$$A := \begin{pmatrix} 0 & 0 & . & 0 & 0 \\ 1 & 0 & . & 0 & 0 \\ 0 & 1 & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & 1 & 0 \end{pmatrix}, \quad \text{and} \quad B := \begin{pmatrix} 1 & 1 & . & 1 & 1 \\ -1 & 0 & . & 0 & 0 \\ 0 & -1 & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & -1 & 0 \end{pmatrix} \tag{2.6}$$

Given an initial forest state $x(0) = x \in \Delta$, a *feasible control policy* for the system (2.5) is a sequence $\mathbf{u} := \{u(t)\}_{t=0}^{T-1}$ of controls $u(t) \in U(x(t))$ for all $t = 0, 1, \cdots, T-1$. The set of feasible control policies is denoted by $\mathcal{U}(x)$.

Income function. Suppose the timber production per unit of area is related to the age of trees through a *biomass vector* $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n$, $\xi_i \geq 0, i = 1, 2, ..., n$, where ξ_i represents the amount of timber produced by i-aged trees occupying a unit of land. Hence, the total amount of timber collected at the end of period t is

$$\langle \xi, u(t) \rangle = \xi_1 u_1(t) + \cdots + \xi_n u_n(t).$$

Now, consider an income function $p : [0, \infty) \to [0, \infty)$; which, as usual, is assumed to be concave. Therefore, given a harvesting plan u, the income obtained for the amount of timber $\langle \xi, u \rangle$ is $p(\langle \xi, u \rangle)$.

Cost function. Here, we assume that the plantation has some costs: in order to sustain a forest configuration, some resource is required (for example water or fertilizer) depending on the tree age. This information will be captured by a vector $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbb{R}^n$, $\gamma_i \geq 0, i = 1, 2, ..., n$, where γ_i denotes the amount of resource needed to maintain a unit of land covered by i-aged trees. Then, the total quantity of resource needed to sustain the forest configuration $x(t)$ in the period t is given by

$$\langle \gamma, x(t) \rangle = \gamma_1 x_1(t) + \cdots + \gamma_n x_n(t).$$

Similarly, consider a cost function $c : [0, \infty) \to [0, \infty)$, which is supposed to be convex. Thus, given a forest state x, the cost of the resource $\langle \gamma, x \rangle$ is $c(\langle \gamma, x \rangle)$.

Profit function. Finally, let us define the profit (or gain) function
$r : \mathbb{K} \to \mathbb{R}$ given by

$$r(x, u) := p(\langle \xi, u \rangle) - c(\langle \gamma, x \rangle). \tag{2.7}$$

Taking into account that p is concave and c is convex, then r is a concave function.

Under the assumption of concavity of the profit function, the analysis of the optimal solution can be made in two cases: when the felicity function is linear (in this case the optimal policy is periodic), or if it is strictly concave (the optimal policy converges to the maximal sustainable stationary forest configuration). Here we assume that the profit function is strictly concave at the maximum sustained yield. For more details and examples on periodic solutions, see Mitra and Wan [13] and Salo and Tahvonen [14].

3. Optimal stationary policies

In this section, we restrict our attention to the search for an optimal forest configuration in the set of stationary ones: that is, to find an optimal forest configuration that does not change over time.

Definition 3.1. A pair $(x, u) \in \mathbb{K}$ is said to be a *stationary state-control* if

$$Ax + Bu = x. \tag{3.8}$$

From (2.6), notice that the matrices $I - A$ and B are non singular and they satisfy the equality $B^{-1}(I - A) = I - A^{\top}$. Here $A^{\top}$ denotes the matrix A transpose and I is the identity $n \times n$ matrix.

Thus, from (3.8), we have $u = (I - A^{\top})x$. This property allows us to characterize the set of stationary states and controls.

Proposition 3.2. *A forest state $x \in \Delta$ is stationary if and only if $x_i \geq x_{i+1}$ for all $i = 1, \cdots, n-1$. Given a stationary state x, it has a unique stationary control given by $u = (x_1 - x_2, \cdots, x_{n-1} - x_n, x_n)$.*

Proposition 3.2 says the stationary forest configurations are those where the portion of land occupied by trees decreases by age, whereas, the stationary controls are harvesting plans in which the differences between adjacent periods are cut in order to maintain the same proportions as in the previous period.

Example 3.3. There is a special class of stationary states: for a given age i, the forest state $x^i := (1/i, ..., 1/i, 0, ..., 0)$, where the first i-coordinates are all equal to $1/i$ and the remaining are equal to 0, is stationary and its stationary control is $u^i := (0, ..., 0, 1/i, 0, ..., 0)$, where $1/i$ is in the i-coordinate. The pair (x^i, u^i) is known as *uniform stationary state-control.*

Remark 3.4. Notice that the amount of timber collected with a uniform stationary control u^i is $\langle \xi, u^i \rangle = \xi_i / i$, which, on the other hand, represents the *average timber production* in a unit area covered by i-aged trees. In its counterpart, the resource needed to sustain a uniform stationary forest configuration x^i is given by $\langle \gamma, x^i \rangle = (\gamma_1 + \gamma_2 + \cdots + \gamma_i)/i$, which, in addition, gives the *average resource invested* in order to grow trees covering a unit area until i years old.

Now, within the set of stationary state-control pairs, we select those that maximize the profit function according to the following definition:

Definition 3.5. A pair $(x^*, u^*) \in \mathbb{K}$ is said to be an *optimal stationary state-control* pair if

$$r(x^*, u^*) = \max\{r(x, u) : (x, u) \in \mathbb{K}, \quad Ax + Bu = x\}. \quad (3.9)$$

In the literature on optimal economic growth, an optimal stationary state is known as a *golden rule*, *von Neumann point*, or *maximum sustained yield.* More discussion on sustainable developments can be found in Amacher et al. [1], Kant and Berry [7] and Samuelson [15].

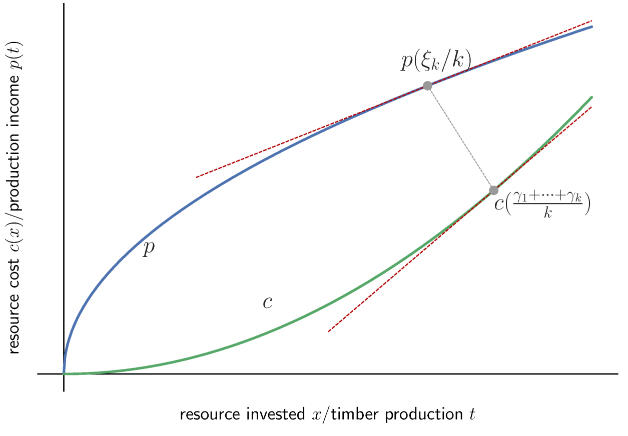
Assumption 3.6. Let us denote by k the age for which the reward function is maximized in the set of uniform stationary states described in the Example 3.3 and Remark 3.4, that is

$$r(x^k, u^k) = \max\left\{p\left(\langle \xi, u^i \rangle\right) - c\left(\langle \gamma, x^i \rangle\right) : i = 1, 2, \cdots, n\right\}. \quad (3.10)$$

Following the classical Brock-Mitra-Wan uniqueness condition Brock [2] and Mitra and Wan [13], we require that (3.10) has a unique solution. In order to have this property, we assume that r is strictly concave at (x^k, u^k).

Recalling Remark 3.4, the right hand side of (3.10) is the maximum profit obtained comparing the average timber production and the average resource invested to grow the trees until that age.

Figure X.2: Income and cost functions



The following two lemmas are technical properties that are derived from the properties of the reward function (2.7) and the matrices in the control system (2.5).

Lemma 3.7. *Consider age k defined in the Assumption 3.6. There are real numbers $\alpha > 0$ and $\beta > 0$ such that*

$$r(x,u)-r(x^k,u^k) \le \alpha\langle \xi, u-u^k\rangle - \beta\langle\gamma, x-x^k\rangle \text{ for all } (x,u)\in\mathbb{K}. \tag{3.11}$$

Lemma 3.8. *For any $(x,u)\in\mathbb{K}$ and $i = 1, 2, ..., n$, the following equalities are satisfied*

1. $\langle(1,2,\cdots,n), x-f(x,u)\rangle = \langle(1,2,\cdots,n), u-u^i\rangle$,
2. $x-x^i = (I-A^\top)^{-1}(u-u^i) + (I-A)^{-1}(x-f(x,u))$.

Lemma 3.9. *For any* $(x, u) \in \mathbb{K}$*, the following inequality is satisfied*

$$r(x, u) - r(x^k, u^k) \leq \langle V, x - f(x, u)\rangle \tag{3.12}$$

where $V := (\alpha\langle\xi, u^k\rangle - \beta\langle\gamma, x^k\rangle)(1, \cdots, n) - \beta(I - A^\top)^{-1}\gamma.$

Proof. From Lemmas 3.7 and 3.8 we have

$$\begin{aligned}
r(x, u) - r(x^k, u^k) &\leq \alpha\langle\xi, u - u^k\rangle - \beta\langle\gamma, x - x^k\rangle \\
&= \langle\alpha\xi, u - u^k\rangle - \langle\beta\gamma, (I - A^\top)^{-1}(u - u^k)\rangle \\
&\quad - \langle\beta\gamma, (I - A)^{-1}(x - f(x, u))\rangle \\
&= \langle\alpha\xi - \beta(I - A)^{-1}\gamma, u - u^k\rangle \\
&\quad - \langle\beta(I - A^\top)^{-1}\gamma, x - f(x, u)\rangle \\
&\leq (\alpha\langle\xi, u^k\rangle - \beta\langle\gamma, x^k\rangle)\langle(1, \cdots, n), u - u^k\rangle \\
&\quad - \langle\beta(I - A^\top)^{-1}\gamma, x - f(x, u)\rangle \\
&= (\alpha\langle\xi, u^k\rangle - \beta\langle\gamma, x^k\rangle)\langle(1, \cdots, n), x - f(x, u)\rangle \\
&\quad - \langle\beta(I - A^\top)^{-1}\gamma, x - f(x, u)\rangle \\
&= \langle V, x - f(x, u)\rangle.
\end{aligned}$$

□

Corollary 3.10. *The unique optimal stationary state-control is the uniform stationary pair* (x^k, u^k)*.*

Proof. This comes from Assumption 3.6 and Lemma 3.9. □

According to Remark 3.4, Assumption 3.6 and Corollary 3.10, the computation of the optimal stationary forest state is reduced to determine the optimal uniform stationary state; that is, the age for which the maximum profit is obtained when comparing the average timber production and the average resource invested to grow trees until that age, see Figure X.2. The age k is known as an *optimal rotational age*.

Given the optimal rotational age k and the vector V defined in Lemma 3.9, the *value loss function* $\delta : \mathbb{K} \to \mathbb{R}$ can be defined as

$$\delta(x,u) := r(x^k, u^k) - r(x,u) + \langle V, x - f(x,u)\rangle. \tag{3.13}$$

From Lemma 3.9, it can be concluded that $\delta(x,u) \geq 0$ for all $(x,u) \in \mathbb{K}$. The value loss function was originally introduced in McKenzie [11] and is also known as *the rotated stage function*, see Diehl et al. [4].

Theorem 3.11. *There exists $\lambda \in \mathbb{R}$ and a function $h : \Delta \to \mathbb{R}$ such that*

$$\lambda + h(x) \geq \max_{u \in U(x)} \{r(x,u) + h(f(x,u))\} \quad \forall x \in \Delta. \tag{3.14}$$

Proof. This result follows from Lemma 3.9 taking $\lambda := r(x^k, u^k)$ and the function $h(x) := \langle V, x\rangle$. □

The inequality (3.14) is called *Average-Reward Optimality Inequality* (AROI) and the pair (λ, h) is known as a solution to the AROI. In this case, for every $\tau \in \mathbb{R}$, the pair $(\lambda, h+\tau)$ is also a solution.

4. Optimal ergodic policies

In this section, we study optimal forest configurations that can change through time; we focus our attention on the infinite horizon and no discounted optimization problem. In other words, given a initial forest state $x(0) = x$ and a feasible control policy $\mathbf{u} = \{u(t)\}_{t=0}^{T-1}$, we are interested in the behavior of the accumulated benefit (performance index),

$$J_T(x, \mathbf{u}) := \sum_{t=0}^{T-1} r(x(t), u(t)) \tag{4.15}$$

when $T \to \infty$.

4.1. Average optimality

First of all, let us consider one of the most widely used criteria for infinite horizon undiscounted optimal control problems: the *long-run average optimality*. For this criterion, the performance index is defined by

$$J(x, \mathbf{u}) := \liminf_{T\to\infty} \frac{1}{T} J_T(x, \mathbf{u}). \tag{4.16}$$

Definition 4.1. Given an initial forest configuration $x \in \Delta$, the *long-run-average value function* is:

$$J^*(x) := \sup\{J(x, \mathbf{u}) : \mathbf{u} \in \mathcal{U}(x)\}. \tag{4.17}$$

A control policy $\mathbf{u}^* \in \mathcal{U}(x)$ is said to be *average optimal* if $J(x, \mathbf{u}^*) = J^*(x)$.

The following theorem shows that there are average optimal policies, and the average optimal value coincides with the value of the stationary optimization problem (3.9).

Theorem 4.2. *Given a forest state $x \in \Delta$, a control policy $\mathbf{u} \in \mathcal{U}(x)$ is average optimal if, and only if*

$$J(x, \mathbf{u}) = p\left(\frac{\xi_k}{k}\right) - c\left(\frac{\gamma_1 + \cdots + \gamma_k}{k}\right) \tag{4.18}$$

where k is the age defined in Assumption 3.6.

Proof. We prove that $J^*(x) = r(x^k, u^k)$. It is not difficult to construct a control policy $\widehat{\mathbf{u}} \in \mathcal{U}(x)$ such that $\widehat{x}(0) = x$ and $(\widehat{x}(t), \widehat{u}(t)) = (x^k, u^k)$ for all $t \geq n$. Then, the control policy $\widehat{\mathbf{u}}$ is such that $J(x, \widehat{\mathbf{u}}) = r(x^k, u^k)$. Therefore $r(x^k, u^k) \leq J^*(x)$.

Conversely, from Lemma 3.9,

$$\begin{aligned} J_T(x,\mathbf{u}) &\le \sum_{t=0}^{T-1}[r(x^k,u^k)+\langle V, x(t)-x(t+1)\rangle] \\ &= Tr(x^k,u^k)+\sum_{t=0}^{T-1}\langle V, x(t)-x(t+1)\rangle \\ &= Tr(x^k,u^k)+\langle V, x(0)-x(T)\rangle. \end{aligned}$$

Thus

$$J_T(x,\mathbf{u}) \le Tr(x^k,u^k)+\langle V, x-x(T)\rangle. \tag{4.19}$$

Because Δ is a compact space, the term $\langle V, x-x(T)\rangle$ is bounded. Multiplying the inequality (4.19) by $1/T$ and taking limits, we obtain
$J(x,\mathbf{u}) \le r(x^k,u^k)$ for all $\mathbf{u}\in\mathcal{U}(x)$, and so $J^*(x)\le r(x^k,u^k)$. □

4.2. Bias optimality

Now, given a forest state $x\in\Delta$ and a policy $\mathbf{u}\in\mathcal{U}(x)$, let us consider the so called *bias function* (Veinott Veinott [17])

$$B_T(x,\mathbf{u}) := \sum_{t=0}^{T-1}[r(x(t),u(t))-J^*(x)], \tag{4.20}$$

and the accumulated value loss function

$$\delta_T(x,\mathbf{u}) := \sum_{t=0}^{T-1}\delta(x(t),u(t)). \tag{4.21}$$

Recalling (3.13), notice that the sequence $\{\delta_T(x,\mathbf{u})\}_{T=1}^{\infty}$ is non-negative and increasing. Moreover, the following equation is satisfied:

$$B_T(x, \mathbf{u}) = J_T(x, \mathbf{u}) - TJ^*(x) = -\delta_T(x, \mathbf{u}) + \langle V, x - x(T) \rangle. \tag{4.22}$$

Combining (4.19) and (4.22), we have: either the sequence a) $\{B_T(x, \mathbf{u})\}_{T=1}^{\infty}$ is bounded or b) $\liminf_{T\to\infty} B_T(x, \mathbf{u}) = -\infty$. From (4.22) we can see that the sequence $\{B_T(x, \mathbf{u})\}_{T=1}^{\infty}$ is bounded if and only if $\{\delta_T(x, \mathbf{u})\}_{T=1}^{\infty}$ is bounded. A policy for which these sequences are bounded is called *good policy* (Gale [6]).

From (4.19), any good policy is average optimal and satisfies a *turnpike property.*

Theorem 4.3. *Given $x \in \Delta$ and r is strictly concave in (x^k, u^k), if $\mathbf{u} \in \mathcal{U}(x)$ is a good control policy, then $\lim_{t\to\infty}(x(t), u(t)) = (x^k, u^k)$.*

Proof. Define the function $\phi : \mathbb{K} \to \mathbb{R}$ as

$$\phi(x, u) := r(x, u) - \langle V, x - x(T) \rangle.$$

Then $\phi(x, u) = -\delta(x, u) + r(x^k, u^k)$. Suppose that $\mathbf{u} = \{u(t)\}_{t=0}^{\infty}$ is a good control policy. Hence $\delta(x(t), u(t)) \to 0$ when $t \to \infty$, which implies that $\phi(x(t), u(t)) \to r(x^k, u^k) = \phi(x^k, u^k)$. If (x^*, u^*) is a cluster point of $\{(x(t), u(t))\}_{t=0}^{\infty}$, then $\phi(x^*, u^*) = \phi(x^k, u^k)$. On the other hand, $\phi(x, u) \le r(x^k, u^k)$ for all $(x, u) \in \mathbb{K}$. Moreover, since r is strictly concave at (x^k, u^k), then so is ϕ and it has a unique maximizer. Therefore, $(x^*, u^*) = (x^k, u^k)$. □

By (4.19), for a given forest state $x \in \Delta$ and any good control policy $\mathbf{u} \in \mathcal{U}(x)$, the following functions are well defined:

$$B(x, \mathbf{u}) := \sum_{t=0}^{\infty} [r(x(t), u(t)) - J^*(x)], \quad \delta(x, \mathbf{u}) := \sum_{t=0}^{\infty} \delta(x(t), u(t)). \tag{4.23}$$

Moreover, from (4.22), both satisfy

$$B(x,\mathbf{u}) + \delta(x,\mathbf{u}) = \langle V, x - x^k \rangle. \tag{4.24}$$

Definition 4.4. Let $x \in \Delta$ be a given initial condition. The *bias value function* is:

$$B^*(x) := \sup\{B(x,\mathbf{u}) : \mathbf{u} \in \mathcal{U}(x)\}. \tag{4.25}$$

A control policy $\mathbf{u}^* \in \mathcal{U}(x)$ is said to be *bias optimal* if $B(x,\mathbf{u}^*) = B^*(x)$.

Given an initial state $x \in \Delta$, and a bias optimal policy $\mathbf{u}^* \in \mathcal{U}(x)$, then for every good policy, $\mathbf{u} \in \mathcal{U}(x)$,

$$J_T(x,\mathbf{u}) - J_T(x,\mathbf{u}^*) = B_T(x,\mathbf{u}) - B_T(x,\mathbf{u}^*) \leq 0, \tag{4.26}$$

therefore,

$$\limsup_{T\to\infty}[J_T(x,\mathbf{u}) - J_T(x,\mathbf{u}^*)] \leq 0 \quad \text{for all } \mathbf{u} \in \mathcal{U}(x). \tag{4.27}$$

We can see that the converse is also true. Hence, (4.25) and (4.27) are equivalent. The policies that satisfy (4.27) are called *overtaking optimal*; this criterion was introduced by Gale [6] and von Weizsäcker [18].

5. Numerical examples

In this section, we illustrate the theoretical results from the previous sections by two examples.

Example 5.1. Assume that we are interested in the forest management of a certain species of trees that lose their economic value after they are 5 years old. Suppose that the biomass vector is given by $\xi = (0, 4, 9, 11, 11)$ and the investment vector by

$\gamma = (4, 3.5, 3, 2.5, 2.5)$, that is, a unit area completely covered by 1 year old trees produces 0 units of timber, and needs 4 units of water to maintain; a unit area completely covered by 2 years old trees produces 4 units of timber, and needs 3.5 units of water to maintain, and so on.

The average timber production by age is $\xi_1 = 0, \xi_2/2 = 2, \xi_3/3 = 3, \xi_4/4 = 2.75, \xi_5/5 = 2.2$. For its counterpart, the average volume of water consumed until each age is $\gamma_1 = 4$, $(\gamma_1 + \gamma_2)/2 = 3.75$, $(\gamma_1 + \gamma_2 + \gamma_3)/3 = 3.5$, $(\gamma_1 + \gamma_2 + \cdots + \gamma_4)/4 = 3.25$, $(\gamma_1 + \gamma_2 + \cdots + \gamma_5)/5 = 3.1$. Notice that the average timber production reaches its maximum at age 3; meanwhile, the minimum average resource invested takes place at age 4.

Now, let us suppose the income function for the timber production is $p(\tau) = 10\tau^{1/2}$ and the cost function for the water resource is $c(\tau) = \tau^2, \tau \geq 0$. Then, as in Assumption 3.6, the optimal rotational age is $k = 4$, see Table X.1. Therefore, from Corollary 3.10, the optimal stationary forest configuration is $x^4 = (1/4, 1/4, 1/4, 1/4, 0)$, with its control $u^4 = (0, 0, 0, 1/4, 0)$, and the optimal stationary value is $r(x^4, u^4) = 6.02$. Thus, the inequality (3.12) from Lemma 3.9 becomes

$$r(x, u) \leq \langle V, x - f(x, u) \rangle + 6.02 \quad \text{for all } (x, u) \in \mathbb{K},$$

where $V = (-113.58, -100.41, -90.50, -83.83, -80.41)$.

Table X.1

	1	2	3	4	5
$p\left(\frac{\xi_i}{i}\right)$	0	14.14	17.32	16.58	14.83
$c\left(\frac{\gamma_1 + \cdots + \gamma_i}{i}\right)$	16	14.06	12.25	10.56	9.61
$r(x^i, u^i)$	-16	0.07	5.07	6.02	5.22

Remark. In case we do not take into account the cost function c from Table X.1, we follow that the optimal stationary state is no longer reached at age 4 but, instead, it takes place at age 3. Hence, in that case, the optimal stationary forest state would be $x^3 = (1/3, 1/3, 1/3, 0, 0)$, with the harvesting plan $u^3 = (0, 0, 1/3, 0, 0)$.

Now, let us consider two non stationary forest configurations determined for the following initial state and control policies:

1. $x = (1, 0, 0, 0, 0), \;\; \mathbf{u}_1 = \{u(t)\}_{t=0}^{\infty}$

$$\begin{array}{ll} x(0) = x & u(0) = (0,0,0,0,0) \\ x(1) = (0,1,0,0,0) & u(1) = (0,1/4,0,0,0) \\ x(2) = (1/4,0,3/4,0,0) & u(2) = (0,0,2/4,0,0) \\ x(3) = (2/4,1/4,0,1/4,0) & u(3) = (0,0,0,1/4,0) \\ x(4) = (1/4,2/4,1/4,0,0) & u(4) = (0,1/4,0,0,0) \\ x(t) = x^4 & u(t) = u^4 \text{ for all } t \geq 5. \end{array}$$

2. $x = (1, 0, 0, 0, 0), \;\; \mathbf{u}_2 = \{u(t)\}_{t=0}^{\infty}$

$$\begin{array}{ll} x(0) = x & u(0) = (0,0,0,0,0) \\ x(1) = (0,1,0,0,0) & u(1) = (0,1/4,0,0,0) \\ x(2) = (1/4,0,3/4,0,0) & u(2) = (0,0,1/4,0,0) \\ x(3) = (1/4,1/4,0,2/4,0) & u(3) = (0,0,0,1/4,0) \\ x(4) = (1/4,1/4,1/4,0,1/4) & u(4) = (0,0,0,0,1/4) \\ x(t) = x^4 & u(t) = u^4 \text{ for all } t \geq 5. \end{array}$$

Both of these policies are good, and in consequence, average optimal and $J(x, \mathbf{u}_1) = J(x, \mathbf{u}_2) = J^*(x) = 6.02$. Nonetheless, they have different bias values: $B(x, \mathbf{u}_1) = -35.61$ and $B(x, \mathbf{u}_2) = -31.07$. Since $\langle V, x - x^4 \rangle = -16.49$, recalling (4.24), the accumulated value losses are $\delta(x, \mathbf{u}_1) = 19.11$, and $\delta(x, \mathbf{u}_2) = 14.58$.

Example 5.2. In this example, we show an average optimal policy that converges to the optimal stationary state, but it is not a good policy. Suppose that (x^k, u^k) is an optimal stationary state-control. For $t \geq k$, consider the following forest configuration and its harvest plan

$$x(t) = \left(\frac{1}{k} + \sum_{i=0}^{k-2} \frac{1}{t+i}, \frac{1}{k} - \frac{1}{t+k-2}, ..., \frac{1}{k} - \frac{1}{t}, 0, ..., 0 \right),$$

$$u(t) = \left(\sum_{i=0}^{k-1} \frac{1}{t+i}, 0, ..., 0, \frac{1}{k} - \frac{1}{t}, 0, ..., 0 \right).$$

We can see that $(x(t), u(t)) \to (x^k, u^k)$ as $t \to \infty$. So, since r is a continuous function, we have that $\lim_{t\to\infty} r(x(t), u(t)) = r(x^k, u^k)$, which means that the policy is average optimal. On the other hand, from Lemma 3.7, for all $t \geq k$ we have

$$\begin{aligned} r(x(t), u(t)) - r(x^k, u^k) &\leq \alpha\langle \xi, u(t) - u^k \rangle - \beta\langle \gamma, x(t) - x^k \rangle \\ &\leq -\tfrac{\alpha\xi_k}{t} + \alpha \textstyle\sum_{i=0}^{k-1} \frac{\xi_1}{t+i} - \beta \sum_{i=0}^{k-2} \frac{\gamma_1 - \gamma_{k-i}}{t+i}. \end{aligned}$$

Now, for simplicity, we can assume that $\xi_1 = 0, \xi_k > 0$ and $\gamma_1 \geq \gamma_i$ for all $i = 1, ..., n$. Hence, $r(x(t), u(t)) - r(\bar{x}, \bar{u}) \leq -\frac{\alpha\xi_k}{t}$ for all $t \geq k$. Thus, $\sum_{t=0}^{\infty} r(x(t), u(t)) - r(\bar{x}, \bar{u}) = -\infty$, and the policy is not good.

6. Concluding remarks

In this work, the Mitra-Wan forestry model is studied as a discrete-time optimal control problem and, in contrast to previous works on this topic, we consider a profit function depending on the timber production and forest maintenance cost. This considera-tion gives us a more precise value of the optimal rotational age (optimal stationary policy): from Lemma 3.9 and Corollary 3.10, this age is the year where the difference between average timber production and average cost of forest maintenance is maximum. On the other hand, for the non stationary optimal control policies, three optimality criteria were studied: average optimality, good policy and bias optimality. The average optimal value coincides with optimal stationary value (Theorem 4.2). The set of average optimality policies includes the good policies, and the set of good policies includes bias optimal policies. Moreover, the good policies converge to the optimal stationary policy (turnpike property), Theorem 4.3.

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XI. EXISTENCE, CHARACTERIZATION AND SIMULATION OF OPTIMAL POLICIES IN A FAMILY OF EPIDEMIC MODELS

Saul Díaz-Infante, Francisco Peñuñuri and David González-Sánchez

Abstract

We survey some theoretical results about a family of optimal control problems that arise in epidemiology. We also implement the so-called forward-backward-sweep method in Python to find approximate optimal control policies via the Pontryagin maximum principle. In addition, four specific models are described and simulated.

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1. Introduction

By the end of the Middle Ages, smallpox cut down the population in centers of Europe and Asia—three of every ten died by smallpox—perhaps that gave it its alias, “speckled monster”. Although experts did not understand the mechanism of transmission of this “monster” until the early 20 century, it represents the first documented disease Bernoulli [3], Bradley et al. [6], and Foppa [14] against which a specific control intervention was available: the inoculation. This process relied on putting material from smallpox sores into healthy people, usually scratching material over an armor or inhaling it through the nose. People developed the symptoms associated with smallpox—fever and a rash. However, the death rate due to inoculation was considerably lower than naturally acquired smallpox.

Then Bernoulli formulated a question in the following manner: What would happen if everybody were inoculated? Here, we address the question: How to inoculate in an optimal way? Throughout the following pages, we try to answer and illustrate the implications of this question.

Optimal control theory is a way to deal with the above question. In the fifties, Pontryagin and Bellman proposed generalizations of the calculus of variations of broader applicability: the Maximum Principle and the method of Dynamic Programming, respectively. Now, these results have applications in the biological sciences and, in particular, in the optimal control of infectious diseases (see Yu et al. [42], Lahrouz et al. [25], Junyoung et al. [21], and Cai et al. [9] for recent literature).

Our approach in this work relies on Pontryagin’s Maximum Principle, Pontryagin et al. [30] and follows the same methodology of Lenhart and Workman [27]. Lenhart’s work makes an accessible optimal control device to describe common epidemic interventions, like vaccination, treatment, quarantine, and isolation among others. Our intention in this work is illustrating the mentioned strategies throughout recent literature and stating results for the existence and characterization of optimal controls for a particular family of epidemic models. Likewise, we present some goals and issues that appear within the theory and numerical approximations.

The paper is organized as follows: we start in Section 2 by introducing a rather simple, but seminal, dynamical system from which most of the epidemic models are derived. In Section 3, we describe four epidemic models, as well as some control policies. In Section 4, we provide the main theoretical results for a family of optimal control problems (OCPs); such a family includes the models in Section 3. Our proofs are based on well known results—stated, for completeness, in the Appendix of Section 8—from optimal control theory. Some numerical methods to solve OCPs are given in Section 5; in particular, we provide the Python implementation code of the forward-backward-sweep method in repository Díaz-Infante et al. [11]. The reader is free to comment, use, improve or do whatever he/she wants about this repository. In Section 6, we run several numerical experiments based on Díaz-Infante et al. [11] for the models described in Section 3. We conclude with some remarks in Section 7.

2. The uncontrolled SIR model

Infectious diseases have struck civilizations in different periods of human history. HIV AIDS, Spanish influenza and Black Death have been the most devastating pandemics; they have killed more than 100 million people. Therefore, understanding the mechanism of spread and control of diseases of this kind is essential. In this line, the SIR structure is a convenient option to model its spread.

The SIR model is a compartmental structure. Primarily, the model consists of three compartments: susceptible S, infected I, and recovered R and transitions functions between compartments.

Practically, all the existing epidemic models are variants of this structure. The variants emerge to describe particular characteristics of a disease, its mechanism of transmission, population dynamics, among others. To fix ideas, consider the classic model of Kermack and Mckendrick [23]

$$\begin{aligned} \frac{dS}{dt} &= -\kappa SI \\ \frac{dI}{dt} &= \kappa SI - \lambda I \\ \frac{dR}{dt} &= \lambda I, \end{aligned} \tag{2.1}$$

with the transition from the susceptible S to the infected class I occurring with a constant rate κ, and from the infected class I to the recovered, happening with rate λ.

In next sections, we provide the main ideas to modify this basic structure with control policies.

3. Control policies in epidemics

Here we present several control models that we consider to be good examples. Before talking about these good examples, we give the core of optimal policy modeling (see, for example, the survey of Wickwire [38], for more details).

First, we require a model to describe the spreading of an uncontrolled disease whose transitions generate a cost. Then, we add a continuous control action to modify the changes from one state to another, but in such a way that the mentioned cost is optimized. A rule that prescribes which control operation to use at each time is a control policy. A control policy which applies only information from the current state of the controlled system to prescribe control actions is a *closed-loop* or *feedback* control. If the current state is not observable, or the control function only depends on the time, we have an *open-loop* policy, the type of policies that we consider in this work.

Here, we consider control policies that affect the bounded rates at which a population moves from one class (e.g., infected) to another (e.g., recovered). In all these problems, the control function is linear in the relevant dynamic. Next, we specify a cost function which assigns the total cost of the control policy implementation. Then, the problem is to determine a policy that optimizes the considered cost strategy.

Now, we present the previously mentioned good examples. In what follows, X denotes a vector with all concerned populations: for example, according to SIR model (2.1), $X = (S, I, R)^\top$.

3.1. Culling

Pathogens that are transmitted between wildlife, livestock, and humans present major challenges for the protection of human and animal health: the economic sustainability of agriculture and the conservation of wildlife. Mycobacterium bovis, the aetiological agent of bovine tuberculosis (TB) is one such pathogen. For example, according to Donnelly et al. [12], the incidence of TB in cattle has increased substantially in parts of Great Britain in the past two decades, adversely affecting the livelihoods of cattle farmers and potentially increasing the risks of human exposure. The control of bovine TB in Great Britain is complicated by the involvement of wildlife, particularly badgers, which appear to sustain endemic infection and can transmit TB to cattle. Between

1975 and 1997, over 20 000 badgers were culled as part of British TB control policy, generating conflict between conservation and farming interest groups.

3.2. Badger bovine tuberculosis

Bolzoni et al. [4] report a controlled model to describe an outbreak of badger bovine tuberculosis. According to this model,

$$\min_{u(t)\in\mathcal{U}} \int_0^T I(t) + P[u(t)]^\theta, \quad \theta \in \{1,2\}, \quad P = B/A$$

subject to:

$$\begin{aligned} \frac{dS}{dt} &= rS\left(1 - \frac{S+I}{K}\right) - \beta SI - u(t)S \\ \frac{dI}{dt} &= \beta SI - (\alpha + \mu + u(t))I. \end{aligned} \tag{3.2}$$

Here, the susceptible class follows a logistic dynamics with net growth rate $r = \nu - \mu$ and carrying capacity K (see Table XI.1 for more details). According to the approach of Driessche [13], the basic reproductive number results:

$$R_0 = \frac{\beta K}{\alpha + \mu}.$$

Our intention with this model is to illustrate the difference between the optimal policies for linear and quadratic costs, though the corresponding infected populations are quite similar. Then, as we can see in Section 6.1 below, the resulting controlled paths depend on the form of the functional cost and the basic reproductive number.

3.3. Vaccination

Usually, public health organizations consider vaccination as a primarily preventive action against infectious diseases, but it incurs a cost. Due to the limited resources associated with vaccination programs, it is imperative to optimize the use of available resources. Using optimal control theory, we formulate a vaccination schedule. The goal is to minimize the number of infected persons and the cost of vaccinating during a fixed time. For this example, we optimize the functional

$$\int_0^T AI(t) + u^2(t).$$

Here u is the vaccination control and denotes the fraction of susceptible individuals to vaccinate per unit of time. Since managing an infected population implies resource consumption, A represents the cost per individual. We also need a spread dynamics. So, let $S(t)$, $E(t)$, $I(t)$, $R(t)$, respectively, be the number of susceptible, exposed, infectious and recovered (immune) individuals at time t. Since the vaccination of the entire susceptible population is impractical, the model considers $0 \leq u(t) \leq 0.9$. Then the whole population N is given by $N(t) = S(t) + E(t) + I(t) + R(t)$, and obeys $\dot{N}(t) = (b-d)N(t) - aI(t)$. Since b is the recruitment rate and d is the natural death, the term $b - d$ denotes the growth of the entire population. Then, the optimal control problem reads

$$\min_u \int_0^T AI(t) + u^2(t)dt,$$

subject to

$$\dot{S}(t) = bN(t) - dS(t) - cS(t)I(t) - u(t)S(t), \quad S(0) = S_0 \geq 0,$$

$$\dot{E}(t) = cS(t)I(t) - (e + d)E(t), \quad E(0) = E_0 \geq 0,$$

$$\dot{I}(t) = eE(t) - (g + a + d)I(t), \quad I(0) = I_0 \geq 0,$$

$$
\begin{aligned}
\dot{R}(t) &= gI(t) - dR(t) + u(t)S(t), \quad R(0) = R_0 \geq 0, \\
\dot{N}(t) &= (b-d)N(t) - aI(t), \quad N(0) = S_0 + E_0 + I_0 + R_0.
\end{aligned} \tag{3.3}
$$

See Table XI.2 for a description of the parameters.

3.4. Case finding and case control

Two-strains of Tuberculosis. Seeking to reduce the latent and infectious groups with the resistant-strain of tuberculosis, in Lenhart et al. [26] the authors use control theory to describe optimal strategies in a tuberculosis model which considers the effect of treatment in two kinds of strains. The controlled version reads:

$$
\begin{aligned}
\frac{dS}{dt} &= \Lambda - \beta_1 S \frac{I_1}{N} - \beta_3 S \frac{I_2}{N} \mu S, \\
\frac{L_1}{dt} &= \beta_1 S \frac{I_1}{N} - (\mu + k_1) L_1 - u_1(t) r_1 L_1 + (1 - u_2(t)) p r_2 I_1 \\
&\quad + \beta_2 T \frac{I_1}{N} - \beta_3 L_1 \frac{I_2}{N}, \\
\frac{I_1}{dt} &= k_1 L_1 - (\mu + d_1) I_1 - r_2 I_1, \\
\frac{L_2}{dt} &= (1 - u_2(t)) q r_2 I_1 - (\mu + k_2) L_2 + \beta_3 (S + L_1 + T) \frac{I_2}{N}, \\
\frac{I_2}{dt} &= k_2 L_2 - (\mu + d_2) I_2, \\
\frac{dT}{dt} &= u_1(t) r_1 L_1 + (1 - (1 - u_2(t))(p + q)) r_2 I_1 - \beta T \frac{I_1}{N} \\
&\quad - \beta_3 T \frac{I_2}{N} - \mu T.
\end{aligned} \tag{3.4}
$$

Lenhart, Jung, and Feng Lenhart et al. [26] consider time dependent optimal control strategies associated with *case holding* and *case finding*. They incorporate the case finding control by adding a term which identifies and cures a fraction of the latent individuals. Case finding, therefore, reduces the rate of disease development by latent individuals. The authors include case holding by adding a term which may decrease the treatment failure rate of individuals with sensitive TB, so this control reduces the incidence of drug resistant TB. In model (3.4), u_1 denotes the fraction of typical TB latent individuals that are identified and put under treatment —case finding control—and $1 - u_2$ represents the effort that prevents the typical TB treatment failure in infectious individuals.

The controls u_1 and u_2 reduce the latent and infected groups with resistant TB. However, the case holding and the case finding strategies produce a cost modeled as

$$\int_0^{t_f} \left[L_2(t) + I_2(t) + \frac{B_1}{2}[u_1(t)]^2 + \frac{B_2}{2}[u_2(t)]^2 \right] dt. \qquad (3.5)$$

3.5. Isolation and quarantine

SARS. If an emergent disease lacks a rapid diagnostic test, therapy, or vaccine, then isolation and quarantine of individuals exposed to the disease seem obvious control policies. For example, Gumel et al. [17] model strategies of this kind for the severe acute respiratory syndrome (SARS). SARS was a highly contagious viral disease that emerged in China late in 2002 and quickly spread to 32 countries and regions, causing more than 774 deaths and 8098 infections worldwide.

Based on the work of Gumel et al. [17], Yan and Zou report in Yan and Zou [40] a control epidemic model for SARS. They use quarantine and isolation as mitigation controls. The authors also propose sub-optimal control policies and perform numerical simulations with genetic algorithms. The controlled version used in the cited reference reads:

$$
\begin{aligned}
\frac{dS}{dt} &= \Lambda - \frac{S\left(\beta I + \mathcal{E}_E \beta E + \mathcal{E}_Q \beta Q + \mathcal{E}_J \beta J\right)}{N} - \mu S, \\
\frac{dE}{dt} &= p + \frac{\beta S\left(\beta I + \mathcal{E}_E \beta E + \mathcal{E}_Q \beta Q + \mathcal{E}_J \beta J\right)}{N} \\
&\quad - (u_1(t) + k_1 + \mu)E, \\
\frac{dQ}{dt} &= u_1(t)E - (k_2 + \mu)Q, \\
\frac{dI}{dt} &= k_1 E - (u_2(t) + d_1 + \sigma_1 + \mu)I, \\
\frac{dJ}{dt} &= u_2(t)I + k_2 Q - (d_2 + \sigma_2 + \mu)J, \\
\frac{dR}{dt} &= \sigma_1 I + \sigma_2 J - \mu R.
\end{aligned}
\tag{3.6}
$$

The control variable u_1 denotes the proportion of people in quarantine who had contact with an infected person inside of a quarantine program or educational campaign. Control u_2 models the proportion of symptomatic population which is in an isolation program. The authors consider the following epidemiological classes.

- S: Susceptible individuals
- E: Asymptomatic individuals who have been exposed to the virus but have not yet developed clinical symptoms of SARS
- Q: Quarantined individuals
- I: Symptomatic
- J: Isolated
- R: Recovered

$N = S + E + Q + I + J + R.$

We enclose a description of the model parameters in Appendix Table XI.4. So, giving the disease dynamics in (3.6), the problem is to minimize the functional cost

$$\int_0^{t_f} \left[B_1 E(t) + B_2 Q(t) + B_3 I(t) + B_4 J(t) + \frac{C_1}{2} u_1^2(t) + \frac{C_2}{2} u_2^2(t)\right] dt. \tag{3.7}$$

Here, parameter B_i denotes the linear cost of the infected class, and C_1, C_2 are the costs for isolation and quarantine controls, respectively. Table XI.4 displays a description of the included parameters.

A common practice to deal with the above control problems follows in the next steps:

(i) Prove that there exists an optimal policy.

(ii) Find necessary conditions for the optimality of a policy.

(iii) From the necessary conditions, determine qualitative properties of the optimal policies and the corresponding state paths.

Usually, these kinds of problems are non linear, and finding a solution is extremely difficult. Therefore, choosing a convenient numerical scheme is very important. In this work, we implement the forward-backward-sweep method Lenhart and Workman [27]. The next sections provide a technique to transform a given optimization problem into solving an ordinary differential equation with boundary values coupled with an optimization problem.

4. Existence and characterization of optimal policies

4.1. Notation

Each element x in $\mathbb{R}^n$ is written as a column vector and $x^\top$ denotes the transpose. We write the gradient of $g : \mathbb{R}^n \to \mathbb{R}$ as a row vector

$$g_x = (\partial g/\partial x_1, \dots, \partial g/\partial x_n).$$

If $\lambda : \mathbb{R} \to \mathbb{R}^n$ is differentiable, the derivative is denoted as $\dot{\lambda} = (d\lambda^1/dt, \dots, d\lambda^n/dt)^\top$. The Jacobian matrix of $f = (f^1, \dots, f^n)^\top : \mathbb{R}^n \to \mathbb{R}^n$ is

$$f_x = \begin{bmatrix} f_x^1 \\ f_x^2 \\ \vdots \\ f_x^n \end{bmatrix}.$$

Given a matrix A, the j-th row of A is denoted as $r_j(A)$.

The non controlled epidemic models described above are of the form

$$\begin{aligned} \dot{X} &= AX + \begin{bmatrix} X^\top B^{(1)} \\ \vdots \\ X^\top B^{(n)} \end{bmatrix} X + k \\ &= \left(A + [X^\top \cdots X^\top] \begin{bmatrix} B^{(1)} \\ \vdots \\ B^{(n)} \end{bmatrix} \right) X + k \end{aligned}$$

where the matrix A represents the linear part of the system, each matrix $B^{(j)}$, $j = 1, \dots, n$, gives the *interaction* part as a quadratic form, and k is a constant vector.

Thus, the j-th row of the above system takes the form

$$\dot{X}_j = r_j(A)X + X^\top B^{(j)} X + k_j.$$

In this section, we consider control policies in both the linear and the interaction parts of the latter system. This family of control systems with a corresponding cost functional includes the above epidemic models.

For such a family of optimal control problems, we state and prove three main results. First, given any control policy, we establish the existence and uniqueness of the associated state path. Second, the existence of an optimal control policy is proven. Finally, by means of the Maximum Principle, sufficient conditions of a control policy and the corresponding state path are also given. Our proofs are based on general and well-known results in optimal control theory which, for completeness, are stated in the Appendix at the end of the paper.

4.2. A family of control systems

Let $\mathbf{X} \subseteq \mathbb{R}^n$ and $\mathbf{U} \subseteq \mathbb{R}^m$ be nonempty and compact sets. The sets $\mathbf{X}$ and $\mathbf{U}$ are, respectively, called the *state space* and the *control space*. The vectors in X have non-negative entries; in particular, we assume that $0 \in X$. The control set $\mathbf{U}$ is convex. We consider the following control system, for $j = 1, \ldots, n$,

$$\dot{X}_j = [r_j(A) + u^\top C^{(j)}]X + X^\top \begin{bmatrix} r_1(B^{(j)}) + u^\top D^{(j1)} \\ \vdots \\ r_n(B^{(j)}) + u^\top D^{(jn)} \end{bmatrix} X + k_j \tag{4.8}$$

where $A \in \mathbb{R}^{n\times n}$, $B^{(j)} \in \mathbb{R}^{n\times n}$, $C^{(j)} \in \mathbb{R}^{m\times n}$, and $D^{(jl)} \in \mathbb{R}^{m\times n}$ for $l = 1, \ldots, n$.

The proof of the following theorem slightly differs from that given in Yong [41, Sect. 2.1] since we consider a weighted norm. This improvement allows us to give a global solution, instead of a local one.

Theorem 4.1. *For each measurable function* $u : [0,T] \to \mathbf{U}$ *and each initial condition* $x_0 \in X$*, there exists a unique absolutely continuous function* $X_u : [0,T] \to \mathbb{R}^n$ *that satisfies the system* (4.8) *almost everywhere.*

Proof. Let $u : [0,T] \to U$ be a measurable function. The control system (4.8) can be written as

$$\dot{X}(t) = f(X(t), u(t)), \quad X(0) = x_0, \quad 0 \le t \le T,$$

where $f : \mathbf{X} \times \mathbf{U} \to \mathbb{R}^n$. Since f is of class $\mathcal{C}^1$ on the compact set $\mathbf{X} \times \mathbf{U}$, there exists a constant $L > 0$ such that

$$\|f(x,u) - f(x_1,u)\| \le L\|x - x_1\| \tag{4.9}$$

$$\|f(0,u)\| \le L \tag{4.10}$$

for every $x, x_1 \in \mathbf{X}$ and $u \in \mathbf{U}$.

Consider the linear space

$$\mathbb{X} = \{X : [0,T] \to \mathbb{R}^n \mid X \text{ is continuous}\}$$

with the norm

$$\|X\|_w := \sup_{t\in[0,T]} \frac{\|X(t)\|}{w(t)},$$

where $w(t) := e^{Lt}$ for each $t \in [0,T]$. It can be shown, with slight modifications in [36, Section 2.1], that the pair $(\mathbb{X}, \|\cdot\|_w)$ is a Banach space. Define the operator $K : \mathbb{X} \to \mathbb{X}$ by

$$K[X](t) := x_0 + \int_0^t f(X(s), u(s))ds.$$

By (4.9) and (4.10), any (x, u) satisfies

$$\|f(x,u)\| \le L(1+\|x\|); \tag{4.11}$$

thus, $f(X(\cdot), u(\cdot))$ is Lebesgue integrable and $K[X]$ is absolutely continuous. We claim that K is a contraction with contraction constant $1 - e^{-LT}$. Indeed,

$$\begin{aligned}
&\|K[X] - K[Y]\|_w \\
&\qquad = \sup_{t\in[0,T]} \frac{|\int_0^t [f(X(s),u(s)) - f(Y(s),u(s))]ds|}{w(t)} \\
&\qquad \le \sup_{t\in[0,T]} \frac{L\int_0^t w(s)[w(s)]^{-1}|X(s)-Y(s)|ds}{w(t)} \\
&\qquad \le L\|X-Y\|_w \sup_{t\in[0,T]} \frac{\int_0^t w(s)ds}{w(t)} \\
&\qquad = L\|X-Y\|_w \sup_{t\in[0,T]} \frac{[e^{Lt}-1]/L}{e^{Lt}} \\
&\qquad = (1-e^{-LT})\|X-Y\|_w.
\end{aligned}$$

Then, by Banach's fixed point theorem [36, Theorem 2.1], there exists a unique $X \in \mathbb{X}$ satisfying

$$X(t) = x_0 + \int_0^t f(X(s), u(s))ds.$$

Therefore, (4.8) holds almost everywhere [28, Corollary 5.4.1]. □

4.3. Existence of optimal policies

Consider the *cost functional* of an admissible control u, given the initial state x_0,

$$V(u, x_0) := \int_0^T g(X(t), u(t))\ , dt, \tag{4.12}$$

where $g : \mathbf{X} \times \mathbf{U} \to \mathbb{R}$ is continuous. The *optimal control problem* (OCP) consists of finding an admissible control u^* such that

$$V(u^*, x_0) = \inf\{V(u, x_0) \mid u \in \mathbb{U}_B\}.$$

If there exists such a control u^*, then it is called an *optimal policy* or *optimal control.* The pair (u^*, X^*), where X^* is given by Theorem 4.1, is called an *optimal pair.*

Theorem 4.2. *Suppose the function* g *is continuous, and, for each* x*, the function* $g(x, \cdot)$ *is convex, i.e.,*

$$\alpha g(x, u_1) + (1-\alpha) g(x, u_2) \geq g(x, \alpha u_1 + (1-\alpha) u_2)$$

for all $u_1, u_2 \in \mathbf{U},\ \alpha \in [0, 1]$*. Then, there exists an optimal pair that minimizes* (4.12) *subject to* (4.8).

Proof. Let us write the control system (4.8) as $\dot{X} = f(X, u)$. By Filippov's Theorem 8.2, it is enough to show that each set

$$\{(z, y) \in \mathbb{R} \times \mathbb{R}^n \mid z \geq g(x, u),\ y = f(x, u),\ u \in \mathbf{U}\}, \quad x \in X,$$

is convex. Fix $x \in X$. Let $z_1, z_2 \in \mathbb{R}$ and $y_1, y_2 \in \mathbb{R}^n$ such that

$$z_j \geq g(x, u_j), \quad j = 1, 2, \tag{4.13}$$

and

$$y_j = f(x, u_j), \quad j = 1, 2, \tag{4.14}$$

for some $u_1, u_2 \in U$. We need to show that for any $\alpha \in [0, 1]$, there exists $u' \in U$ such that

$$\alpha z_1 + (1-\alpha) z_2 \geq g(x, u') \tag{4.15}$$

and

$$\alpha y_1 + (1-\alpha) y_2 = f(x, u'). \tag{4.16}$$

Let $u' := \alpha u_1 + (1-\alpha)u_2$. Then (4.15) follows from (4.13) and the convexity of $g(x,\cdot)$. On the other hand, (4.16) holds because $f(x,\cdot)$ is affine, i.e.,

$$f(x, \alpha u_1 + (1-\alpha)u_2) = \alpha f(x,u_1) + (1-\alpha) f(x,u_2).$$

□

4.4. Sufficient conditions for optimality

Consider the Hamiltonian $\mathcal{H} : \mathbf{X} \times \mathbf{U} \times \mathbb{R}^n \to \mathbb{R}$, defined as

$$\mathcal{H}(x,u,\lambda) := g(x,u) + \lambda^\top f(x,u),$$

and

$$\mathcal{H}^*(x,\lambda) := \inf_{u \in \mathbf{U}} \mathcal{H}(x,u,\lambda),$$

where g determines the cost functional (4.12) and f is given by the right-hand side of the control system (4.8). The function $\lambda : [0,T] \to \mathbb{R}^n$ and the admissible pair (u,X) are said to satisfy the necessary conditions of the *Maximum Principle* (MP) if they satisfy the *adjoint equation*

$$\dot{\lambda}(t) = -\mathcal{H}_x(X(t), u(t), \lambda(t))^\top, \quad \lambda(T) = 0, \tag{4.17}$$

and the *optimality condition*

$$\mathcal{H}^*(X(t), \lambda(t)) = \mathcal{H}(X(t), u(t), \lambda(t)). \tag{4.18}$$

Definition 4.3. The function w from $[0,T]$ to some Euclidean space is *piecewise continuous* if

(a) w is continuous on $[0,T]$ except at a finite number of points, and

(b) if w is discontinuous at t, then

$$\lim_{s\to t^-} w(s) \text{ and } \lim_{s\to t^+} w(s)$$

are finite.

Theorem 4.4. *Let $\lambda : [0,T] \to \mathbb{R}^n$ be a continuous function and let (u^*, X^*) be an admissible pair such that*

(a) *u^* is piecewise continuous,*

(b) *$\dot{X}^*$ exists and is piecewise continuous,*

(c) *(λ, u^*, X^*) satisfies the optimality condition* (4.18) *and,*

(d) *except at the points of discontinuity of u^*, the adjoint equation* (4.17) *holds.*

If, for each t, the function $\mathcal{H}^*(\cdot, \lambda(t))$ is convex on $\mathbf{X}$, then (u^*, X^*) is an optimal pair.

Proof. The conclusion follows from Theorem 8.8 whenever Assumptions 8.6 and 8.7 hold. If u^* is piecewise continuous, then X^* is absolutely continuous by Theorem 4.1, and so continuous. Thus, Assumption 8.7 holds. Assumption 8.6 also holds since $\mathcal{H}$ and $\mathcal{H}_x$ are clearly continuous. This completes the proof. □

Remark 4.5. In general, the convexity of $H^*(\cdot, \lambda(t))$ does not hold for the whole family (4.8) even if g is convex. However, the above models meet this assumption.

5. Numerical analysis

5.1. Direct and indirect methods

Since we can transform the problem of optimal control into a two-point boundary ODE problem, the methods designed for this sort of problem are applicable; see Keller [22], Ascher et al. [1], and Stoer et al. [34] for classic references. In this line, Caetano and Yoneyama [8] and Yan and Zou [40] use multiple shooting methods to solve the resulting extended two-value boundary ODE.

Multiple shooting method. Consider the controlled dynamics and corresponding adjoint equations given by

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \qquad x(0) = x_0 \\ \dot{\lambda}(t) &= -\mathcal{H}_x(x(t), u(t), \lambda(t))^\top, \quad \lambda(T) = 0. \end{aligned} \tag{5.19}$$

Roughly speaking, the multiple shooting method follows the next algorithm. Given a partition of the interval $[0, T]$ with uniform step h,

$$\tau_h^n := \{t_k = kh,\ k = 0, \dots n\}.$$

The multi shooting method is described in Algorithm 1.

However, the *forward-backward-sweep method* is the most popular method in works on optimal control epidemic models, perhaps for its simple implementation and acceptable convergence. All simulations presented in this work run with this scheme. Hackbusch [18] proposes this numerical scheme to solve a class of optimal problems that encloses the models of Section 3. Lenhart and Workman [27] provide MATLAB code for much of their work in biological models.

5.2. Evolutionary algorithms

Evolutionary algorithms are kinds of heuristic algorithms well suited for global optimization. Such algorithms emulate natural evolution introducing operators for mutation (**M**), crossover (**C**) and selection (**S**). One of the earliest works on evolutionary methods was developed by George E. P. Box Box [5]. Nevertheless, it can be said that the first evolutionary algorithm, at least as they are known today, was the so-called Genetic Algorithm (GA) introduced by Holland [20]. Many variations of evolutionary algorithms have been developed with Differential Evolution (DE), introduced by Storn and Price [35], being one of the simplest, yet efficient and effective, optimization algorithms.

Algorithm 3 shows the general form of Evolutionary Algorithms for optimizing the objective function fob. There, an initial population Y of size N_p, generated in the search space $\mathcal{V}$ by the operator $\mathbf{Y}_0$, is subject to the evolutionary process until a certain stopping criterion is met. Then the best individual ($\mathbf{y}_{best}$), i.e., the individual who optimizes fob, is selected by introducing the operator $\mathbf{Best}$. In Algorithm 3 the variable M stores a mutated population; the variable C stores the results of the crossover operator. The selection operator selects from C and Y the individuals which will form the new generation of individuals of Y. This selection is based in some criteria usually dictated by the objective function. For instance, if $\mathbf{y}$ is an element of Y and $\mathbf{c}$ an element of C, a common criterion for minimization is to select $\mathbf{y}$ if $fob(\mathbf{y}) < fob(\mathbf{c})$.

A detailed explanation for constructing the main operators $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{S}$ can be found in Bagchi [2] for GA and in Price et al. [31] for the DE algorithm.

Regarding the optimal control policies problem, authors frequently apply the evolutionary method by using piecewise constant functions for the controllers $u_k, k = 1, 2, \ldots, n$. For instance, the optimization of a quantity of the form

$$V(u, x_0) = \int_0^T g(X(t), u(t))dt, \tag{5.20}$$

can be conducted by discretizing the interval $I = [0, T]$ in disjoint subintervals I_j and choosing

$$u_k(t) = \begin{cases} u_k^j & \text{if } t \in I_j \\ 0 & \text{otherwise.} \end{cases} \tag{5.21}$$

Usually, the function $X(t)$ under the sign of integral in Equation (5.20) obeys a specific dynamical model which needs to be solved, but in such a way that V is optimized. Now, the numbers u_k^j will be part of an individual who will be subject to the evolutionary process. Yan and Zou [40] and Junyoung et al. [21] follow this approach for the GA and the DE algorithms, respectively.

5.3. Optimal Control Software

In addition to the implementation of the schemes discussed above, we provide a list with useful software and some of its references. We follow the list reported in Rodrigues et al. [32]; see this reference for code examples and more details.

OC-ODE The OC-ODE, Gerdts [16], Optimal Control of Ordinary-Differential Equations, by Matthias Gerdts, is a collection of Fortran 77 routines for optimal control problems subject to ordinary differential equations. It uses an automatic direct discretization method for the transformation of the OC problem into a finite-dimensional non linear problem. OC-ODE includes procedures for numerical adjoint estimation and sensitivity analysis.

DOTcvp Hirmajer et al. [19], provide the MATLAB Toolbox DOTcvp. Giving a piecewise solution for the control, the toolbox uses the control vector parametrization approach for the calculation of the optimal control profiles.

Muscod-II MUSCOD-II is a robust and efficient optimization tool that allows one to quickly implement and solve very general optimal control problems in differential-algebraic equations (DAE). This package relies on the Multiple Shooting method for the solution of mixed integer nonlinear ODE or DAE. The authors provide the code and a reference manual, Kühl et al. [24].

Ipopt [37] provide the software package Ipopt (Interior Point OPTimizer). Ipopt implements a primal-dual interior point method and uses a line search strategy based on filter method and is written in `Fortran` and `C`.

Knitro Byrd et al. [7] report `Knitro 5.0`, a C-package for nonlinear optimization that combines complementary approaches to nonlinear optimization to achieve robust performance over a wide range of application requirements. The package is designed for solving large-scale, smooth nonlinear programming problems, and it is also effective for the following special cases: unconstrained optimization, nonlinear systems of equations, least squares, and linear and quadratic programming. Various algorithmic options are available, including two interior methods and an active-set method.

6. Numerical experiments

6.1. *Culling in badger bovine tuberculosis*

In this numerical experiment, we go back to the culling control model (3.2). Our main objective is to contrast two kinds of controls that produce quite similar paths for the infected population under different cost schemes. All simulations run with the forward-backward-method and with the parameters enclosed in Table XI.5.

6.2. *Vaccination*

Now, we come back to the vaccination control presented in model (3.3). According to Table XI.6, we illustrate the effect of vaccination control in Figure XI.4. The simulation shows that the optimal-vaccination policy diminishes almost to zero the infected population. We plot the state solution without control in black to stress the impact of the optimal policy.

6.3. *Case finding and case control in a two strain tuberculosis model*

Figure XI.5 shows the effect of case finding and case holding controls. The combination of these strategies diminishes the multi-drug resistant population. Table XI.7 compiles the parameters and their values used to produce this figure with $N = 30\,000$, $\beta_3 = 0.29$. To minimize the resistant TB population, L2 + I2, the simulation suggests that the case holding strategy u_2 would be at the upper bound during almost $4.3\,\mathrm{years}$ and then would decrease to the lower bound. Meanwhile, case finding is applied over most of the simulated time, $5\,\mathrm{years}$. The total number of infected resistant TB $L2 + I2$ at the final time $t_f = 5\,\mathrm{years}$ would be 1123. This same number, but without control, results in 4176. So this policy prevents $3053\,\mathrm{cases}$ of resistant TB.

According to Table XI.7 and taking different values for the parameter β_3, in Figure XI.6 we display the effect of parameter β_3 over controls. At the top, both controls experience small variations at the beginning, but reach almost the same level after $5\,\mathrm{years}$. The simulation suggests that lower values of β_3 just delay the control profile for few months. At the bottom, we enclose a zoom frame to emphasize the small difference for case holding. Summarizing, the simulation shows that parameter β_3 modifies the case finding control in a wider way.

Figure XI.7 illustrates case finding and case holding strategies with populations of different sizes. The simulation suggests that with relatively small populations, it is more important to prioritize case finding, while for relatively bigger populations, case holding is the more important strategy.

All figures run with `forward-backward-sweep method`, see Díaz-Infante et al. [11] to check a Python implementation code.

6.4. Isolation and quarantine for SARS

Since SARS lacked effective treatments or vaccines, measures to control the spread of SARS had to take two major forms: isolating symptomatic individuals and quarantining symptomatic individuals under close observation WHO [39]. We return to model (3.6) and obtain, via the forward-backward-sweep method, the optimal policies. We use the value parameters listed in Table XI.8.

The left side of Figure XI.8 shows the simulations of total infected populations $E+Q+I+J$. Here we contrast the resulting dynamics using a constant policy (solid green), $\widehat{u}_1 = 0.2, \widehat{u}_2 = 0.2$ with the optimal quarantine and isolation control (dash orange) policies u_1, and u_2. At left, we see that in order to minimize the number of total infected individuals, optimal control quarantine u_1 is at its upper bound during more than $150\,\mathrm{days}$, then u_1 is steadily decreased to the lower bound. Optimal control isolation u_2 stays at its upper bound about $70\,\mathrm{days}$ and then steadily decreases to the lower bound over the rest simulated time.

Figure XI.9 contrasts the evolution of the dynamics controlled with the optimal policy, a constant policy $\widehat{u}_1 = \widehat{u}_2 = 0.2$ and the lower bound policy, $\bar{u_1} = \bar{u_2} = 0.05$.

7. Concluding remarks

We end by mentioning some points that should be kept in mind when dealing with OCPs in epidemics.

Uniqueness of optimal policy The proof of the uniqueness of the state path X_u, given a policy u, is fairly standard (Theorem 4.1). However, the uniqueness of an *optimal policy* is not trivial and it can be established in a sufficiently short time interval; see, for instance, Gaff and Schaefer [15] and the references therein.

Numerical Issues According to the forward-backward-sweep and multi shooting methods, both schemes need an ODE solver for one of its steps. However, sometimes this solver generates spurious solutions as a result of numeric instability. We see an opportunity to apply nonstandard numerical schemes which are consistent with the underlying conservation laws (see for example the work of Mickens [29]).

Regarding the application of genetic algorithms, it would be interesting to address the problem considering controls as general functions, and not only restricted to piecewise constant functions in the interval I. As far as we know, there is no work addressing the optimal control policies problem in this manner, and thus, this paragraph intends to motivate further research in this direction.

Maximum principle vs. Dynamic programming The same approach is followed in almost all the related literature on optimal control of epidemics/diseases. As an alternative, the so-called Dynamic programming approach can be used to analyze these kinds of problems. With the Maximum principle, we need to solve a system of ordinary differential equations (ODEs), whereas in Dynamic programming, a partial differential equation (PDE) arises. In addition, both approaches involve an optimization problem. The Maximum principle is mostly used because there are plenty of methods to numerically solve ODEs.

By following the DP approach, the optimal policies are obtained in *feedback* (or *Markov*) form; i.e., the control policy is a function of the state of the system. Thus DP is a natural approach to solve stochastic models.

8. Appendix: Deterministic OCPs in continuous time

Consider the following *control system*

$$\dot{X}(t) = f(t, X(t), u(t)), \quad X(0) = x_0, \quad 0 \leq t \leq T, \tag{8.22}$$

where $f : [0,T] \times \mathbf{X} \times \mathbf{U} \to \mathbb{R}^n$ and $u : [0,T] \to U$. For each u, it is assumed that there exists a unique solution X_u to (8.22) (ensured, for instance, by Theorem 4.1). In some applications the terminal state $X_u(T)$ is constrained to belonging to a given set $\mathbf{B}$. Then the set of *admissible controls* is defined as

$$\mathbb{U}_{\mathbf{B}} := \{u : [0,T] \to \mathbf{U} \mid u \text{ is measurable and } X_u(T) \in \mathbf{B}\} \tag{8.23}$$

and $\mathbb{U}_{\mathbf{B}}$ is also assumed to be nonempty. A pair (u, X_u), where $u \in \mathbb{U}_{\mathbf{B}}$, is called an *admissible pair*. To simplify notation, we write (u, X).

The following performance index is said to be in the *Bolza form*

$$V(u, x_0) := \int_0^T g(t, X(t), u(t))dt + h(X(T)), \tag{8.24}$$

where $g : [0, T] \times \mathbf{X} \times \mathbf{U} \to R$ and $h : \mathbf{X} \to \mathbb{R}$ are measurable. When $g = 0$ and $h \neq 0$, it is said to be in the *Mayer form*. Another form occurs when $h = 0$ and $g \neq 0$; in such a case (8.24) is said to be in the *Lagrange form*. These three forms are equivalent; see, for instance, Cesari [10, Sect. 1.9].

In Section 3 we considered minimization problems; in contrast, in this appendix, we consider maximization problems. The reason is due to the name, *Maximum principle*, which appears in (8.28). Then the OCP consists of finding an admissible control u^* such that

$$V(u, x_0) = \sup\{V(u, x_0) \mid u \in \mathbb{U}_{\mathbf{B}}\}.$$

The elements of the OCP can be given in the following seven-tuple

$$(\mathbf{X}, \mathbf{U}, \mathbf{B}, f, g, h, T). \tag{8.25}$$

Assumption 8.1. The sets $\mathbf{X}$, $\mathbf{U}$, and $\mathbf{B}$ are compact. The functions f, g, and h are continuous.

A proof of the following theorem can be found, for instance, in Cesari [10, Sect. 9.3.] or Yong [41, Theorem 2.2.1].

Theorem 8.2 (Filippov). *Assume the OCP* (8.25) *satisfies Assumptions 1. If for almost every t in $[0, T]$, each set*

$$F(t, x) := \{(\alpha, y) \in \mathbb{R} \times \mathbb{R}^n \mid \alpha \leq g(t, x, u),\ y = f(t, x, u),\ u \in \mathbf{U}\}, \tag{8.26}$$

with $x \in X$ is convex, then there exists an optimal pair (u^*, X^*).

We define the *Hamiltonian*, for each $(t, x, u, \lambda_0, \lambda)$ in $[0, T] \times \mathbf{X} \times \mathbf{U} \times \mathbb{R} \times \mathbb{R}^n$, as

$$H(t, x, u, \lambda_0, \lambda) := \lambda_0 g(t, x, u) + \lambda^\top f(t, x, u).$$

Assumption 8.3. (a) The function h is of class $\mathcal{C}^1$.

(b) For every $(t, u, \lambda_0, \lambda)$, the function $H(t, \cdot, u, \lambda_0, \lambda)$ is of class $\mathcal{C}^1$.

(c) For every $(t, x, \lambda_0, \lambda)$, the functions

$$H(t, x, \cdot, \lambda_0, \lambda) \text{ and } H_x(t, x, \cdot, \lambda_0, \lambda)$$

are continuous.

The following theorem is proven in Yong [41, Theorem 2.3.1].

Theorem 8.4 (Maximum Principle). *Suppose the OCP* (8.25) *satisfies Assumptions 8.1 and 8.3. Suppose also that the set* $\mathbf{B}$ *is convex. Let* (u^*, X^*) *be an optimal pair. Then there exists a constant* $\lambda_0 \geq 0$ *and an absolutely continuous function* $\lambda : [0, T] \to \mathbb{R}^n$*, with*

$$(\lambda_0)^2 + \|\lambda(T) - \lambda_0 h_x(X^*(T))^\top\|^2 = 1, \tag{8.27}$$

that satisfy

(a) *the maximum condition, for almost every* $t \in [0, T]$,

$$\mathcal{H}(t, X^*(t), u^*(t), \lambda_0, \lambda(t)) \geq \mathcal{H}(t, X^*(t), u, \lambda_0, \lambda(t)) \;\forall u \in \mathbf{U}, \tag{8.28}$$

(b) *the adjoint equation, for almost every* $t \in [0, T]$,

$$\dot{\lambda}(t) = -H_x(t, X^*(t), u^*(t), \lambda_0, \lambda(t))^\top, \tag{8.29}$$

(c) *and the transversality condition*

$$[\lambda(T)^\top - \lambda_0 h_x(X^*(T))][y - X^*(T)] \geq 0 \quad \forall y \in \mathbf{B}. \quad (8.30)$$

Remark 8.5. As pointed out by Yong [41, p. 43], if $\mathbf{B} = \mathbb{R}^n$, then (8.30) implies

$$\lambda(T) - \lambda_0 h_x(X^*(T))^\top = 0$$

and so $\lambda_0 = 1$ by (8.27). In such a case, the Hamiltonian takes the form

$$\mathcal{H}(t, x, u, \lambda) := g(t, x, u) + \lambda^\top f(t, x, u) = H(t, x, u, 1, \lambda).$$

Then the form of the Hamiltonian used in Section XX is justified. Further, when $h = 0$, the adjoint equation (8.29) and the transversality condition (8.30) become

$$\dot{\lambda}(t) = -g_x(t, X^*(t), u^*(t))^\top - [f_x(t, X^*(t), u^*(t))]^\top \lambda(t), \ \lambda(T) = 0$$

as in (4.17).

Consider the OCP (8.25) with $\mathbf{B} = \mathbb{R}^n$ and $h \equiv 0$. Define

$$\begin{aligned} \mathcal{H}^*(t, x, \lambda) &:= \sup_{u \in \mathbf{U}} \mathcal{H}(t, x, u, \lambda) \\ &= \sup_{u \in \mathbf{U}} \{g(t, x, u) + \lambda^\top f(t, x, u)\}. \end{aligned}$$

Assumption 8.6. The functions $\mathcal{H}$ and $\mathcal{H}_x$ are continuous.

Assumption 8.7. The functions $u^* : [0, T] \to \mathbf{U}$ and $X^* : [0, T] \to \mathbf{X}$ satisfy the following:

(a) u^* is piecewise continuous on $[0, T]$,

(b) X^* is continuous on $[0, T]$,

(c) $\dot{X}^*$ exists and it is piecewise continuous on $[0, T]$.

The following theorem is proven in Seierstad and Sydsæter [33, Theorem 3].

Theorem 8.8. *Suppose that Assumption 8.6 holds. Let (u^*, X^*) be an admissible pair that satisfies Assumption 8.7. Suppose that there exists a continuous function $\lambda : [0, T] \to \mathbb{R}^n$ such that*

$$\mathcal{H}(t, X^*(t), u^*(t), \lambda(t)) \geq \mathcal{H}(t, X^*(t), u, \lambda(t)) \quad \forall u \in \mathbf{U}, \tag{8.31}$$

and, except at the points of discontinuity of u^*,

$$\dot{\lambda}(t) = -\mathcal{H}_x(t, X^*(t), u^*(t), \lambda_0, \lambda(t))^\top, \quad \lambda(T) = 0. \tag{8.32}$$

If the set $\mathbf{X}$ is convex and, for each t, the function $\mathcal{H}^*(t, \cdot, \lambda(t))$ is concave on $\mathbf{X}$, then (u^*, X^*) is an optimal pair.

9. Appendix: Algorithms

Algorithm 1 Multi shooting method

Input: t_0, T, x_0, h, tol, λ_f, n_{max}
Output: x^*, u^*, λ

procedure MULTI_SHOOTING($g, \lambda_{\text{function}}, u, x_0, \lambda_f, h, n_{max}$)
  while $\epsilon >$ tol **do**
    Choose $y_i := [x(t_i), \lambda(t_i)], \quad i = 1, \ldots, n.$

    Integrate (5.19) for each sub-interval $[t_i, t_{i+1})$ using y_i as the initial conditions
    and obtain $y(t_{i-1}) = [x(t_{i-1}), \lambda(t_{i-1})], i = 2, \ldots, n.$

    $\mathcal{Y} \leftarrow [y_i - y(t_i)], i = 0, \ldots, n$
    Actualize initial condition y_i for next iteration using for a example a
    Newton's method.
    $\epsilon \leftarrow |\mathcal{Y}|$
  end while
end procedure

Algorithm 2 Forward Backward Sweep

```
Input: t_0, t_f, x_0, h, tol, λ_f
Output: x*, u*, λ

procedure FORWARD_BACKWARD_SWEEP(g, λ_function, u, x_0, λ_f, h, n_max)
    while ε > tol do
        u_old ← u
        x_old ← x
        x ← RUNGE_KUTTA_FORWARD(g, u, x_0, h)
        λ_old ← λ
        λ ← RUNGE_KUTTA_BACKWARD(λ_function, x, λ_f, h)
        u_1 ← OPTIMALITY_CONDITION(u, x, λ)
        u ← αu_1 + (1 − α)u_old,    α ∈ [0, 1]      ▷ convex combination
        ε_u ← ||u − u_old|| / ||u||
        ε_x ← ||x − x_old|| / ||x||                  ▷ relative error
        ε_λ ← ||λ − λ_old|| / ||λ||
        ε ← max {ε_u, ε_x, ε_λ}
    end while
    return x*, u*, λ                                  ▷ Optimal pair
end procedure
```

Algorithm 3 Evolutionary Algorithms

$Y \leftarrow \mathbf{Y}_0(Np, \mathcal{V})$
while (the stopping criterion has not been met) **do**
  $M \leftarrow \mathbf{M}(Y)$
  $C \leftarrow \mathbf{C}(Y, C)$
  $Y \leftarrow \mathbf{S}(Y, C, fob)$
end while
$\mathbf{y}_{best} \leftarrow \mathbf{Best}(Y, fob)$

10. Appendix: Tables

Table XI.1: Parameter description of the control model (3.2).

Parameter	Description
ν	Natural fertility rate
μ	Natural mortality rate
K	Carrying capacity
α	Disease-induced mortality rate
R_0	Basic reproductive number
β	Transmission coefficient
P	Relative cost per unit culling effort over the cost of a single infection

Table XI.2: Parameters and simulation values of the epidemic model (3.3).

Parameter	Description
b	Recruitment rate
a, d	Disease and natural death rates
c	Incidence of disease
e	Rate at which the exposed individuals become infectious
g	Recovering rate
A	Vaccination cost
T	Final time

Table XI.3: Parameter descriptions for the control problem (3.4).

Parameter	Description	Parameter	Description
β_1	Probability that a susceptible individual becomes infected.	r_1	Treatment recovery rate of individuals with latent TB.
β_2	Probability that a recovered individual becomes infected.	r_2	Treatment recovery rate of individuals with infectious TB.
β_3	Probability that an uninfected individual becomes infected by resistant-TB.	p, q	Proportion of infectious individuals that don't complete the treatment for TB or MDR-TB, respectively.
μ	Natural per-capita death rate.	N	Total population size.
d_1	Per-capita death rate from TB.	Λ	Recruitment rate.
d_2	Per-capita death rate from MDR-TB.	t_f	Final time.
k_1	Rate at which a latent TB individual becomes infectious.	B_1	Systematic cost of the case finding control.
k_2	Rate at which a latent individual with MDR-TB becomes infectious.	B_2	Cost of the case holding strategy

Table XI.4: Parameter description for the SARS model (3.6).

Parameter	Description
β	Transmission coefficient
$\varepsilon_E, \varepsilon_Q, \varepsilon_J$	Modification parameter for exposed, quarantined and isolated classes
μ	Natural death rate
Λ	Constant recruitment rate
p	Net inflow of asymptomatic individuals
k_1	Transfer rate from class of asymptomatic to symptomatic
k_2	Transfer rate from the quarantined class to isolated
d_1, d_2	Per-capita disease induced death rates for the symptomatic individuals and isolated individuals
σ_1, σ_2	Per-capita recovery rates for the symptomatic individuals and isolated individuals
t_f	Final time
B_1, B_2, B_3, B_4	Cost for E,Q,I,J classes
C_1, C_2	Costs for Isolation and Quarantine policies

Table XI.5: Parameter values of model (3.2) to reproduce Figure XI.1, XI.2, and XI.3.

Parameter values		Initial Conditions
ν	0.6	
μ	0.4	$S(0) = K, I(0) = 1$
K	0.4	
α	0.05	
R_0	6.0, 3.5	**Control bound**
β	$\dfrac{R_0(\alpha+\mu)}{K}$	$u_{max} = 0.1$
P	70.0, 110.0	

Table XI.6: Parameters and simulation values of the epidemic model (3.3).

Parameter values		Initial conditions
b	0.525	$S(0) = 1000$, $E(0) = 100$
a, d	0.2, 0.5	$I(0) = 50$, $R(0) = 15$
c	0.0001	
e	0.5	
g	0.1	
A	0.1	
T	20.0	

Table XI.7: Simulation values for the control problem (3.4).

Parameter values				Initial Conditions
β_1	13.0	β_2	13.0	$S(0) = (76/120)N$
β_3	0.0131, 0.0217, 0.029, 0.0436			$L_1(0) = (36/120)N$
				$L_2(0) = (2/120)N$
				$I_1(0) = (4/120)N$
μ	0.0143			$I_2(0) = (1/120)N$
d_1	0.0	d_2	0.0	$T(0) = (1/120)N$
k_1	0.5	k_2	1.0	
r_1	2.0	r_2	1.0	
p	0.4	q	0.1	
N	6000, 12 000, 30 000	Λ	μN	**Control Bounds**
				Lower 0.05
				Upper 0.95
t_f	5.0 years			
B_1	50.0	B_2	500.0	

Table XI.8: Parameter description for the SARS model (3.6).

Parameter values			
β	0.2	d_1, d_2	0.0079, 0.0337
$\varepsilon_E, \varepsilon_Q, \varepsilon_J$	0.3, 0.0, 0.1	k_1, k_2	0.1, 0.125
μ	0.000 034		
Λ	μN		
p	0.0		
σ_1, σ_2	0.0337, 0.0386		**Initial conditions**
			$S(0) = 12 \times 10^6$, $E(0) = 1565$,
t_f	1.0 year		$Q(0) = 292$, $I(0) = 695$,
Step size	$dt = 1.0$ day		$J(0) = 326$, $R(0) = 20$
u_i bounds	0.05, 0.5		
B_1, B_2, B_3, B_4	1.0, 1.0, 1.0, 1.0		
C_1, C_2	300, 600		

11. Appendix: Figures

Figure XI.1: State solutions without control, under optimal quadratic control and with linear (bang-bang) control.

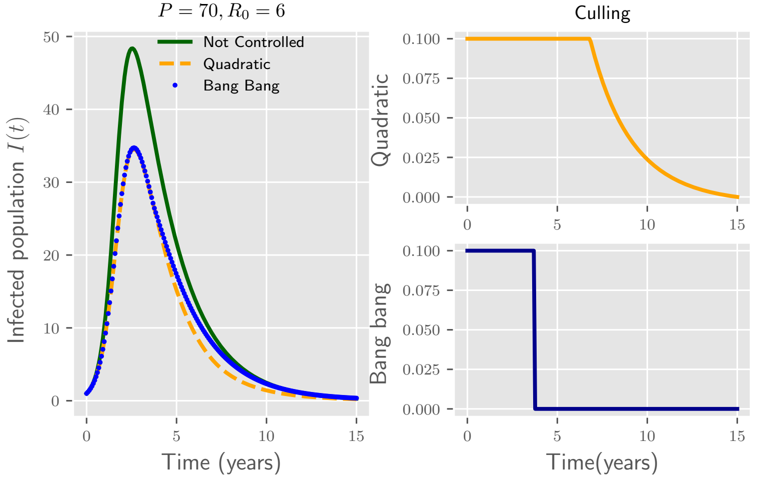


Figure XI.2: State solutions without control, under optimal quadratic control and with linear (bang-bang) control.

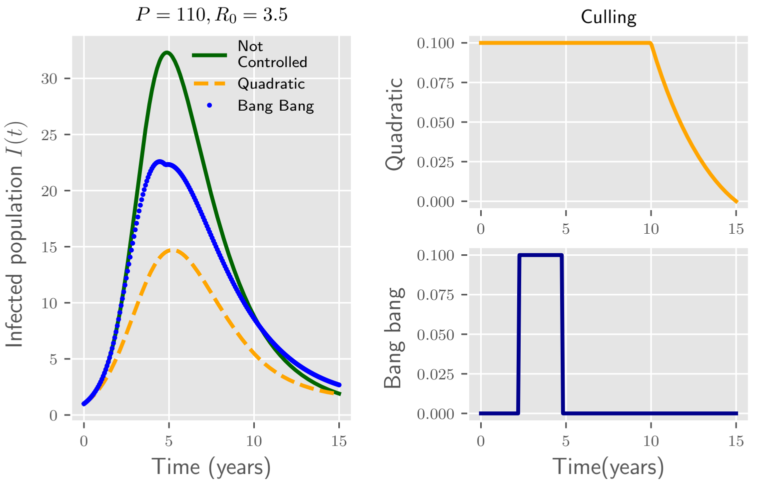


Figure XI.3: Costs for the linear and quadratic controls under two scenarios. Upper $P = 70$, $R_0 = 6$, bottom, $P = 110$, $R_0 = 3.5$, and the rest of parameters as in Table XI.5.

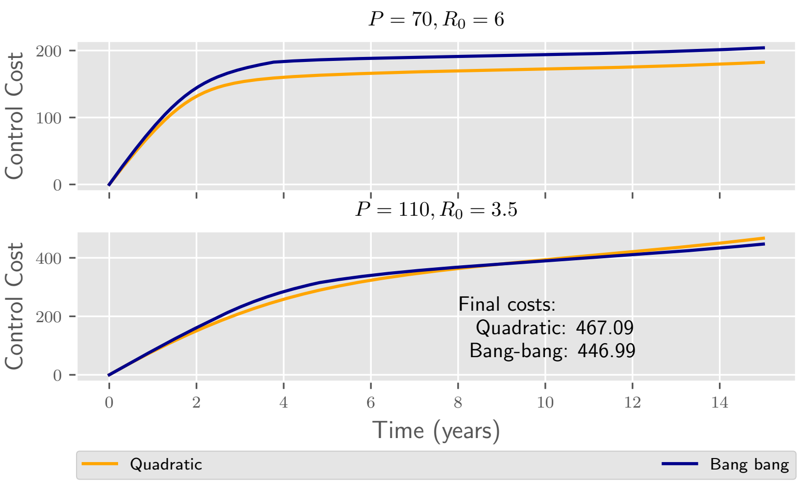


Figure XI.4: Comparison between the controlled and uncontrolled infected population. On the left, we show the optimal infected state against the dynamics without control. On the right, we present the corresponding adjoint function λ and the optimal control.

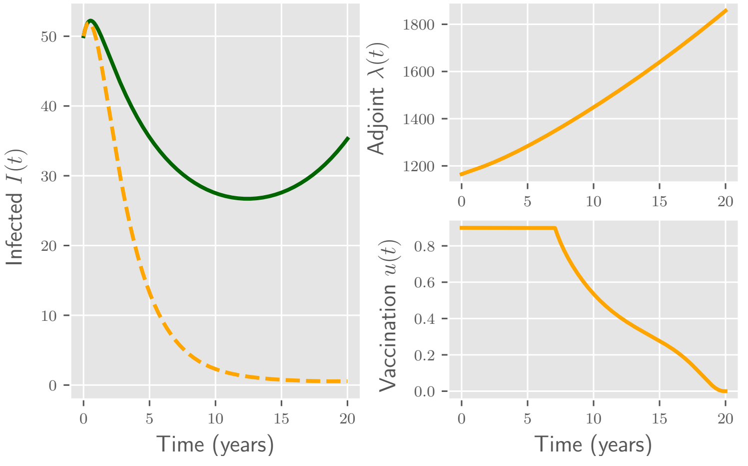


Figure XI.5: Normalized infected population according to parameters of Table XI.7. Here, the black line represents the infected population without control. As we see, combining case finding $u_1(t)$ and case holding $u_2(t)$, dramatically diminishes the density of infection with resistant TB.

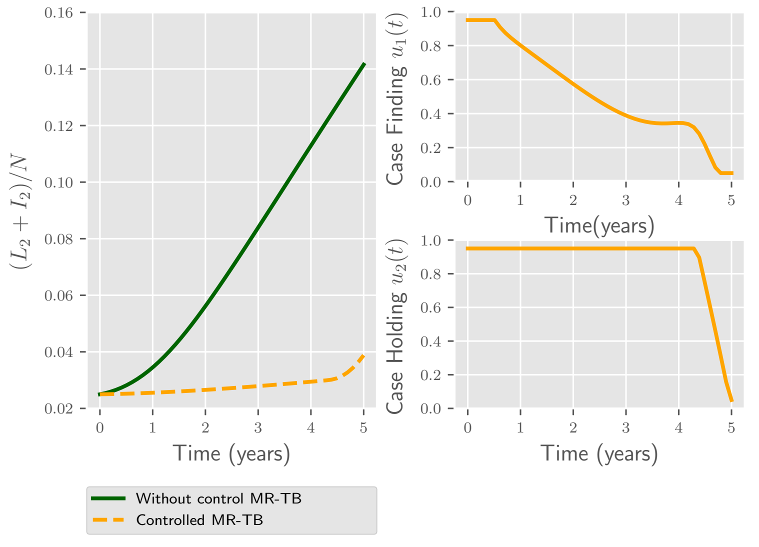


Figure XI.6: At the top, case finding and case holding controls are shown with the parameters listed in Table XI.7 with different values for parameter β_3. At the bottom, we capture a smaller region to illustrate the variations regarding case holding. This simulation suggests that case holding results have almost the same profile, while case finding delays these same results by only a few months.

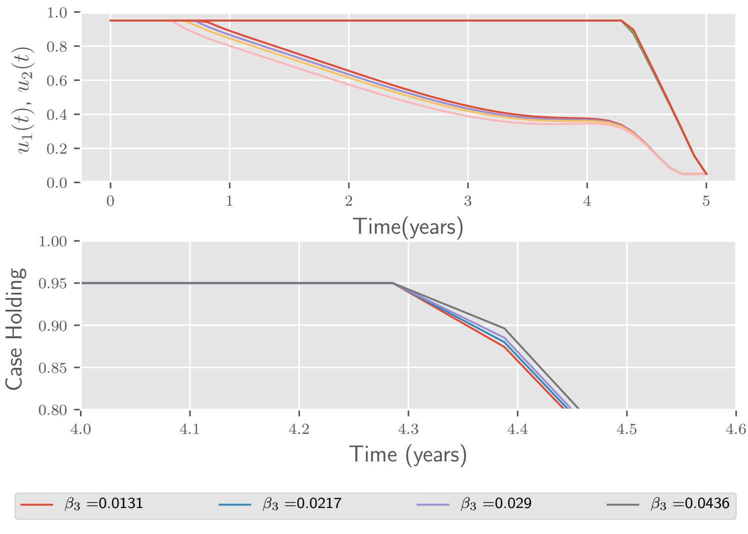


Figure XI.7: The effect of different sizes of population. For relatively small populations, the case finding strategy is more important than case holding, while for bigger populations, the case holding plays a more important role. The rest of the parameters are shown in Table XI.7.

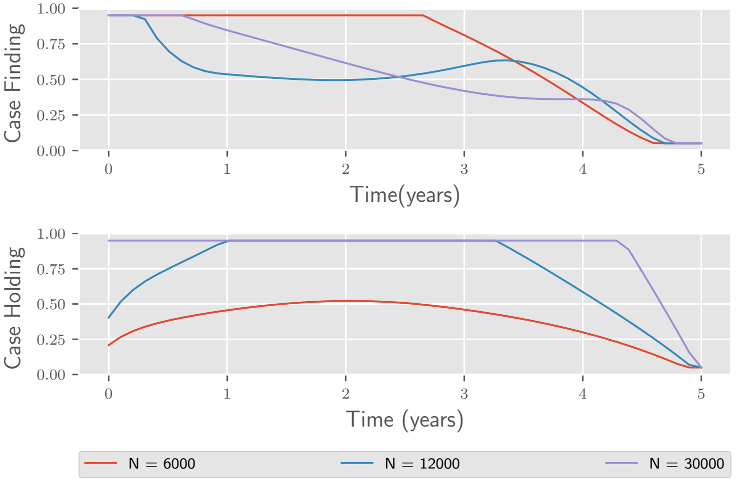


Figure XI.8: On the left, the simulations of the whole infected population without constant control and under the optimal control policy.

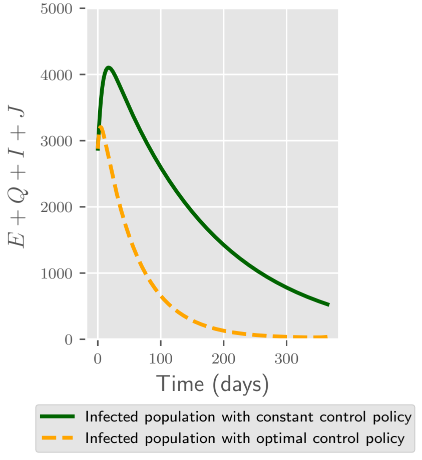


Figure XI.9: Susceptible and recovered populations under optimal, constant and lower bound control policies.

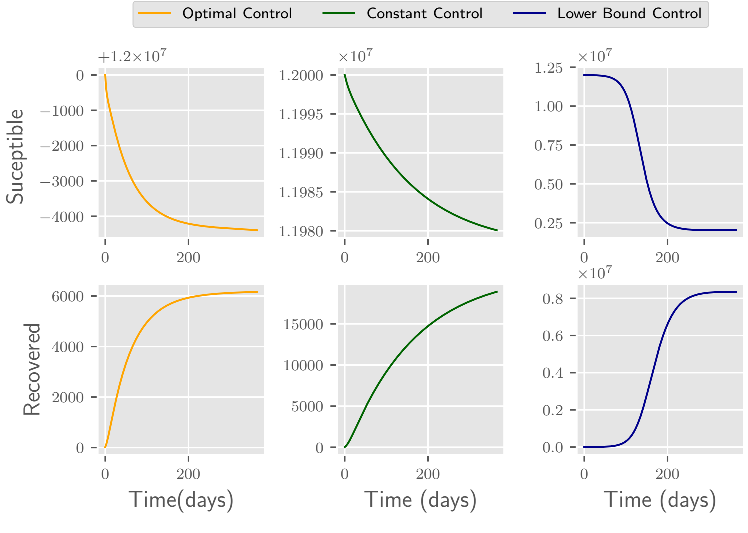
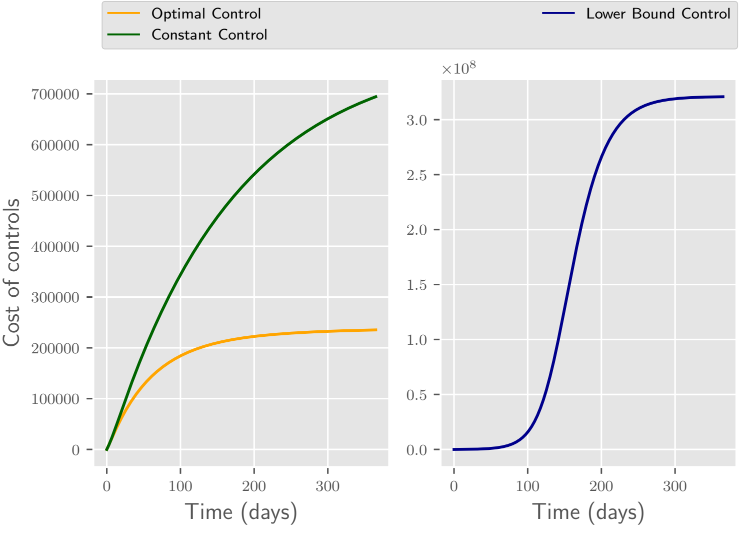


Figure XI.10: Cost of disease control for the optimal, constant and lower bound policies, see Equation (3.7).



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XII. POLICE OBEDIENCE IN LOCAL GOVERNMENTS: A NORMAL GAME MODEL FROM PUBLIC POLICY PERSPECTIVE

Mónica Naime and Itza Tlaloc Quetzalcoatl Curiel Cabral

Abstract

The forced disappearance of forty-three students in Ayotzinapa, Guerrero in September 2014 brought international attention to gross human rights violations in Mexico. Even if disappearances committed by state agents are not new -notably the disappearances during the dirty war from the 1960's until the 1980's - they have been on the rise since 2007. Currently, disappearances do not respond exclusively to a logic of elimination of political opposition, but no studies have been undertaken about the probable causes of the rise.

The present work seeks to propose an explanation from a public policy perspective, since more than 96% of disappearances occur at the local level. Towards that end, we develop a normal

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game model using parameters to understand the interaction between municipal presidents and their policemen.

We find that the hierarchical structure of police officers subservient to municipal presidents sets the stage for the development of dominant strategies between each other. This creates a dominant equilibrium: municipal presidents will seek their private interests, and policemen will obey. Also, by themselves, police salaries are insignificant. More relevant are the payoffs for corruption and obedience.

1. Introduction

The disappearance of forty-three students in Ayotzinapa, Guerrero in September 2014 gave rise to national and international attention about human rights violations in Mexico. Since then, forced disappearances—the disappearance of persons by agents of the State—has been a relevant issue in the national public agenda.

Disappearances in Mexico are not novel: from the end of the 1960's through the beginning of the 1980's, hundreds of persons were disappeared by agents of the State during the so-called *dirty war*, UNHRC [29]. During this period, the police and military forces undertook a policy of systematic repression of students, peasants and social activists, amongst others. The causes of the disappearances during the dirty war are well known: the government adopted forced disappearance as a strategy to wipe out any alleged member of the opposition.

This strategy was put into practice for the first time during the national-socialist regime of the Third-Reich of Germany. In December 1941, the Fuhrer issued a decree which instituted the Night and Fog program. The decree established that most of the prisoners of the occupied territories accused of offenses against the German State should be transferred to Germany without a trial. Only if the officers believed it was highly probable that the offender would receive capital punishment, could she stay in the occupied territory NND [18]. Likewise, it envisaged that if anyone would ask for information about the prisoners taken to Germany, officers would only say that they had been arrested, but no more information should be disclosed. They were going to be disappeared in the night and in the fog, this is, without information.

These disappearances had a two-fold objective: on the one hand, to prevent prisoners from asking for the protection of the law and, on the other hand, to intimidate the rest of the family and the population, due to the uncertainty of the whereabouts of the prisoner, Finucane [6]. Forced disappearance was employed as a hierarchical, systematic and intentional strategy of terror.

This same strategy was implemented by different military dictatorships in Latin America, notably in Brazil (1964-1979), Argentina (1976-1983) and Chile (1973-1990)[1]. In the Brazilian case, it was possible to determine the intentionality and systematization of the government in the more than 450 forced disappearances because of the existence of a registry of their actions, Informe-Brazil [13]. Those responsible haven't been subjected to trial, due to an amnesty.

Regarding Argentina, in the frame of the program "National Reorganization Process", acts of torture and forced disappearances were undertaken systematically, under the argument of the fight against guerrilla groups. In 1983, Argentinian President Raul Alfonsin - on the fifth day of his presidency- created the National Commission on the Disappearance of Persons. In its final report,

[1] Military dictatorships of Bolivia, El Salvador, Guatemala, Nicaragua, Peru and Uruguay, during the 1960's and 1980's, present similar cases to those herein described (Molina Theissen [17]).

"Never More" CONADEP [4], this commission verified the disappearance of 8960 people, the appropriation of new born babies, and proved the existence of a deliberate government plan to disappear its citizens. Afterwards, in 1985, members of the Military Juntas would be judged and sentenced for these crimes.

In Chile, during the military dictatorship of General Pinochet, 1192 people disappeared in the hands of the government, specifically the Army, the *carabineros* and the National Intelligence Office. In 1991, president Patricio Aylwin established the National Commission for Truth and Reconciliation. In its final report, the Rettig Report, the commission recognized the will to exterminate certain categories of persons based on political reasons, Rettig-Report [24]. Afterwards, several mechanisms were instituted to compensate victims and their families. Like Brazil, an amnesty has prevented judging those responsible.

It is no coincidence that Latin America, a region where the implementation of forced disappearances in a planned and systematic way by repressive regimes was once common, is the region that first proposed the adoption of a conventional instrument in the matter: the Inter-American Convention on The Forced Disappearance of Persons OAS [19].

Nonetheless, the forced disappearance of the students of Ayotzinapa does not fit well into the causal logic. Even worse, disappearances in Mexico have been on the rise since 2007 without anyone knowing why.[2] Furthermore, victims of disappearances

[2]The number of direct victims of forced disappearances in Mexico is unknown. The most important source of quantitative information is the National Registry of Missing and Disappeared Persons (RNPED) which contains raw data without distinguishing between categories of abandonment by migration, natural disasters, forced disappearances, etc. This registry shows that in 2007, 0.5 people of every 100,000 inhabitants were disappeared, while in 2016, the number rose to 4 of every 100,000. This represents a 800% rise. The National

seem to no longer be those that make the government uncomfortable, but rather they have diversified: now migrants, students, architects, teenagers, etc., without any relation to any expression of opposition to the government.

Even if some actions have been adopted in an effort to tackle this crime—ranging from the passing of laws, to the creation of commissions—no action will be effective unless the phenomenon is understood from its roots. The lack of understanding implies that the risk of any action taken being unfruitful arises, and thus, forced disappearances continue to rise.

It is possible that some cases of forced disappearances are attributable to individual causes. The theory of rotten apples could explain that public officials, unilaterally, and departing from their public duty, commit such crimes (Griffin and Ruiz [10] and Zimbardo [32]). Also, it is possible that some forced disappearances are committed by officials under threats by members of organized crime. Nevertheless, these theories are not enough to explain why countries like Italy or South Korea, with high presence of criminal organizations (WEF [31]), do not face a similar problem. The explanation about the rise of forced disappearances has to respond, additionally, to something other than the theories previously exposed.

It is important to recognize individual studies that try to explain forced disappearances for specific events. Mastrogiovanni [15] considers that forced disappearances in the north of Mexico are part of a wider strategy by the State to allow transnational

Human Rights Commission, the United Nations and the Organization of American States have all pointed out the constant rise of forced disappearances in their reports. Also, the General Prosecutors' Office has declared a 60% increase in the investigation of forced disappearances between October 2016 and March 2018 (Reforma [23]).

companies to exploit natural resources, like shale gas or oil (p. 200). Illades [12] argues that the disappearances of Ayotzinapa are explained by negligence, criminal control and generalized violence (p. 14).

Nevertheless, evidence shows that the phenomenon of forced disappearances is generalized United-Nations [30][3]. This strongly manifests the need to move from individual explanations, to trying to identify a more general causality that may give rise to the design of a national public policy. Hence, to this day, there is no study at the national level that explains why forced disappearances have been on the rise since 2007.

Most of the literature on forced disappearances is from the legal and anthropological perspective. From the legal perspective, studies like the one from Anderson [3] and Genovese and Wilt [8] analyze forced disappearances from the perspective of International Law, pointing out the importance for States to regulate this nationally. Ramos Koprivitza [22] analyzes different legal definitions of this concept in Mexico, shedding light on the diversity in these regulations. Finally, there are more specific studies, like the one from González Ramírez et al. [9] about its statute of limitations.

From the anthropological perspective, studies center on the viewpoint of the victims of forced disappearances and their families, and the effect on mourning Zorio [33], Huffschmid [11], and Palma [21], and also, on identity at the individual or family level Gatti [7] and Alvis-Rizzo et al. [2], as well as on social disintegration, Endo [5].

Nonetheless, literature on the causes of forced disappearances is basically non-existent in Mexico, as well as in the rest of the

[3]It is possible that part of the rise may be related to the increase in reporting.

world. There are two studies that analyze its effects: the first study on forced disappearances is from Meadowcroft et al. [16], in which the authors analyze this type of violence in a transition period: the change from a democracy to an autocracy during Pinochet's government in Chile. This study, rather than challenging or verifying a hypothesis, offers a statistical description of gross human rights violations between 1973 and 1989.

More recently, Osorio et al. [20] have done a second study on forced disappearances. These authors analyze the long-term consequences of forced disappearances in Mexico during the dirty war, and its effects on the State's consolidation. This is, forced disappearance is the independent variable that tries to explain effects on different categories of State capacity, such as legal, fiscal and territorial.

The purpose of this work is to develop, using game theory, a parametric model that explains why local police officers in Mexico are the ones that most often commit forced disappearances - more than 96% of disappearances occur at the local level. We hope that this study allows us to understand the phenomenon from its causes, leaving behind individual explanations that presuppose corruption or some sort of evil from public officers. This would allow the design and development of public policy strategies that tackle forced disappearances effectively.

2. Description of the model and its parameters

The Mexican State is a federal state. This means that it is composed of different units of government: the federal, state and municipal levels. The Political Constitution of the United Mexican States makes every level responsible for the provision of public security:

> "Public security is a function of the Federation, the states and the municipalities, and it comprises crime prevention, their investigation and prosecution, as well as administrative sanctions (Article 21, paragraph nine)..."

With this provision, municipalities are co-responsible for public security tasks. Nonetheless, it is at this level that most forced disappearances occur, RNPED [25].

The present study is focused on forced disappearances at the local level. We do not reject that a similar study may be developed at the federal level, but that determination is left for future research. In this manner, the model we propose comprises two players ($N = 2$): a municipal president (MP) and a police officer (P).

The action set of each player is specified according to the following: the municipal president, as a political agent, has been elected to pursue public interest. Nonetheless, this municipal president has a latent incentive to employ public institutions for personal gain and to obtain personal benefits in the short-run, given that reelection is impossible. According to this, such a municipal president can either choose to pursue public interest (PuI) or private interest (PrI).

Regarding the police officer, the National Constitution establishes that municipalities shall have their own police forces, and each municipality will be responsible for its own regulation (Article 115). The municipal president is in charge of this police force. Hence, a police officer may chose between obeying (O) or not obeying (nO) any specific instruction given by the municipal president.

It is important to consider that the model proposed is a strategic game in which information is perfect for both players. Also, it is a static game, played once, considering that each decision is autonomous from the next one.[4]

2.1. Model specifications

The model has the following parameters:

Table XII.1: Model parameters

Municipal President		Police officer	
Salary	S_e	Salary	S_p
Corruption benefit	λS_e	Punishment for not obeying	O_s
Corruption cost	C_e	Prize for obeying	O_p
Reputation	R	Moral bonus	M

The parameters assigned to the Municipal President are specified according to the following:

- Salary: the economic remuneration that they periodically get in exchange for the performance of their functions
- Corruption benefit: the economic payoff received for committing acts of corruption in pursuance of private interests. It is specified as a proportion of their salary
- Corruption cost: the expenses and risks they undertake while committing acts of corruption
- Reputation: the prestige or status that the public official has. This parameter is relevant, even if reelection is prohibited,

[4]The Mexican constitution was reformed in 2014 to allow municipal presidents to be reelected for a consecutive period. In this case, the game would be repeated.

since she can have an interest in running for any other popular election post, in the legislative, as well as in the executive branch.

The parameters assigned to the police officer are specified according to the following:

- Salary: the economic remuneration that they periodically get in exchange for the performance of their functions
- Incentives for obeying or not obeying: given the reward system where loyalty is rewarded, police officers will receive a prize if they obey, and punishment if they do not
- Moral bonus: we presuppose a particular public service motivation in the police officers when choosing their career, wanting to contribute to society and to the prevention of crimes, thus they get benefits for fulfilling their functions

The payment functions of the model are the following:

Table XII.2: Game's bimatrix

		P			
		O		nO	
MP	Pu I	$S_e + R$	$Sp + O_p + M$	$S_e - R$	$S_p - O_S$
	Pr I	$S_e + \lambda S_e - C_c$	$Sp + O_p$	$S_e - C_c$	$S_p - O_S + M$

2.2. *Hypothesis*

Based on theoretical and empirical propositions, we propose three hypothesis for the model. First, because of historical reasons, mainly the Mexican Revolution that started in 1910 after the Mexican President, Porfirio Díaz, remained in power for more than thirty years, Mexican executive authorities at every level of government have had a constitutional prohibition for reelection. At the municipal level, this means that municipal presidents have a once-in-a-lifetime opportunity to govern for three years. This short period has been considered as too short to actually implement public policies (Albiter Gonzalez [1]), and thus, incentives are only considered in the short-term.

Additionally, since 1995, Mexico has constantly been in the lowest 25% of the Corruption Perception Index, TI [27]. This, in a context of generalized impunity—the country fourth highest in rate of impunity UDLAP [28]—implies that for municipal presidents the social cost of their corruption is bigger than the benefit of their reputation. Hence, $\lambda S_e - C_c > R$.

Second, closely related to the previous hypothesis, impunity reduces the cost of corruption—the risk of being caught and sanctioned is close to zero—thus we can establish $C_c < R$.

Third, municipal police institutions, like many other Mexican police institutions, have been characterized by loyalty to political authorities, complicity, impunity and autonomy, (Lopez Portillo Vargas [14]). Even if it is recognized that subjective elements influence the decision to become a police officer, once part of the institution, they are subjected to formal authority by the chain of command and are obedient, Suarez de Garay [26]. This means that the benefits for obedience become more important than the subjective compensation of being a member of the police force. Hence $O_p + O_s > M$.

In summary, the following three conditions are proposed:

$$\lambda S_e - C_c > R \tag{2.1}$$

$$C_c < R \tag{2.2}$$

$$O_p + O_s > M. \tag{2.3}$$

3. Game analysis

If we consider the three hypotheses previously explained, the game has a dominant equilibrium, because the municipal presidents, as well as the police, have dominant strategies: pursuing private interests and obeying, respectively (see Appendix 5). This explains that if municipal presidents believe it is convenient for them to order the commission of forced disappearances, the police will obey.

An interesting finding is that salaries, by themselves, are not important, either for the municipal president, or the police. More relevant are the payoffs received for corruption and for obedience, respectively.

Of course, the question becomes, how can we break this strategy? To answer this, it is necessary to analyze each player's strategy and conditions.

For the municipal president, it would be necessary to create a new condition for the model. Specifically, it would be necessary that the net benefit of corruption be lower than the one obtained from reputation. This is, transforming hypothesis 1 into $R > \lambda Se - Cc$.

For the police officer, it would also be necessary to modify one condition. Specifically the third one: invert the relation $O_p + O_s > M$ to $O_p + O_s < M$. This means that the sum of the payoff for obeying and not obeying is lower than the benefit of the moral bonus.

With these transformations to the conditions, the model has mixed equilibriums: when municipal presidents pursue public interests, police will obey; and when they pursue private interests, police will not obey.

4. Conclusions

The parametric model we developed offers an explanation about why forced disappearances happen inside police institutions in Mexico at the municipal level. The hierarchical structure of police departments, with police being subordinate to municipal presidents, as well as incentives municipal presidents have to pursue their private interests, set the stage for the development of dominant strategies between each other.

The conclusions herein proposed can also be generalized to other type of crimes committed by the interaction of these players, like corruption or abuse of authority.

Future studies can analyze a dynamic model, this is, in the long run. Notably, how are incentives located in a model that allows reelection of local authorities? This would shine light on what mechanisms contribute—or not—to more stability, and less corruption, from local authorities.

5. Equilibrium

Let's begin the analysis of the associated strategic game's equilibrium by defining the expected utilities of both players for each of their possible actions.

On one hand, for the Municipal President, we have:

$$\begin{aligned} \mathbb{E}[PuI] &= 2R\beta + S_e - R, \\ \mathbb{E}[PrI] &= \lambda S_e \beta + S_e - C_c; \end{aligned}$$

where β is the probability of the police officer obeying. On the other hand, for the police officer we have

$$\begin{aligned} \mathbb{E}[o] &= \alpha M + S_p + O_p, \\ \mathbb{E}[no] &= -\alpha M + S_P - O_S + M; \end{aligned}$$

where α represents the probability that the Municipal President pursues public interest.

Given conditions (2.1) and (2.2) we have

$$\mathbb{E}[PuI] \prec \mathbb{E}[PrI].$$

Thus, $\alpha = 0$ and $\beta < \frac{R-C_c}{2R-\lambda S_e}$. This is, the Municipal President has a dominant strategy: he will always prefer to pursue private interests over public interests.

Also, if it were the case that $\mathbb{E}[PuI] \succeq \mathbb{E}[PrI]$, then it must be that $0 < \beta < 1$; this is not true.

Aditionally, given (2.3) we have

$$\mathbb{E}[o] \succ \mathbb{E}[no].$$

This means that $\beta = 1$ and $\alpha > \frac{M-O_S-O_p}{2M}$. This is, $\mathbb{E}[o] \preceq \mathbb{E}[no]$ is not feasible because α is a probability distribution: $0 < \alpha < 1$. In short, the police officer has obedience as a dominant strategy.

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XIII. ON THE OPTIMAL CONCESSION TO KEEP A COUNTRY UNIFIED

Julen Berasaluce Iza

Abstract

Three different fiscal strategies are compared to prevent the formation of a majority in favor of independence in a region: changing the common tax, fiscal autonomy and a fiscal premium. Fiscal autonomy can always be combined with any of the other two and reduces the sacrifice that the median voter of the state needs to make. In order for a fiscal premium to dominate the common tax strategy, the region must have a greater proportion of poor citizens, and it must be relatively small.

1. Introduction

The formation of countries has been a topic of great importance in the social sciences. Even when one country has already been formed, there might be territories belonging to that country willing to form new countries by themselves. Among the reasons that may explain such divisions are cultural, historical, or economic differences. Whatever the reasons to be considered, it is undeniable that the division of countries is an important phenomenon when explaining the current political picture of the globe. Among

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the many historical examples, we have the former Yugoslavia, Czechoslovakia, the Former Soviet Union, Timor, Sudan, etc., as well as much tension in several regions of Spain, Italy and the United Kingdom, among others.

Let us call a division of a former country a secession, although the concept is formally used only in the case of federal unions. If the secession of a region is an optimal result for both the region and the rest of the former country, it is hard to argue against it. However, many times conflicts arise over this issue, making it obvious that secession may not be a desirable result for all the parties involved. Most of the time, citizens of a region, or at least one subset of them, favor secession, with the the rest of the citizens of the country being opposed to it. This paper studies this phenomena and what peaceful strategies can be adopted in order to avoid the existence of such conflicts. In addition, those strategies are compared in order to conclude which of them are preferable.

The question that we want to analyze is of great importance all over the world. Fearon [5] lists 708 minorities in 161 countries conforming at least 1% of the population of the country. More than 10% of them experienced some kind of violence as a strategy of some subsets of those minorities, in order to obtain secession or greater autonomy. The existence of such violence shows, at least, that some subsets of the minorities are looking for a different political status and that the governments of the respective countries are not willing to concede this.

In this paper, violence is not included as a tool or a threat. We are going to consider a region which is part of a bigger country, and we are going to describe when a majority of the citizens in the region favors secession. In order for a majority to favor secession, such a majority must expect to obtain enough benefits to compensate for the fixed cost of the creation of a new institutional apparatus. We assume that when such a majority exists, secession automatically happens. However, a majority of the citizens not belonging to the region may prefer to prevent it. In order to reach that objective, we analyze different strategies. Those strategies would imply a change in the public provision of some private good through proportional taxes.

The first strategy that we are going to consider is changing taxation, and therefore, the public provision of the private good to all the citizens of the country. We analyze how much, and in which direction, taxes should be changed in order to avoid secession. This first strategy is referred to as the strategy of the common tax.

The second strategy, which is named fiscal autonomy, consists of allowing the government of the region to impose an extra tax in order to finance some extra provision of the public good for the citizens of the region. Such extra provision of public good is made through the existing institutional apparatus so that the creation of a new regional apparatus is not needed. This is a simplifying assumption, because it is obvious that some additional regional institutional apparatus is needed, and that would imply some fixed cost. However, if that cost is smaller than the one needed to finance the creation of a new institutional apparatus under secession, the simplification seems reasonable.

The third strategy is named fiscal premium, and consists of allowing the citizens of the region to pay a smaller tax rate for the same public provision of the private good.

All three strategies can be combined. For instance, we can allow fiscal autonomy, but also change the common tax.

We compare the different strategies according to how they reduce the country's median voter's welfare. We find that fiscal autonomy can prevent secession without altering the welfare of the median voter of the country. Moreover, in case fiscal autonomy is not enough, its combination with any of the other two strategies reduces their negative impact on the welfare of such voters. We also find that the common tax strategy can only be helpful to prevent the formation of majorities in favor of secession in regions with a small percentage of poor citizens. Finally, we show that in order for the fiscal premium strategy to dominate, the common tax strategy, two necessary requirements must be fulfilled: the region must be relatively poor and small. If not, changing the common tax is less harmful for the median voter of the state than conceding a fiscal premium to the citizens of the region.

The study of the breakup and unification of nations in political economy was started by Casella and Feinstein [4], Wei [6] and Alesina and Spolaore [1], which consider Hotteling's spatial model to analyze the heterogenous preferences of voters over the provision of some public good.

Casella and Feinstein [4] describe nations who gain from international trade, but can also unite to reduce trading costs. However, unification also implies political costs, since the median voter of the united nation may be located far from the ideal preferences of certain groups of voters. These two effects, combined, result in nations' optimal intermediate sizes.

Wei [6] models the limitations of big nations through greater inefficiency costs when providing the public good. In Wei's model, contrary to that of Casella and Feinstein [4], political integration is needed to make trade possible. Therefore, nations are only ready to unite when they develop enough so as to enjoy the benefits from trade.

Alesina and Spolaore [1] also develop a Hotelling location model to explain how different political institutions affect the number of nations at equilibrium and compare it to a social optimum. They find that with democratic institutions, the number of optimal nations exceeds the optimum.

This model is closer to the one by Bolton and Roland [2], who explain how differences in wealth, instead of location, itself, are the origin of conflicts between regions. Unlike their assumptions, no trade is included in this analysis, neither is it assumed that citizens obtain higher income under integration. Instead, we assume that any independent country must face fixed institutional costs. Therefore, the benefit from integration comes from sharing those fixed costs between a greater number of citizens.

We do not consider two regions which are ex-ante identical and are deciding about integrating or keeping separated. Instead, we assume that a region within a country is deciding whether or not to become independent. The benchmark at which we analyze such a decision is similar to the work by Bolton and Roland. However, we focus on a different topic. We compare different strategies that can prevent the formation of majorities in favor of secession.

The rest of the paper is structured as follows. In Section 3, the basic model is explained. In Section 4, we analyze which are the optimal policies when the central government does not seek to avoid secession, as well as the conditions for secession to arise. In Sections 5, 6 and 7, we analyze, respectively, the common tax, fiscal autonomy and fiscal premium strategies. We end up with conclusions in Section 8.

2. The Model

Let us consider a country of size 1 and a region of size R, where $0 < R < 1$, within that country. Every individual in the country, whether or not they belong to the region, receives an income w. A proportion α of the citizens of the region has its income uniformly distributed over $[0, 1]$, while a proportion $(1 - \alpha)$ of the citizens of the region receives an income $\bar{w}_R(\alpha)$ which is calculated to keep the mean income of the region equal to 1, so that the only difference that we consider is a distributional one. Thus, $\bar{w}_R = \frac{2-\alpha}{2(1-\alpha)}$, and the total income of the region is equal to its size R. Notice that it has to be the case that $\alpha < 1$; otherwise, all the citizens would have their income uniformly distributed over $[0, 1]$, their mean income being $\frac{1}{2}$.

Similarly, let us consider that the rent of those citizens not belonging to the region follows a similar distribution, where a proportion β of them receives an income uniformly distributed over $[0, 1]$, while a proportion $(1 - \beta)$ obtains $\bar{w}_{1-R}(\beta)$. In order to keep the mean rent equal among regions, $\bar{w}_{1-R} = \frac{2-\beta}{2(1-\beta)}$.

We will restrict ourselves to cases where the following two conditions hold:

Condition 2.1.

$$\alpha > \frac{1}{2} \tag{2.1}$$

Condition 2.2.

$$\beta > \frac{1}{2} \tag{2.2}$$

Conditions 2.1 and 2.2 are necessary and sufficient for the mean income to be above the median, both in the region and outside of it. These conditions hold for most of the income distributions around the globe and are necessary for the median voter to ask for positive redistribution.

We will consider different institutional arrangements, such as a government for the whole country, or independent ones. These governments can impose a proportional tax $t \in [0, 1]$ to finance a public good g, which is a perfect substitute for private consumption. That is, tax policy is purely redistributive. Therefore, given t and g, the utility function of an individual with income w is given by

$$U(w, t, g) = (1 - t)w + g. \tag{2.3}$$

The provision of the public good by any of the governments considered implies a deadweight loss equal to $\frac{t^2}{2}$ and public debt is not allowed. Moreover, any independent country must pay some fixed costs P in order to finance its institutions. Since savings do not yield any utility, and governments are not allowed to incur debt, they exhaust their budget constraint. Therefore, if the central government imposes a tax t, it is able to generate an amount of public good equal to

$$g_C = t - \frac{t^2}{2} - P. \tag{2.4}$$

Notice that (2.4) is constructed for the government of the whole country, which imposes taxes and distributes the public good among a population of size 1. If the region became independent, we would have two different countries. Even if we keep the marginal deadweight loss invariant, the two independent countries would each need to face a higher institutional cost per capita. Therefore, the budget constraints of each of their governments would, respectively, imply

$$g_R = t - \frac{t^2}{2} - \frac{P}{R} \tag{2.5}$$

$$g_{1-R} = t - \frac{t^2}{2} - \frac{P}{1-R}. \tag{2.6}$$

Thus, we have two basic features that will explain the costs and benefits of a secession. On the one side, due to the difference in distribution, the citizens of a region may prefer a secession so that they can select a distributional policy on their own. However, in order to benefit from it, they need to consider the greater institutional costs per capita, which accounts for the scale economies of a bigger country.

Having said that, let us first analyze which are the different possible equilibrium outcomes at which a majority of the citizens of the region are willing to become independent.

3. The Benchmark

First of all, let us analyze a central government that does not seek the integrity of the country. We will consider the individual preferences of voters in that country. Therefore, let us obtain the optimal tax rate for an individual with given income w. This is calculated by solving the following problem:

$$\max_t \ (1-t)w + t - \frac{t^2}{2}. \tag{3.7}$$

Since the preceding problem is clearly concave, we can obtain the optimal tax rate for a citizen with income w, ($t^\star(w)$), from the first order condition, which is

$$t^\star(w) = 1 - w. \tag{3.8}$$

That is, a citizen's optimal tax rate is decreasing with its income. Moreover, her preferences are single peaked with respect to w. Therefore, let us focus on the tax rate that two identical office seeking parties would offer at a Nash equilibrium, i.e., the one preferred by the median voter. Notice that the income of the median voter of the country (w_C^{me}) is equal to

$$w_C^{me} = \frac{1}{2A}, \tag{3.9}$$

where

$$A = R\alpha + (1-R)\beta. \tag{3.10}$$

Then, a central government that did not care about the integrity of the country would impose a tax rate $t_C^\star$ equal to

$$t_C^\star = \frac{2A-1}{2} > 0. \tag{3.11}$$

The preceding would be true as long as the referred tax rate yielded enough income to the central government to cope with the fixed institutional cost P. Otherwise, the government would need to impose a large enough tax rate so that the selected tax rate would actually be:

$$t_C^{\star\star} = max\left\{\frac{2A-1}{2}, 1-(1-2P)^{\frac{1}{2}}\right\}. \tag{3.12}$$

In order to analyze conflicts about the issue of secession, we need to obtain which taxes would be imposed, both in the region and outside of it, in case the region becomes independent, which would automatically imply that the rest of the country becomes independent, as well. As before, the single peaked preferences are maintained, so let us focus on an independent government in the region which seeks to maximize the welfare of its median voter. The tax rate imposed by an independent region, $t_R^{\star\star}$, would be the solution to the following problem:

$$\begin{aligned} \max_{t_R} \quad & (1-t_R)\left(1-\frac{2\alpha-1}{2}\right)+t_R-\frac{t_R^2}{2}-\frac{P}{R} \\ \text{s. t.} \quad & t_R-\frac{t_R^2}{2} \geq \frac{P}{R}. \end{aligned} \tag{3.13}$$

By solving (3.13), we obtain that

$$t_R^{\star\star} = max\left\{\frac{2\alpha-1}{2\alpha}, 1-(1-2PR)^{\frac{1}{2}}\right\}. \tag{3.14}$$

Equivalently, in case the region becomes independent, the citizens outside the region will enjoy a tax rate equal to

$$t^{\star\star}_{1-R} = max\left\{\frac{2\beta-1}{2\beta}, 1-[1-2P(1-R)]^{\frac{1}{2}}\right\}. \qquad (3.15)$$

Although the tax rates for the country and for each of the regions in case of independence, are, respectively, $t^{\star\star}_C$, $t^{\star\star}_R$ and $t^{\star\star}_{1-R}$, we will first focus on those cases in which the institutional cost restriction is not active for the median citizen. Thus $t^\star_R$ and $t^\star_{1-R}$ would be defined as $t^\star_C$ in Equation (3.11).

Then, we can calculate the induced utility of a citizen in the region with income w in each of the two systems, that is $U(w, t^\star_C, g(t^\star_C))$ and $U(w, t^\star_R, g_R(t^\star_R))$. Therefore, if we calculate the difference between these two terms, we can state when a citizen from the region obtains a higher utility at independence, rather than in a state that does not care about secession. If we calculate the difference in the two utility functions, we obtain that a citizen with income w prefers to have independence when

$$U(w, t^\star_R, g_R(t^\star_R)) - U(w, t^\star_C, g_C(t^\star_C)) \geq 0. \qquad (3.16)$$

We may ask how this inequality changes with respect to w. Thus, we can conclude that:

Proposition 3.1. *If $\alpha < \beta$ (resp. $\alpha > \beta$), the greater w, the greater (resp. smaller) the utility a citizen of the region obtains at independence compared to the one it gets when the government is a central one. If $\alpha = \beta$, this difference does not change with respect to w.*

The preceding proposition is very useful to determine which segment of the population of the region is going to support independence. If $\beta = \alpha$, the distribution in the region exactly replicates that of the state. So, if one citizen favored independence, everyone in the region would. If $\alpha < \beta$, the proportion of poor people is smaller in the region than outside of it. Therefore, $t_R^\star < t_C^\star$, so if someone with income w favors independence, anyone with income $w' \geq w$ will do so. That is, the richer a citizen from the region is, the happier she is under secession. If $\alpha > \beta$, the argument works in the opposite direction.

By proposition 3.1, if the median voter of the region favors independence, a majority of the citizens of the region also do. The income of the median voter of the region is given by $w_R^{me} = \frac{1}{2\alpha}$, so if we substitute w_R^{me} at (3.16) and reorganize, we can conclude the next proposition.

Proposition 3.2. *A majority of citizens in the region favors secession if and only if*

$$P \leq \frac{R(1-R)(\beta-\alpha)^2}{8\alpha^2 A^2} = P_R^I. \tag{3.17}$$

Notice that the existence of a majority in the region favoring independence is not a problem by itself. Indeed, if a majority of citizens outside the region favor independence, they will simply realize they both prefer to be separated rather than in the same country. As a result, we will have a happy divorce. Focusing on Proposition 3.1, we notice that as long as the only differences in income are distributional, the model predicts that if secession is supported by a majority in both regions, it is favored by the richest citizens of one of them, and the poorest of the other. This is true in the model because the only political conflict we assume lies in the differences between the respective income distributions so that being in the same country may represent a cost for the median voter of the respective regions. Thus, if a happy divorce occurs, this is enforced by two opposite political groups which suffer the greatest political gap between them.

By repeating the exercise for the median voter outside the region, we can re-write Proposition 3.2 for those citizens outside the region.

Proposition 3.3. *A majority of citizens outside the region favors secession if and only if*

$$P \leq \frac{R(1-R)(\beta-\alpha)^2}{8\beta^2 A^2} = P^I_{1-R}. \tag{3.18}$$

By contrasting Propositions 3.2 and 3.3, we can notice that as long as there exist distributional differences ($\alpha \neq \beta$), if the institutional cost is small enough, secession is optimal for a majority both in, and outside the region. Proposition 3.3 predicts that diminishing institutional costs would increase the number of secessions. An interesting issue is to examine, without loss of generality, when a majority in the region favors secession, while a majority outside of it does not. We would need (3.17) to hold, while (3.18) did not. In order for some intermediate institutional cost P that allows for that case to exist, it must be true that

$$\alpha < \beta. \tag{3.19}$$

That is, the only possibility in which a majority of the citizens of a region want to become independent, while a majority of the citizens outside the region prefer not to, is when the ones that prefer it have a smaller proportion of poor people and, therefore, are choosing a weaker distribution policy.

By doing some comparative statics on P^I_R we find that

$$\frac{\partial P^R_I}{\partial R} = \frac{(\alpha-\beta)^2[\alpha R^2 + \beta(1-R)^2]}{8\alpha^2 A^2} > 0, \tag{3.20}$$

so that the larger a region is, the larger institutional costs allow for secession, so that the easier secession would be supported by a majority. On the other hand, if we focus on the effect of α on P_R^I, we observe that:

$$\frac{\partial P_I^R}{\partial \alpha} = \frac{R(1-R)}{8} \left\{ \frac{-2(\beta-\alpha)\alpha^2 A - (\beta-\alpha)^2[3\alpha^2 R + \beta(1-R)]}{\alpha^4} \right\}. \tag{3.21}$$

Although the sign of the previous equation depends on the relation between α and β, since we mainly care about what happens when a majority in the region favors secession and a majority outside of it does not, i.e., $\alpha < \beta$, increasing α reduces P_I^R, because it reduces the differences between the preferred distributional policies of each of the median citizens.

Finally, with respect to the effect of β on P_R^I, we obtain that

$$\frac{\partial P_I^R}{\partial \beta} = \frac{R(1-R)}{8\alpha^2} \left\{ \frac{2(\beta-\alpha)A - (1-R)(\beta-\alpha)^2}{A^2} \right\}, \tag{3.22}$$

which is clearly strictly positive when $\alpha < \beta$, yielding the same effect that we discussed above.

In the following sections, we restrict ourselves to cases in which a majority in the region favors secession, while a majority outside of it does not. In each of the sections, we examine a different fiscal strategy that prevents secession.

4. Avoiding secession with a common tax

The first measure that we are going to analyze is assessing the same tax on all citizens and making it so attractive to the median voter of the region that he no longer favors independence. Such a tax must be good enough for the median voter outside of the region, so that she will not favor secession, either. In this case, the sacrifice in the fiscal policy is preferred to the splitting the country. In case there exists more than one tax for which the median voter of the region prefers not to favor independence, let us select the one that minimizes the political cost of the median voter outside of the region. This would be a perfect nash equilibrium of a three stage game. The third stage only occurs if a majority favors secession in the region and corresponds to the determination of fiscal policies as the result of a competition between two office seeking parties. The second stage refers to the independence referendum in the region. In the first stage, the political competition refers to the whole country, but forward looking voters may not vote for their preferred tax rate if that prevents an undesired secession.

If the state imposes a tax t on all citizens of the state, the median voter of the region will not favor independence if and only if

$$U(w, t, g_C(t)) - U(w, t_R^\star, g_R(t_R^\star)) \geq 0. \tag{4.23}$$

Rearranging the previous inequality, it becomes

$$-\frac{t^2}{2} + t\frac{2\alpha - 1}{2\alpha} + P\frac{1-R}{R} - \frac{(2\alpha-1)^2}{8\alpha^2} \geq 0. \tag{4.24}$$

Notice that the LHS of Inequality (4.24) always has some root. Therefore, there exist a pair of tax rates $(\hat{t}_{R^-}, \hat{t}_{R^+})$, such that $\forall t \in [\hat{t}_{R^-}, \hat{t}_{R^+}] \cap [0, 1]$ and the median voter of the region does not favor independence. These are given by

$$\hat{t}_{R^-} = \frac{2\alpha - 1}{2\alpha} - \sqrt{\frac{2P(1-R)}{R}}, \tag{4.25}$$

$$\hat{t}_{R^+} = \frac{2\alpha - 1}{2\alpha} + \sqrt{\frac{2P(1-R)}{R}}. \tag{4.26}$$

Notice that, by construction, if there exists a majority that favors independence at $t_C^\star$, then $t^\star \notin [\hat{t}_{R^-}, \hat{t}_{R^+}]$. If that is the case, which, from all the taxes that avoid a majority in favor of independence, is the one preferred by the median voter outside of the region?

First of all, we should replicate the analysis of the common tax made for the median voter of the region, with the median voter outside of the region. Similarly, we can find a pair of tax rates $(\hat{t}_{(1-R)^-}, \hat{t}_{(1-R)^+})$, such that $\forall t \in [\hat{t}_{1-R^-}, \hat{t}_{1-R^+}] \cap [0,1]$, so the median voter outside of the region prefers not to favor secession. These are given by

$$\hat{t}_{(1-R)^-} = \frac{2\beta - 1}{2\beta} - \sqrt{\frac{2PR}{1-R}}, \tag{4.27}$$

$$\hat{t}_{(1-R)^+} = \frac{2\beta - 1}{2\beta} + \sqrt{\frac{2PR}{1-R}}. \tag{4.28}$$

Having said that, we need to find if there is any common tax that prevents secession. Let us focus on the only case in which a majority in the region wants to become independent and a majority outside of it prefers to prevent that situation; that is, when $\alpha < \beta$, which implies that $\alpha < A$. Therefore, it follows that $t^\star R < t^\star_C$. So, if the median voter of the region favors independence, we can conclude that

$$\frac{2A-1}{2A} > \frac{2\alpha - 1}{2\alpha} + \sqrt{\frac{2P(1-R)}{R}}, \tag{4.29}$$

which can be simplified to

$$\frac{\beta - \alpha}{2A\alpha} > \sqrt{\frac{2P}{R(1-R)}}. \tag{4.30}$$

Moreover, the maximum sacrifice to be made by the median voter outside of the region must be attractive enough for the median voter of the region to favor unification. This implies that

$$\frac{2\beta - 1}{2\beta} - \sqrt{\frac{2PR}{1-R}} \leq \frac{2\alpha - 1}{2\alpha} + \sqrt{\frac{2PR}{1-R}}, \qquad (4.31)$$

which becomes

$$\frac{\beta - \alpha}{2\beta\alpha} \leq \sqrt{\frac{2P}{R(1-R)}}. \qquad (4.32)$$

It is straightforward that the inequalities (4.30) and (4.32) may be true at the same time. If that is the case, the median voter outside of the region would be ready to reduce the common tax up to $\hat{t}_R^+$ and prevent the formation of a majority that favors secession.

Proposition 4.1. *Changing the common tax prevents the formation of a majority in a region when there is conflict. In that case, the formation of a majority can be avoided by reducing the common tax to the most preferred one by the median voter of the country.*

Notice that preventing a majority in favor of independence in the region in the only case that matters to us implies the reduction of the common tax in the whole country. So, the median citizen would favor that strategy with the support of those citizens who have a greater income than themselves. In fact, the reduction of the common tax would lower the common tax to the ideal one of the citizens whose income is greater than the median.

5. Avoiding secession through fiscal autonomy

Let us consider a different fiscal regime and compare it to the one with a common tax. With fiscal autonomy, the central government sets a tax for all the citizens in the country, including those in the region. With the revenues obtained with that tax, the institutional fixed costs are covered, and some public good is offered to all citizens. Then, given a tax t, the central government offers an amount of public good equal to the one given in Equation (2.4).

However, after the central government states t, the citizens of the region are allowed to impose an extra tax rate t_R on themselves. This can be modelled by a four stage game, where the last two stages are the same as before. In the first stage, voters in the whole country would decide the common tax, while in the second, voters in the region would decide t_R. With the revenue obtained from t_R, no extra institutional fixed cost is paid, since it is assumed that, without independence, there is no need to replicate the institutions. However, new taxes imply greater inefficiency costs given the convex deadweight loss function that is being assumed. There can be different considerations about how this deadweight loss is shared between the central and regional institutions. Let us consider the most restrictive case for the regional government; that is the one in which the extra deadweight loss is applied to the regional government. So, if the central government imposes a tax t for all the population and t_R for those citizens in the region, the latter ones receive an extra amount of public good given by $\tilde{g}_R(t_R, t)$, where:

$$\tilde{g}_R(t_R, t) = (1 - t)t_R - \frac{(t + t_R)^2}{2} + \frac{t^2}{2}. \tag{5.33}$$

With such a fiscal regime, when would a majority of citizens of the region want an extra tax to obtain an extra amount of public good? Each citizen of the region would decide her optimal autonomic tax rate $\tilde{t}_R$ given the tax rate of the central government

t_C, as a solution to the problem

$$\max_{t_R \in [0,1]} (1-t_C)(1-t_R)w + t_R(1-t_C) - \frac{(t_C+t_R)^2}{2} + \frac{t_C^2}{2}. \quad (5.34)$$

As long as the solution is interior, the first order condition of this concave optimization problem shows that citizens still have single peaked preferences, so that we can focus on the most preferred tax rate of the median voter. Let us name it $\tilde{t}_R$, which is given by:

$$\tilde{t}_R(t_C) = (t - t_C)\frac{2\alpha - 1}{2\alpha} - t_C. \quad (5.35)$$

Since we are not allowing the regional government to impose any negative tax rate, it must be the case that

$$(1 - t_C)\frac{2\alpha - 1}{2\alpha} > t_C, \quad (5.36)$$

which becomes

$$\frac{2\alpha - 1}{2\alpha} > \frac{t_C}{1 - t_C}. \quad (5.37)$$

Since $(1 - t_C) \in (0, 1)$, in order for Equation (5.37) to hold, it must be true that:

$$\frac{2\alpha - 1}{2\alpha} > t_C. \quad (5.38)$$

That is, fiscal autonomy, considered as the possibility for the citizens of the region to increase redistribution, is only a possibility when the citizens of the region want to impose a higher tax rate than the one of the country. Therefore,

Proposition 5.1. *Whenever a majority in the region favors independence while a majority outside of the region does not, a majority of citizens in the region does not favor increasing redistribution between them.*

Therefore, when the only difference between the citizens in the region and those outside of it is distributional, fiscal autonomy does not work as a tool to avoid independence. However, fiscal autonomy may be considered with a different set of variables that induce a higher tax rate in the region. For instance, if the mean income is higher in the region, citizens there may favor a greater redistribution. Such an increase in income would be enough to compensate the larger tax deadweight loss from a higher tax. Equivalently, we may have a higher tax rate in the region if the deadweight loss in the region is smaller as a result of, for instance, a better administration. Thus, although fiscal autonomy is not a useful fiscal tool to avoid secession when the only difference is distributional, it might be so when there are differences in the mean income or the quality of institutions.

6. Avoiding secession through fiscal premium

Finally, let us consider a third strategy in which the citizens of the region would have to pay lower taxes than the rest of the country. Although citizens would be contributing differently, all of them would be receiving the same amount of public good. Hence, the citizens of the region, who would pay lower taxes, would enjoy a fiscal premium. It is straightforward that decreasing the tax to be paid by the citizens of the region increases their utility and, therefore, can help to avoid the appearance of a majority in favor of independence.

There is no need to compare the effectiveness of a fiscal premium with fiscal autonomy, because the latter, as we have seen, is not useful in our framework. If fiscal autonomy was a useful strategy under some other circumstances, fiscal premium could be combined with fiscal autonomy. The interesting issue would be comparing using a fiscal premium with the strategy of a common tax. The question is whether it is better to reduce the tax of everybody, and adjust the expenditure in public good, or to only reduce the taxes of the citizens in the region, while increasing the taxes of those citizens outside of the region to keep constant the expenditure in public good. We will answer that question for the median voter of the rest of the country.

Reducing t, while keeping the public provision of the private good constant, increases the utility of a citizen in w; that is, it increases the utility of the median citizen of the region in $\frac{1}{2\alpha}$. Therefore, in order to increase an infinitesimal unit of utility of the median citizen in the region, the reduction of taxes in the region must be equal to 2α. The reduction of one unit of taxes in the region decreases the total revenue of the government in R, which requires an increase in the tax rate outside of the region equal to $\frac{R}{1-R}$. Thus, in order to increase one unit of utility of the median citizen in the region, the decrease in the utility of the median citizen outside of the region must be equal to $\frac{R2\alpha}{(1-R)2\beta}$.

On the other hand, reducing the common tax one (infinitesimal) unit in order to make it closer to the preferred tax of the median citizen of the region, increases her utility by $\left(t + \frac{1-2\alpha}{2\alpha}\right)$. Therefore, in order to increase one unit the utility of the median citizen of the region, the reduction in the utility of the median citizen outside of the region must be $\left(\frac{2\beta-1}{2\beta} - t\right) / \left(\frac{1-2\alpha}{2\alpha} + t\right)$.

By comparing the previous two, we can conclude that allowing some fiscal premium is an interesting strategy with respect to imposing a common tax if and only if

$$\frac{2\beta - 1 - 2\beta t}{2\alpha - 1 + 2\alpha t} > \frac{R}{1-R}. \tag{6.39}$$

Proposition 6.1. *In order for a fiscal premium to be a more interesting strategy than common tax from the point of view of the* *median voter of the state, it is necessary that* $\alpha < \beta$.

Proof in the appendix.

Therefore, a fiscal premium might be an interesting strategy in the most important case of an undesirable secession from the point of view of the median voter outside of the region. Moreover, in order to consider the fiscal premium as a desirable strategy, the size of the region must be relatively small, so that (6.39) holds.

7. Non interior tax rates

Although the results above have been given for $t^\star_R$, $t^\star_{1-R}$ and $t^\star_C$, they are easily generalizable for $t^{\star\star}_R$, $t^{\star\star}_{1-R}$ and $t^{\star\star}_C$.

When the fixed cost restriction is active, both in the region and in the country, no citizen in the region would prefer to become independent. Either of the two tax rates would imply a zero redistribution, while the tax in the country would be lower.

When the fixed cost restriction is active in the country, but not in the region, since the fixed cost is more restrictive in the region, it must be the case that $\alpha > \beta$. This case could be analyzed as if the distribution of income outside of the region responded to some $\beta' > \beta$. However, it needs to be the case that $\alpha > \beta'$, so there will not be a majority in favor of secession in the region and a majority outside of the region against it.

If the fixed cost restriction is only active in the region, we may have two cases. In the first one, $t^{\star\star}_R \geq t^{\star\star}_C$, while there is zero redistribution in the region in case of independence. Clearly, in this case, there would not be any majority in the region in favor of independence. On the other hand, if $t^{\star\star}_R \geq t^{\star\star}_C = t^{\star\star}_C$, together with the fixed cost restriction being active in the region, which implies $t^\star_R < t^{\star\star}_R$, the results we have given for $t^\star_R$, would be easily extensible to $t^{\star\star}_R$, by considering an α' such that $\alpha < \alpha' < \beta$.

8. Conclusions

Up to now we have compared three different fiscal strategies to prevent the formation of a majority favoring independence in a region. We have restricted our considerations to cases where the mean rent is constant and, therefore, the only variables of the model are the size of the region and the differences between the distributions.

We have seen that fiscal autonomy, as described, should never be considered a useful strategy with respect to avoiding majorities in favor of secession when these are only motivated by distributional differences. However, fiscal autonomy can be considered as an interesting strategy if secession is motivated by differences in some of the variables than have been considered equal among regions, such as the mean rent or the quality of institutions. Those variables should be considered when extending the current paper.

As we have seen, a fiscal premium can only be an optimal strategy for small regions with a large set of relatively poor citizens. That could be the case, for instance, with Greenland, as long as they do not extract oil from it. Many citizens of Greenland receive great subsidies from the Danish Government.

Although we have focused the welfare comparison on the median voter outside of the region, we must point out that when comparing the fiscal premium and the common tax strategy, preferences may differ among individuals with different incomes. For instance, those citizens with greater income, that is, those who are commonly related to rightist political parties, may favor the common tax strategy, that is, reducing the tax rate of all the citizens in the country. That strategy may be justified by principles such as the equal fiscal treatment of all the citizens in the country. The citizens with a lower income and, therefore, commonly related to leftist parties, may favor fiscal premium strategies, which may be argued as adapting fiscal regulation to the specificities of each region.

9. Appendix: Proof of proposition 6.1.

Since the denominator of the RHS of Inequation (6.39) is positive, and $\alpha > \frac{1}{2}$, which is a basic feature of the model, the numerator must be positive. If we applied that condition to the preferred tax of the median citizen of the country, it must be the case that $\beta > A$, which is true if and only if $\beta > \alpha$. Alternatively, we may not consider the preferred tax of the median citizen of the country, but a lower one as a result of a reduction of the common tax in line with the first considered strategy. The preceding argument could be applied to some A', where the size of the region is greater, and the argument explained above still applies.

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