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FINANCIAL HEDGE AND STRATEGIC INTERACTION

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Financial Hedge and Strategic Interaction

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Abstract

In this work, we will focus on a problem that could be very common in the transactions that are carried out on a daily basis in the Financial Markets. We start with the decision of a Company about the proportion of a certain monetary amount that she needs to hedge against changes in the exchange rate. On the other hand, we will have another player who is interested in obtaining a profit through the variation of the exchange rate. First, we will assume that all operations that the players perform will be costless. In this situation, we will have two Nash equilibria. They correspond to corner solutions. In previous literature, on which our work is based, it is assumed that there are no operating costs. We extend this literature by relaxing this assumption, and find that the Nash equilibrium will depend on the costs. These costs can be interpreted as a disincentive to operate in markets, because at higher costs, the agents will not have the incentives to participate in these operations. This will reduce the number of transactions and the markets will suffer stagnation.

Keywords: Game Theory, Financial Derivatives, Financial hedge, Nash equilibrium.

Contents

1 Introduction

In recent years, there have been several events that have generated an increase in the volatility of the markets. Examples of these events are the exit of Great Britain from the European Union (Brexit) in 2016 and, the US elections in 2016, among others. As a consequence, hedging is a priority for international companies.

The financial markets are divided into some branches such as: Equities, Commodities, Fixed income, and Foreign Exchange. In this work, we will focus mainly on the last market.

As we mentioned earlier, diverse market factors can produce unexpected movements in the Financial Markets, in our case, we will see variations in the exchange rate and the way this bring profits for some agents. For example, we present the variation of the US dollar[1] versus Mexican peso in the last ten years.[2]

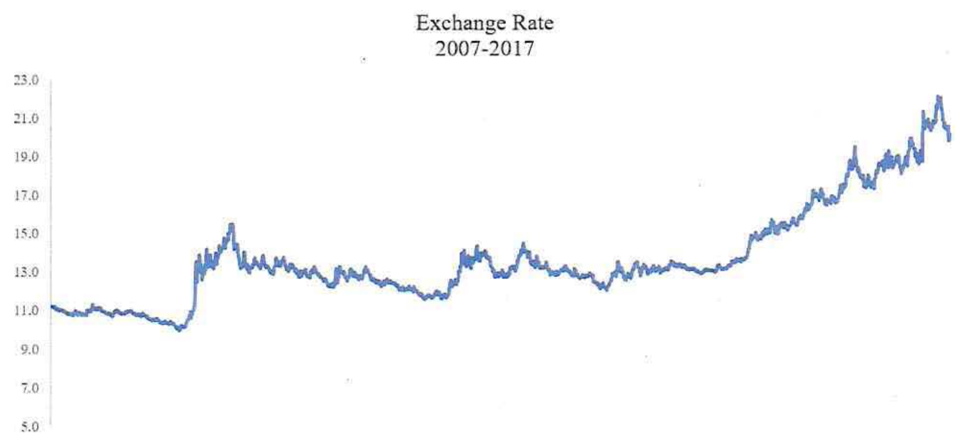

Figure 1: USDMXN rate from March 2007 to March 2017.

We can see how the dollar has a bullish trend, so hedging against an increase in the USDMXN rate is rational.

[1] In this work we will use US dollar like dollar.
[2] Banco de México data.

When we talk about Financial Markets we must consider complex interactions between brokers, financial institutions, multinational companies, among others. This is the main idea to understand why we want to develop this paper. For example, consider the next operation that is very common in Financial Markets.

A company has a debt in foreign currency (for example dollars). She wants to contract to hedge a portion of her debt against unexpected increases in the foreign currency, therefore she decides to buy a forward contract to buy a foreign currency at a fixed price for a determined amount. On the other hand, a Financial Institution decides to intervene in the spot market to speculate on the exchange rate (buying or short-selling foreign currency), after she must enter in a forward contract with an opposite position.

The problem for the Company is how much she must hedge with a forward contract, and for the Financial Institution how much she must participate in the spot market.

2 Literature Review

In this work, we will study a relationship between agents and brokers. The research about such a relationship in the Derivatives Markets performed by Carfi and Musolino (2011), gave us the foundations for our approach. We will follow the same line of research. Next, we would like to mention their findings from previous work.

Carfi and Musolino (2011) established a similar problem with the difference that the underlying asset was commodities. Also, they consider various behaviors about the players, for example, if they behave in an offensive, cooperative, or friendly manner. These behaviors bring about different equilibria. Next, Musolino (2012) continued this research and suggested a greater emphasis on obtaining a cooperative equilibrium.

Later, Carfi and Musolino (2012) took over the interaction between agents with the modification that the instrument they use are Government Bonds, as a consequence they had to add another player. Also, the different in valuation of the products change the approach and the equilibrium.

Finally, Carfi and Musolino (2015) analyzed the case where the underlying asset is the exchange rate. Nevertheless, the development in our work is different to the one used by them because we will relax several assumptions.

3 Background

In the introduction, we talk about financial derivatives and forward contracts. What are those? How can we value them? Can we use only some of them? How about other types of derivatives like call option to protect against the variation in the exchange rate? Well, in this section we develop the concepts that we will use for the approach of the game and how to find the Nash Equilibrium. The next definitions are obtained from Hull (1993).

Definition 1 (Derivative). *Is an instrument which value depends of the value of other or others underlying asset.*

Definition 2 (Forward Contract (Long position)). *Is a contract that gives its holder the obligation to buy an underlying asset at an agree price (strike price K) in a future date agreed (maturation date T).*

Definition 3 (Forward Contract (Short position)). *Is a contract that gives its holder the obligation to sell an underlying at an agree price (strike price K) in a future date agreed (maturation date T).*

Definition 4 (Discount Factor). *We assume that for any time T, the value of a foreign exchange promised at time t is given by*

$$e^{-d(T-t)}$$

For some constant $d > 0$. d is the continuously compounded interest rate for this period and is the interest rate for our currency.

Now, we are going to develop the valuation of a contract forward. Nevertheless, before we start is important do the following observations.

Definition 5(No arbitrage). *By no arbitrage, we refer to the fact that is not possible obtain a profit without risk when performing simultaneous transactions in two or more markets.*

Note that a derivative is characterized by its payoff, that is, it tells us the transactions performed when the contract is settled.

We assume that in our economy arbitrage is not allowed, as a consequence, if two assets have the same payoff their price must be the same at any previous time.

Using the previous definition, we can build the following portfolios that will help us for the valuation of the contract forward.

Portfolios 1) Contains at time t a long forward contract with exercise price K and maturity date T, also, it will have $Ke^{-d(T-t)}$ units of the domestic currency, the reason of this particular amount is because, we just need K units of the domestic currency at time T, then, we need discount that amount.

Portfolios 2) Contains at time t one unit of underlying asset, in this case, contains a foreign exchange discount, the purpose for this choice is that a maturity we will have a foreign exchange unit.

The payoffs of these portfolios are:

Portfolios 1) $S_T - K + Ke^{-d(T-t)}e^{d(T-t)} = S_T$

Portfolios 2) $S_T e^{-f(T-t)}e^{f(T-t)} = S_T$

We can observe that both portfolios have the same payoff, then, for the previous observations, we have that both must cost the same at time t. We will denote the forward contract price at time t like f_t.

$$\begin{aligned} f_t + Ke^{-d(T-t)} &= S_t e^{-f(T-t)} \\ f_t &= S_t e^{-f(T-t)} - Ke^{-d(T-t)} \end{aligned}$$

This will be the forward contract price at time t with an agreed price K. But, if we ask us what would be the strike price in such a way that at time t, the price of the contract is zero? We have the following:

$$\begin{aligned} 0 &= S_t e^{-f(T-t)} - Ke^{-d(T-t)} \\ Ke^{-d(T-t)} &= S_t e^{-f(T-t)} \\ K &= S_t e^{-f(T-t)}e^{d(T-t)} \\ K &= S_t e^{(d-f)(T-t)} \end{aligned}$$

With this we have that the forward price that makes the forward contract worth zero is:

$$F_t = S_t e^{(d-f)(T-t)}$$

Note that, this price is the best estimator for the underlying asset price at time T.

The previous is just for a forward contract, now we enunciate the concepts for the valuation of a call option.

Definition 6 (Call option). *Is a contract that gives its holder the right and not the obligation of buy an underlying asset at an agree price (strike price K) in a future date agreed (maturation date T).*

Lemma 1

Suppose that $Y \sim N(\mu, \sigma^2)$. Then, for $a, b \in \mathbb{R}$,

$$\int_b^\infty e^{ay} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(y-\mu)^2}{2\sigma^2}} dy = e^{a\mu + \frac{1}{2}a^2\sigma^2} N(d)$$

Where:

$d = \frac{-b+\mu+a\sigma^2}{\sigma}$

$N(d)$ is the standard normal cumulative distribution until d.

Now, we develop the way of valuating an option, especially a call option. We must note that the process will be different because, we should consider the optionality. First, we present a model that will help us.

Black-Scholes Model[3]

Suppose we have a Black-Scholes model for a continuously tradable stock and bond, that is assuming the following: the existence of a constant $d, \mu,$ *and* σ, a complete market, not consider brokerage costs, and the underlying asset follows the next dynamic:

$$S_t = S_0 e^{(d-f-\frac{1}{2}\sigma^2)t + \sigma \hat{w}_t}$$

Where:

d is the domestic interest rate.

[3]We obtained of Baxter & Rennie (2003).

f is the foreign interest rate.

$\hat{w}_t$ follows a $\mathbb{Q}$ -Bowniano movement. $\mathbb{Q}$ is the neutral risk measure.

Using the previous, we will see that the price that we should have is:

$$X_t = E_{\mathbb{Q}}[Xe^{-rT}|\mathcal{F}_t]$$

Where:

X is the payoff of the derivative.

In other words, this model tells us that the price of our derivative is the present value of its payoff.

Now, we will use the Black and Scholes model to obtain the call option price. We will denote the price like $c(S_0, K, T)$.

$$\begin{aligned} c(S_0, K, T) &= E_Q[Xe^{-dT}] \\ &= E_Q[(S_T - K)_+ e^{-dT}] \\ &= e^{-dT} E_Q[(S_0 e^{(d-f-\frac{1}{2}\sigma^2)T + \sigma\hat{w}_T} - K)_+] \\ &= e^{-dT} E_Q[(F_0 e^{-\frac{1}{2}\sigma^2 T + \sigma\hat{w}_T} - K)_+] \\ &= e^{-dT}[F_0 N(d1) - KN(d2)] \end{aligned}$$

Where:

$d1 = \frac{ln(\frac{F_0}{K}) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$

$d2 = \frac{ln(\frac{F_0}{K}) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$

4 Problem

The main problem has been stated previously, but in this section, we will explain step by step the operations undertaken by each player and when they make them.

4.1 Company (first player)

We already saw that the Company wants to hedge a portion of her debt. We will denote this portion by x. Also, we must note that, although the contract is agreed today (at time 0) the amount that it refers to is not receive in this moment because the Company needs such amount until the next period (at time 1), for this reason the maturity of the forward contract is at t=1.

Now, let us analyze the unhedged position of the Company. She needs at time 1 the unhedged amount of her financial needs, and she buys this amount at the current exchange rate (denoted by $S_1(x, y)$).

With these elements, we can establish the next payoff function.

$$f_1(x, y) = -xM_1F_0 - (1 - x)M_1S_1(x, y)$$

Where:

M_1 is the amount of her debt.

F_0 is the forward price at time 0.

We can note that this payoff function is at time 1 because at this time the Company has to exercise the forward contract and buy the remaining amount needed to repay her debt.

Previously, in the background we saw how to determine the forward price in the absence of arbitration:

$$F_0 = S_0 e^{(d-f)T}$$

Where:

S_0 is the exchange rate at time 0.

T is the period between 0 and 1.[4]

e^d is the rate of return of the domestic currency.

e^f is the rate of return of the foreign currency.

Moreover, we defined the exchange rate at time 1 like:

$$S_1(x, y) = S_1^* + nye^{dT} + mxe^{dT}$$

Where:

S_1^* follows the dynamic described in the background section.

n is the marginal coefficient representing the effect of y in $S_1(x, y)$, in other words, it is the change in the exchange rate produced by the purchase of dollar.

m is the marginal coefficient representing the effect of x in $S_1(x, y)$, in other words, it is the change in the exchange rate produced by the purchase of a forward contract.

y is the proportion which with the Financial Institution will take part in the spot market.

Although we defined the exchange rate at time 1 in this way because is a random variable, we will work with its best estimator. The actions of the Company and the Financial Institution have repercussion on the spot market. Remember that the Financial Institution buys or short-sells y proportion of dollars in the spot market at time 0, this is the reason of ny. On the other hand, the operation of the Company also affects this market because the Institution who sold the forward contract to the Company has to buy m proportion of

[4]We assume this distance is equal to 1.

dollars to hedge that contract, hence, the hedged has an impact in the spot market, this is the reason of mx.

Now, we obtain the best estimator for $S_1(x, y)$

$$\begin{aligned} E_{\mathbb{Q}}[S_1(x,y)] &= E_{\mathbb{Q}}[S_1^* + nye^{dT} + mxe^{dT}] \\ &= E_{\mathbb{Q}}[S_1^*] + nye^{dT} + mxe^{dT} \\ &= E_{\mathbb{Q}}[S_0 e^{(d-f-\frac{1}{2}\sigma^2)t+\sigma\hat{w}_t}] + nye^{dT} + mxe^{dT} \\ &= S_0 e^{(d-f)T} E_{\mathbb{Q}}[e^{-\frac{1}{2}\sigma^2 T+\sigma\hat{w}_T}] + nye^{dT} + mxe^{dT} \\ &= S_0 e^{(d-f)T} + nye^{dT} + mxe^{dT} \\ &= F_0 + nye^{dT} + mxe^{dT} \end{aligned}$$

If we substitute the above in the payoff function of the Company, we have:

$$\begin{aligned} f_1(x,y) &= -xM_1F_0 - (1-x)M_1S_1(x,y) \\ &= -M_1[xF_0 + (1-x)S_1(x,y)] \\ &= -M_1\left[x(S_0e^{(d-f)T}) + (1-x)\left(S_0e^{(d-f)T} + nye^{dT} + mxe^{dT}\right)\right] \\ &= -M_1\left[S_0e^{(d-f)T} + (1-x)\left(nye^{dT} + mxe^{dT}\right)\right] \\ &= -M_1e^{dT}\left[S_0e^{-fT} + (1-x)(ny+mx)\right] \end{aligned}$$

4.2 Financial Institution (second player)

We will explain the operations made by the Financial Institution and the time they are made.

As we mentioned earlier, the Financial Institution decides buying or short-selling dollars with speculative purposes, this operations is made at time 0. After that, at time 1 the Financial Institution will sell or buy forwards contracts with an opposite position that she carried out at the time 0. For example, if she bought dollars at time 0, she will sell dollars through forwards contracts. We should note that when the Financial Institution sells or buys the forwards contracts, she has the obligation to liquidate her position at time 2.

Thereby, the payoff function of the Financial Institution is:

$$f_2(x,y) = -yM_2S_0e^{dT} + yM_2e^{fT} + yM_2F_1e^{2fT}e^{-dT} - yM_2e^{fT}$$

Where:

M_2 is the amount that she can buy or short-sell of the foreign currency.

$F_1(x,y)$ is the forward price at time 1.

As we discussed earlier, y represents the proportion of a certain amount that the Financial Institution will take part in the spot market, whence, y is between -1 and 1. If she buys dollars y is positive but if she short-sells dollars y is negative.

We observe that payoff function is at time 1, furthermore, dollars that Financial Institution buys or short-sells generate a profitability during two periods because she has to deliver them until time 2 once forward contracts have matured.

As in the payoff function of the Company, we calculate the forward price as follows:

$$F_1 = S_1(x,y)e^{(d-f)T}$$

If we replace the previous definitions, we can reduce the payoff function in the next way:

$$
\begin{aligned}
f_2(x,y) &= -yM_2S_0e^{dT} + yM_2e^{fT} + yM_2F_1e^{2fT}e^{-dT} - yM_2e^{fT} \\
&= yM_2\left[F_1e^{2fT}e^{-dT} - S_0e^{dT}\right] \\
&= yM_2\left[S_1(x,y)e^{(d-f)T}e^{2fT}e^{-dT} - S_0e^{dT}\right] \\
&= yM_2\left[(S_0e^{(d-f)T} + nye^{dT} + mxe^{dT})e^{fT} - S_0e^{dT}\right] \\
&= yM_2\left[S_0e^{dT} + nye^{(d+f)T} + mxe^{(d+f)T} - S_0e^{dT}\right] \\
&= yM_2\left[nye^{(d+f)T} + mxe^{(d+f)T}\right] \\
&= yM_2e^{(d+f)T}(ny + mx)
\end{aligned}
$$

4.3 Adding operating costs

In this part, we are going to add operating cost for both institutions. When we consider this assumption the payoff function change for both players, as now we explain.

The operations that the Company perform will be the same, with the only difference that for each operation the Company has to pay a percentage. This percentage applies to the amount of the forward contracts that she will acquire and the dollars which she will buy at time 1. Then, we define that costs as follows.

$c_1 = zxM_1F_0e^{dT}$

$c_2 = z(1-x)M_1S_1(x,y)$

Where:

z is a percentage that represents the cost of one operation.

Taking this into consideration, the new payoff function is:

$$
\begin{aligned}
f_1(x,y) &= -xM_1F_0 - c_1 - (1-x)M_1S_1(x,y) - c_2 \\
&= -xM_1F_0 - zxM_1F_0e^{dT} - (1-x)M_1S_1(x,y) - z(1-x)M_1S_1(x,y) \\
&= -M_1[xF_0 + (1-x)S_1(x,y)] - zM_1\left[xF_0e^{dT} + (1-x)S_1(x,y)\right] \\
&= -M_1e^{dT}\left\{\left[S_0e^{-fT} + (1-x)(ny+mx)\right] + z\left[xS_0e^{(d-f)T} + (1-x)(S_0e^{-fT} + ny + mx)\right]\right\}
\end{aligned}
$$

On the other hand, when we talk about the Financial Institution the analysis is a little more complicated because like the Company's case the costs are related to the amount of the operations, nevertheless, the Financial institution could sell forward contracts and dollars, and that would cause that the costs as just defined not work properly, because y would be negative.

To solve this problem, we will define the costs like those of the Company but with the difference that y will be always positive.

$c_1 = z|y|M_2S_0e^{dT}$

$c_2 = z|y|M_2F_1$

We will assume the percentage is the same for both, this is just to simplicity. Now, with this definition we calculate the new payoff function.

$$
\begin{aligned}
f_2(x,y) &= -yM_2S_0e^{dT} + yM_2e^{fT} - c_1 + yM_2F_1e^{2fT}e^{-dT} - c_2 - yM_2e^{fT} \\
&= yM_2\left[F_1e^{2fT}e^{-dT} - S_0e^{dT}\right] - c_1 - c_2 \\
&= yM_2e^{(d+f)T}[ny+mx] - z|y|M_2\left[S_0e^{dT} + F_1\right] \\
&= yM_2e^{(d+f)T}[ny+mx] - z|y|M_2e^{dT}\left[S_0 + (S_0e^{-fT} + ny + mx)e^{(d-f)T}\right]
\end{aligned}
$$

4.4 Using a call option

In this section, we will develop the case where the Company can hedge using options, specifically a call option. Also, we will assume that only the Company can acquire this type of contract and we will not have operating costs.

We must note that the payoff function of the Financial Institution will not change in comparison with the section 4.2. Thus, we only focus in the payoff function of the Company.

The movements of the Company are the same, we just present them in a different way, because the payments changes:

$$Z = -M_1 \left\{ xc(S_0, K, T)e^{dT} + xK\mathbb{I}_{S_1>K} + (1-x)S_1 + xS_1\mathbb{I}_{K>S_1} \right\}$$

Where:

$\mathbb{I}$ is a characteristic function and denotes if the call option was exercised or not.

But like we said before, we need to work with the best estimator of the exchange rate at time 1. For this reason, the payoff function of the Company will be the expected value of those payments. The payoff function is[5]:

$$\begin{aligned}
f_1(x,y) &= E[Z] = E\left[-M_1 \left\{ xc(S_0, K, T)e^{dT} + xK\mathbb{I}_{S_1>K} + (1-x)S_1 + xS_1\mathbb{I}_{K>S_1} \right\}\right] \\
&= -M_1 \left\{ xc(S_0, K, T)e^{dT} + xKE[\mathbb{I}_{S_1>K}] + (1-x)E[S_1] + xE[S_1\mathbb{I}_{K>S_1}] \right\} \\
&= -M_1 \left\{ xc(S_0, K, T)e^{dT} + xK(1-N(b)) + (1-x)(F_0 + (ny+mx)e^{dT}) \right\} \\
&\quad -M_1 \left\{ x(F_0(1-N(d)) + (ny+mx)e^{dT}N(b)) \right\} \\
&= -M_1 \left\{ xc(S_0, K, T)e^{dT} + xK(1-N(b)) + F_0(1-xN(d)) + (ny+mx)e^{dT}(1-x(1-N(b))) \right\}
\end{aligned}$$

[5]The calculations of expected value are in the annexes.

Where:

$$b = \frac{ln\left(\frac{K-(ny+mx)e^{dT}}{F_0}\right)+\frac{1}{2}\sigma^2 T}{\sigma}$$

$$d = \frac{ln\left(\frac{F_0}{K-(ny+mx)e^{dT}}\right)+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

5 Nash Equilibrium

In this section, we will get the Nash Equilibrium using the payoff functions that we obtained previously for the different cases. Before we analyze the best response functions let us make the following observations.

Remember that, m is the effect of x in the spot price, then, if $x > 0$ it means that the Company bought dollars in consequence the spot price increase, for this reason, m is positive.

On the other hand, n also is the effect of y in the spot price, but y can be positive or negative. If $y < 0$ it implies the Financial Institution short-sold dollars, in consequence the spot price decreases, but if $y > 0$ it implies she bought dollars, in consequence the spot price increases, then, n is positive.

5.1 Without operating costs

The Company wants to reduce her payment, which will be achieved if she maximizes her payoff function. To obtain what would be the maximum of the payoff function we will calculate the first and second derivative.

Remembering her payoff function:

$$f_1(x,y) = -M_1 e^{dT}\left[S_0 e^{-fT} + (1-x)(ny+mx)\right]$$

We will see the first derivative with respect to x.

$$\begin{aligned}\frac{\partial f_1(x,y)}{\partial x} &= -M_1 e^{dT}[(1-x)(m)+(ny+mx)(-1)] \\ &= -M_1 e^{dT}[m-2mx-ny]\end{aligned}$$

Now, we will calculate the second derivative.

$$\begin{aligned}\frac{\partial^2 f_1(x,y)}{\partial x^2} &= -M_1 e^{dT}[-2m] \\ &= 2mM_1 e^{dT}\end{aligned}$$

The second derivative is always positive, which indicates that we will find a minimum. If we consider the interval of x is a closed set and $f_1(x,y)$ is a continuous function, then, her maximum is reached in the extremes. We just need to check what would be her best respond when she decides to fully hedged or she decides to unhedged.

The Company wants to reduce $S_0 e^{-fT} + (1-x)(ny+mx)$. If $y > 0$, this part $((1-x)(ny+mx))$ is positive, the best response for the Company is to do $x = 1$. If $y < 0$, this part $((1-x)(ny+mx))$ is negative, the best response for the Company is to do $x = 0$.

On the other hand, the Financial Institution wants to maximize her profits, namely, she wants to maximize her payoff function, then, we calculate the first and second derivative.

Remembering her payoff function is:

$$f_2(x,y) = yM_2 e^{(d+f)T}(ny+mx)$$

Her first and second derivative are respectively:

$$\begin{aligned}\frac{\partial f_2(x,y)}{\partial y} &= M_2 e^{(d+f)T}[(ny+mx)+ny] \\ &= M_2 e^{(d+f)T}[2ny+mx]\end{aligned}$$

$$\frac{\partial^2 f_2(x,y)}{\partial y^2} = M_2 e^{(d+f)T}(2n)$$

We can see that the derivative is always positive, which means that, as in the case of the Company, we have a minimum. Moreover, we must note the interval of y is a closed set and $f_2(x,y)$ is a continuous function, then, her maximum is reached in the extremes. Under this idea, we will obtain her best response contemplating only the cases where the Financial Institution decides to buy all her available capacity of dollars or short-selling all her available capacity of dollars.

If $x = 1$, the Financial Institution must maximize her payoff function, then, her best response is $y = 1$, but, if $x = 0$, her payoff function is a quadratic function, then her best response is $y = 1$ or $y = -1$.

In summary, we have:

$$BR_C(FI) = \begin{cases} x = 1 & \text{if} \quad y = 1 \\ x = 0 & \text{if} \quad y = -1 \end{cases}$$

$$BR_{FI}(C) = \begin{cases} y = 1 & \text{if} \quad x = 1 \\ y = 1 \ or \ -1 & \text{if} \quad x = 0 \end{cases}$$

In this case, we see the Nash Equilibria are:

$$\{x = 1, y = 1\} \ and \ \{x = 0, y = -1\}$$

5.2 With operating costs

Now, we analyze the best response functions considering the costs, but before that we must note the following.

When we add the costs, we must consider some implications about that, because we need to analyze for which values of z it is optimal that players use these financial instruments to realize their hedge and their intervention in the market. First, we start with the Financial Institution.

Remembering her payoff function with cost is:

$$f_2(x,y) = yM_2e^{(d+f)T}[ny+mx] - z|y|M_2e^{dT}\left[S_0 + (S_0e^{-fT} + ny + mx)e^{(d-f)T}\right]$$

If we want the Financial Institution realizes her operations in this game, we need that her profit would be greater than zero, namely, her costs should not excess the profit. Whereby, we have the next condition.

$$\Leftrightarrow \quad yM_2e^{(d+f)T}(ny+mx) - z|y|M_2e^{dT}[S_0 + (S_0e^{-fT} + ny + mx)e^{(d-f)T}] \geq 0$$

$$\Leftrightarrow \quad yM_2e^{(d+f)T}(ny+mx) \geq z|y|M_2e^{dT}[S_0 + (S_0e^{-fT} + ny + mx)e^{(d-f)T}]$$

$$\Leftrightarrow \quad ye^{fT}(ny+mx) \geq z|y|[S_0 + (S_0e^{-fT} + ny + mx)e^{(d-f)T}]$$

$$\Leftrightarrow \quad \frac{ye^{fT}(ny+mx)}{|y|[S_0 + (S_0e^{-fT} + ny + mx)e^{(d-f)T}]} \geq z$$

This condition tells us that, for values of z greater than this level, the Financial Institution will do nothing in the market because she would have losses, in other words, this is the maximum percentage that the Financial Institution is willing to accept to perform operations.

On the other hand, analyzing the payoff function of the Company we can see that she always incurs costs because she must buy dollars to repay her debt no matter if she does this through forward contracts or directly in the spot market.

In addition to that, remember that the percentage that represent the costs is the same for the Company and the Financial Institution, for this reason we will consider like maximum quota the percentage that the Financial Institution is willing to accept to perform the corresponding operations in both markets.

Another aspect that we must consider is that the response of both players influences the determination of this quota, but remember that the optimal movement for each player is in the extreme values, because we just aggregated a linear term to the payoff function without costs, that is, when the Company decides to fully hedged or not and when the Financial Institution decides to intervene with all her available capacity. Given this, we value the quota that we find for z and take the maximum value.

The player's response depends of the parameters, then, we propose some values for these parameters, obtained from some data of the Mexican Market. Consider the next values.[6]

$S_0 = 18.7192$
$e^{dT} = 1.07477$
$e^{-fT} = 0.98847$
$n = 0.5\%$
$m = 0.4\%$

[6]Bloomberg data.

$M_1 = 10,000,000$

$M_2 = 100,000,000$

$z = 0.02\%$

These values were chosen of the following way:

S_0 this value represents the exchange rate at the time the forward contract is agreed to, then, this is the value of the exchange rate as on April 6, 2017.

M_1 and M_2 these are the amount of the Company's debt and the amount that the Financial Institution can dispose to operate in the respectively markets. These amounts can be arbitrary with the only condition that they must be large amounts to provoke a movement in the markets.

e^{dT} this is the rate at which money accumulates in domestic currency. This data is obtained directly from the prices in the Mexican Market on April 6, 2017, valid for one year.

e^{fT} this is the rate at which money accumulates in foreign currency. This data is obtained directly from the prices in the Mexican Market on April 6, 2017, valid for one year.

n is the marginal coefficient representing the effect in the spot market for the intervention of the Financial Institution. This data is proposed.

m is the marginal coefficient representing the effect in the futures market for the intervention of the Company. This data is proposed.

z this is the percentage for operating costs and is obtained through the analysis performed previously by the condition of the Financial Institution.

Now, we will obtain the best respond of the Company, if the Financial Institution decides to buy all her capacity of dollars, the Company must fully hedged, because no matter the operating costs the Company always must do this operations either before or after. On the other hand, if the Financial Institution decides to short-sold all her available capacity of dollars, the Company must wait for the exchange rate to decrease and with that the best she can do is buy all her debt at the time 1.

For the best response of the Financial Institution, we should note that in this case the operating cost have a relevance, because if these are greater than her profits the Financial Institution will never perform any operation in the market. Considering this, we suppose a cost of the 0.02% for each dollar that was sold or bought. If the Company decides to fully hedged, the Financial Institution should buy all her available capacity of dollars generating that the exchange rate increases and, in the next period she will have to sell through forward contracts getting a profit. Nevertheless, considering the case when the Company decides to not hedged, the Financial Institution would have losses if she does any operation in the market no matter if it is a sale or a purchase, because the costs are greater than her profits.

In summary, the best response function for the Company is:

$$BR_C(FI) = \begin{cases} x = 1 & \text{if} \quad y = 1 \\ x = 0 & \text{if} \quad y = -1 \end{cases}$$

On the other hand, the best response function for the Financial Institution is:

$$BR_{FI}(C) = \begin{cases} y = 1 & \text{if} \quad x = 1 \\ y = 0 & \text{if} \quad x = 0 \end{cases}$$

In this case, the only Nash Equilibrium is:

$$\{x = 1, y = 1\}$$

Note that the operating costs determine how many Nash equilibrium we will have, because if the costs are very small we have two Nash equilibria because the Financial Institution always does something in the market. Let us see the next example when we have a cost of the 0.01%.

If the Company decides to fully hedge, the Financial Institution should buy all her available capacity of dollars like in the case when we have a cost of 0.02%, but in this case, if the Company decides to unhedged, the Financial Institution should sell all her available capacity of dollars because in this case the costs are low, causing that the Financial Institution does not have losses for that operation. We must note that the payoff function of the Company is the same.

In summary, the best response function for the Company is:

$$BR_C(FI) = \begin{cases} x = 1 & \text{if } y = 1 \\ x = 0 & \text{if } y = -1 \end{cases}$$

On the other hand, the best response function for the Financial Institution is:

$$BR_{FI}(C) = \begin{cases} y = 1 & \text{if } x = 1 \\ y = -1 & \text{if } x = 0 \end{cases}$$

The Nash Equilibria are:

$$\{x = 1, y = 1\} \; and \; \{x = 0, y = -1\}$$

5.3 With a call option

In this part, we will solve the problem without operating cost and using a call option instead of a forward contract, but this type of contract is available just for the Company, that is the reason why we focus in the payoff function of the Company.

Remembering the payoff function of the Company is:

$$f_1(x,y) = -M_1\left\{xc(S_0,K,T)e^{dT} + xK(1-N(b)) + F_0(1-xN(d)) + (ny+mx)e^{dT}(1-x(1-N(b)))\right\}$$

We need to maximize that function; therefore, we must obtain the partial derivative with respect x, but note that is not possible to solve x analytically. For this reason, we will use some data of the Mexican market for to solve the problem, but first we present a payoff function graph.[7]

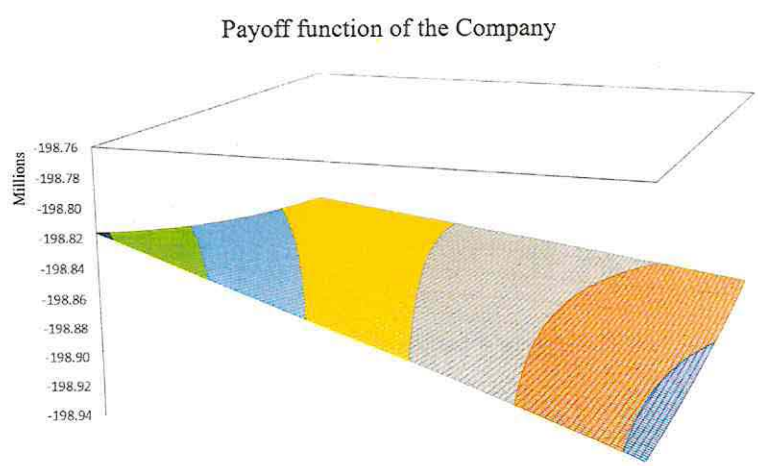

Figure 2: Payoff function of the Company

With this graph, we can note the maximum value of the payoff function is in the extreme values of x. In the case of the payoff function of the Financial Institution, we will have the

[7] Bloomberg data.

same best response as in the section 5.1.

Using the previous values and a volatility of $\sigma = 13.265\%$ obtained of the Mexican market at April 6, 2017,[8] we will have the next best response function for the Company.

If the Financial Institution chooses $y = 1$, the Company must choose $x = 1$, that is, she decides to fully hedged because the variation caused by the Financial Institution increases the exchange rate price at time 1. On the other hand, if the Financial Institution chooses $y = -1$, the Company must choose $x = 0$, that is, she decides to unhedged because the variation caused by the Financial Institution decreases the exchange rate price at time 1. On the other hand, the best response for the Financial Institution is the same as in the section 5.1.

In summary, we have:

$$BR_C(FI) = \begin{cases} x = 1 \text{ if } y = 1 \\ x = 0 \text{ if } y = -1 \end{cases}$$

$$BR_{FI}(C) = \begin{cases} y = 1 & \text{if } x = 1 \\ y = 1 \; or \; -1 & \text{if } x = 0 \end{cases}$$

In this case, we see the Nash Equilibria are:

$$\{x = 1, y = 1\} \; and \; \{x = 0, y = -1\}$$

[8]Bloomberg data.

6 Discussion

In this section, we want to discuss which are the main differences between this work and Musolino's paper (2012). As a first difference, we mention that the underlying asset that Musolino considers is a commodity, that means, the underlying asset does not generate a rate of return and, he assumes there are not storage costs. In the case of this work, we consider that the underlying asset is a foreign currency, then, we cannot assume the underlying asset does not pay any yield, because when the Financial Institution buys or short-sells the foreign currency (in this case dollars) we must note that the dollars are not static, they are generating a certain yield, namely, when we use a different underlying asset we are relaxing the assumption that they do not generate any yield.

The next difference is related to the previous, because when we talk about of a different underlying asset, the way of calculating the forward price changes, in addition to this, we must consider the yield that the dollars are generating, this creates a modification in the payoff functions of each player making that the Nash equilibrium changes.

On the other hand, Musolino takes a specifies form for the price at time 1, whereas in this work we suggest the exchange rate is a random variable, as detailed in the background and problem sections. Also, in this work we aggregate the participation of the operation carried out by the Company due to the hedge of forward contracts in the exchange rate at time 1.

In addition, we aggregate the operation costs applying to each operation that players perform, because we will have a cost applied to the amount that they decide to operate, this contribution is made with the purpose of resembling the problem as much as possible to what happens in the Financial Market.

Moreover, in this work we add the way to solve this problem using a call option. This change is relevant because we provide another instrument as alternative that a player has to the previous instrument to operate in the market.

Finally, in Musolino's paper, he purposes a solution in a cooperative way, however, in this work we do not study this approach, nor do we consider the tax for the speculative operation carried out by the Financial Institution.

7 Conclusion

Throughout the development of this work we have analyzed a problem studied in previous literature under new scenarios. Also, we have added the fact that the underlying asset generates certain rate of return, and this modifies the way of valuation of financial derivatives.

In our first scenario which did not consider operating costs, we obtained two Nash equilibria. One of them is the equilibrium $\{x = 1, y = 1\}$. The interpretation of this equilibrium is as follows. If the Company decided to fully hedged, the best thing that the Financial Institution could do is to buy all her available capacity of dollars, because she would be buying when the dollar is cheap and she would be selling at time 1 when the dollar would be expensive due to the variation in the exchange rate at time 0. This operation does not harm the Company because she hedged the exchange rate for all her debt when it was cheap and she will not need to go out to buy dollars at time 1.

For the next equilibrium $\{x = 0, y = -1\}$ the interpretation is that, if the Company does not want to hedge the exchange rate, the best thing the Financial Institution could do is short-sell all her available capacity of dollars, which would cause the exchange rate to decrease. At time 1, the Financial Institution and the Company would go out and buy the dollars that each one needs, the Financial Institution would have a profit because she is short-selling when the dollar is expensive and buying when it is cheap, on the other hand, the Company would buy her dollars when they are cheap.

In the case where we introduce operating costs we see that they play a very important role because, if they are too high they can cause that one of the players does not have incentives to participate, this can be translated to the following. Assume that the cost can be seen as the premium that an intermediary will charge for performing the operation that the players demand, in the case of the Company, she will not have other option more than to participate in the game because she has to pay her debt, however, if we look at the Financial Institution, she wants to participate with the only purpose of making profits, but if the costs are too high this profit would be small or even negative, which would imply that the Financial Institution will not make any operation. This gives the incentive to lower the costs until the Financial Institution wants to participate to get the profit generated for her operation. The impact of this decision is a bit more complex because of the reduction in the operations that we can make in the market and we will have a reduction in its dynamics.

We see that our Nash equilibrium will depend on the costs that we suggest, because it will not only determine if the Financial Institution will want to participate in the game, also it will determine how many Nash equilibria we will have. When considering costs of 0.02%, the explanation for the elimination of the Nash equilibrium $\{x = 0, y = -1\}$ that we obtained in the model without costs is that if the Company does not want to hedge, then she does not generate a variation in the exchange rate and makes the Financial Institution not to obtain high enough revenues to compensate the cost from short-selling the dollars. This causes that this is no longer a Nash equilibrium.

In the case when the Company uses a call option, we will have again two Nash equilibria, because we do not have a restriction like in the case where we add operating cost. The first Nash equilibrium that we will discuss is $\{x = 1, y = 1\}$, in this case, the Company decides to fully hedge, with this operation she just needs to watch the exchange rate at time 1 and decides if it is correct to exercise or not, but her debt has a limit because the worse price that she have to buy the dollars is K. On the other hand, the Financial Institution buys all her available capacity of dollars that causes that the exchange rate increases and when she has to sell them, she will obtain a profit.

The second Nash equilibrium is $\{x = 0, y = -1\}$, in this case, the Company decides to unhedged, that means, she has to buy all the dollars that she needs at time 1, on the other hand, the Financial Institution decides short-sell all her available capacity of dollars, that causes that the exchange rate to decrease and, at time 1, the exchange rate will be cheaper than at time 0, that represent a profit for the Financial Institution, and for the Company a smaller loss.

A Expectation Value

In the section 4.4 we calculate the payoff function of the Company, in this annexes we present the calculations of each expectation value.

$$
\begin{aligned}
E\left[\mathbb{I}_{S_1>K}\right] &= P[S_1 > K] \\
&= P\left[F_0 e^{-\frac{1}{2}\sigma^2 T + \sigma \hat{w}_T} + (ny+mx)e^{dT} > K\right] \\
&= P\left[F_0 e^{-\frac{1}{2}\sigma^2 T + \sigma \hat{w}_T} > K - (ny+mx)e^{dT}\right] \\
&= P\left[e^{-\frac{1}{2}\sigma^2 T + \sigma \hat{w}_T} > \frac{K - (ny+mx)e^{dT}}{F_0}\right] \\
&= P\left[-\frac{1}{2}\sigma^2 T + \sigma \hat{w}_T > ln\left(\frac{K - (ny+mx)e^{dT}}{F_0}\right)\right] \\
&= P\left[\sigma \hat{w}_T > ln\left(\frac{K - (ny+mx)e^{dT}}{F_0}\right) + \frac{1}{2}\sigma^2 T\right] \\
&= P\left[\hat{w}_T > \left\{ln\left(\frac{K - (ny+mx)e^{dT}}{F_0}\right) + \frac{1}{2}\sigma^2 T\right\}\frac{1}{\sigma}\right] \\
&= 1 - P\left[\hat{w}_T \leq \left\{ln\left(\frac{K - (ny+mx)e^{dT}}{F_0}\right) + \frac{1}{2}\sigma^2 T\right\}\frac{1}{\sigma}\right] \\
&= 1 - P\left[\frac{\hat{w}_T}{\sqrt{T}} \leq b\right] \\
&= 1 - N(b)
\end{aligned}
$$

With

$$
b = \frac{ln\left(\frac{K-(ny+mx)e^{dT}}{F_0}\right) + \frac{1}{2}\sigma^2 T}{\sigma}
$$

Now, other expected value that we need is:

$$\begin{aligned}
E\left[S_1 \mathbb{I}_{K>S_1}\right] &= E\left[\left\{F_0 e^{-\frac{1}{2}\sigma^2 T + \sigma \hat{w}_T} + (ny+mx)e^{dT}\right\} \mathbb{I}_{K>S_1}\right] \\
&= E\left[\left\{F_0 e^{-\frac{1}{2}\sigma^2 T + \sigma \hat{w}_T}\right\} \mathbb{I}_{K>S_1}\right] + E\left[\left\{(ny+mx)e^{dT}\right\} \mathbb{I}_{K>S_1}\right] \\
&= F_0 e^{-\frac{1}{2}\sigma^2 T} E\left[e^{\sigma \hat{w}_T} \mathbb{I}_{K>S_1}\right] + (ny+mx)e^{dT} E\left[\mathbb{I}_{K>S_1}\right] \\
&= F_0 e^{-\frac{1}{2}\sigma^2 T} \int_{-\infty}^{\infty} e^{\sigma \hat{w}_T} \mathbb{I}_{K>S_1} f_{\hat{w}_T}(w_T) dw_T + (ny+mx)e^{dT} P\left[K > S_1\right] \\
&= F_0 e^{-\frac{1}{2}\sigma^2 T} \int_{-\infty}^{b'} e^{\sigma \hat{w}_T} f_{\hat{w}_T}(w_T) dw_T + (ny+mx)e^{dT} N(b) \\
&= F_0 e^{-\frac{1}{2}\sigma^2 T} \left[\int_{-\infty}^{\infty} e^{\sigma \hat{w}_T} f_{\hat{w}_T}(w_T) dw_T - \int_{b'}^{\infty} e^{\sigma \hat{w}_T} f_{\hat{w}_T}(w_T) dw_T\right] + (ny+mx)e^{dT} N(b) \\
&= F_0 e^{-\frac{1}{2}\sigma^2 T} \left[e^{\frac{1}{2}\sigma^2 T} - e^{\frac{1}{2}\sigma^2 T} N(d)\right] + (ny+mx)e^{dT} N(b) \\
&= F_0 e^{-\frac{1}{2}\sigma^2 T} \left[e^{\frac{1}{2}\sigma^2 T}(1 - N(d))\right] + (ny+mx)e^{dT} N(b) \\
&= F_0(1 - N(d)) + (ny+mx)e^{dT} N(b)
\end{aligned}$$

With

$$d = \frac{ln\left(\frac{F_0}{K-(ny+mx)e^{dT}}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

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