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ON THE INCENTIVES TO INVEST IN SCHOOL QUALITY

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Abstract

We present a problem in which colleges choose quality in a continuum two-sided matching market, where a finite set of colleges is matched to a continuum of students. We consider the case when this choice is constrained, that is, when each college has a budget to invest in improving. We show that there exists a Nash Equilibrium when all of them choose quality simultaneously. Moreover, we analyze the case of the duopoly in order to make our results clear, it also shows how the mechanism works and it allows us to get some comparative statics.

Contents

1 Introduction

This paper develops a model of school quality choice inspired by the article of Azevedo and Leshno [6], which consists of choosing a school quality parameter for each school in a continuum matching market, that is, where a finite set of colleges is matched with a continuum of students. The choice of quality is taken before the assignment takes place, so that the mechanism assigns a continuum of students having observed the quality parameter of each one.

We take an important and innovative result by Azevedo and Leshno [6], which establishes the equivalence between i) finding a stable matching using the continuous version of the deferred acceptance algorithm of Gale & Shapley [10], and ii) solving a system of equations of supply and demand (finding the market-clearing cutoff vector). The main issue of this work is to model the choice of school quality by taking into account two aspects:

1. The strategic behavior of colleges, and
2. A budget constraint faced by each college to carry out the investment in quality.

This approach proposes a maximization problem subject to a budget constraint by each college, where the decision variable is precisely school quality. We use the Lagrange method to incorporate the budget constraint into the maximization of the entering class, which we define as the quality of students that are assigned to each college by the mechanism. We show that there exists at least one Nash quilibrium. Moreover, we analyze the case of duopoly (i.e. only two schools), in order to explain how the mechanism works and to get some comparative statics.

This model has two parts. The first one is related to the continuum matching model of Azevedo and Leshno [6], from which we take the terms to analyze the incentives for colleges to invest in school quality (with basically the same notation). The second part is related to a model of a constrained maximization problem in order to capture the strategic behavior of the colleges.

We find basically two kinds of results. The first one is theoretical, in which we show that there exists a Nash equilibrium in the game where colleges choose quality simultaneously. It is well known that even in the basic Cournot model, the sufficient conditions for equilibrium to exist are quite restrictive (Roberts and Sonnenschein [15]), so we had to assume a quasi-concave function of the quality of the entering class in order to use the Kakutani's fixed point theorem to prove existence. The second result is related to comparative statics, where, by assuming a truncated distribution function, we show that the market clearing cutoffs of the colleges (defined as a threshold above which students are acceptable for colleges), are indeed positive in some interval, and, with income restrictions, that the quality of the entering class of a college increases with the college's budget to invest in quality.

Since this paper considers the case of an economy where each of a finite number of colleges is matched to a continuum mass of students, it is based on Aumann [2] insight that markets with a continuum of traders may be considerably simpler that those with a finite number of traders. As mentioned by Azevedo and Leshno [6], the model follows Gale and Shapley [10] college admissions model, which allows for complex heterogeneous preferences. In the mechanism design literature there are several papers that study the properties of large markets (see, for example, Swinkels [19]). In the market design literature, many recent papers have focused on large markets (Che and kojima [8] and Azevedo and Budish [5]). More related to this model, are the contributions in market design that study large matching markets (Roth and Peranson [17] and Kojima and Pathak [12]). Besides, Avery and Levin [3] consider noncooperative models of college admissions, where colleges use admissions thresholds.

In the context of the incentives for colleges to invest in school quality, Hatfield et al. [11] consider this problem, in particular whether competition for the best students gives colleges incentives to improve their quality. They show that no stable mechanism respects improvements of school quality. However, in large markets, any stable mechanism approximately respects improvements, so the incentives are nonnegative but we do not know anything about the magnitude of them. Finally, for the last part, we follow the work of Azevedo [4] who introduce imperfect competition in two-sided matching markets.

2 The Model

We start with some definitions in order to analyze the two parts of the model. We use the same notation of Azevedo and Leshno [6] to make our extension clear.

2.1 Definitions

Consider an economy with a set of public colleges $C = \{c_1, ..., c_C\}$ on one side of the market, where we denote a generic college as c. Each college c has capacity s_c, which is the measure of the maximum number of students that college c can admit. The preferences of the colleges over students are defined by the vector $e^c \in [0,1]^C$ which are the scores that students obtain at college c. We need some measure of the quality of colleges. Let δ be the vector of qualities of the C colleges, so that $\delta_c \in \mathbb{R}$ represents the quality of college c; students prefer higher δ_c to lower δ_c.

On the other side, I is the set of students, generically denoted i. In this case we say that e^i_c is the score of student i at college c. It is assumed that the vectors e^i are distributed according to a distribution function $G(\cdot)$ in $[0,1]^C$, with an associated continuous density $g > 0$. Let $\tau : I \longrightarrow \Theta$ be a type function (where Θ is the set of student types), such that $\tau(i) = \theta$, that is, τ assigns a type $\theta \in \Theta$ to each student $i \in I$. We denote the type of students $\theta = (\succ^\theta, e^\theta)$, where $\succ^\theta$ represents the student's strict preference ordering over colleges, and e^θ is the vector of scores obtained by θ at every single college. Colleges prefer students with higher scores, that is, c prefers θ over θ' if $e^\theta_c > e^{\theta'}_c$, or equivalently, $i \succ^c j$ iff $e^{\tau(i)}_c > e^{\tau(j)}_c$, for all $i, j \in I$.

Students' preferences depend on the quality of each college δ_c. Nevertheless, there is the possibility that different students may be affected differently by δ_c.

This will be useful when competition for the best students is taken into account. Student i has utility $u_c^i(\delta_c) > 0$ of attending college c, increasing in δ_c, and utility 0 of being unmatched. Strict preferences are assumed, that is, the measure of students who are indifferent between two colleges is 0 for any value of δ; even when δ_c is the same for all colleges, they are not perfect substitutes, it means that even if $\delta_c = \delta_{c'}$, it is possible that $u_c^i(\delta_c) \neq u_{c'}^i(\delta_c^{'})$, and also when $\delta_c \neq \delta_{c'}$, it could be the case that $u_c^i(\delta_c) = u_{c'}^i(\delta_c^{'})$. Intuitively, if δ_c measures the quality of an applied econometrics course in a masters program, students with interest in working at Sedesol[1] will be more sensitive to changes in δ_c than those who prefer to study a PhD. Given δ, colleges' preferences induce a distribution η_δ over student types Θ, this distribution is assumed to be smooth in δ and θ, and to have a density $f_\delta > 0$.

The decision of colleges of admitting students depends on a threshold of the students' scores, above which a student is acceptable. Azevedo and Leshno [6] name this threshold a *cutoff* and assume dependence on quality, $P(\delta) = (P_{c_1}(\delta), ..., P_{c_1}(\delta))$, so that P_c is the college c's cutoff. For a given δ and $P(\delta)$, a student's demand is defined as her favorite college among those she can afford (i.e. those where $P_c \leq e_c^\theta$). Dependence on δ will be omitted when there is no risk of confusion ($P = (P_{c_1}, ..., P_{c_C})$).

Now, since Θ is continuous, we say that a **continuum economy** is given by $E = [\eta, S]$, where η is a probability measure over Θ and $S = (s_1, s_2, ..., s_C)$ is a vector of strictly positive capacities for each college. As stated above, strict preferences have been assumed, so that every college's indifference curves have η-measure 0. That is, for any college c and real number x, $\eta(\{\theta : e_c^\theta = x\}) = 0$.

Definition 1. (*Azevedo and Leshno* [6]) A **matching** describes an allocation of students to colleges. Formally, in a continuum economy $E = [\eta, S]$, it is a function $\mu : C \cup \Theta \longrightarrow 2^\Theta \cup (C \cup \Theta)$, such that

1. For all $\theta \in \Theta : \mu(\theta) \in C \cup \{\theta\}$.

[1] Secretaría de Desarrollo Social.

2. For all $c \in C : \mu(c) \subseteq \Theta$ is measurable, and $\eta(\mu(c)) \leq s_c$.

3. $c = \mu(\theta)$ iff $\theta \in \mu(c)$.

4. (Right continuity) For any sequence of student types $\theta^k = (\succ, e^k)$ and $\theta = (\succ, e)$, with e^k converging to e, and $e_c^k \geq e_c^{k+1} \geq e_c$ for all k, c, we can find some large K so that $\mu(\theta^k) = \mu(\theta)$ for $k > K$.

Conditions (1)-(3) are analogous to those in the discrete model. Condition (1) states that each student is matched to a college or to herself (unmatched); (2) that colleges are matched to sets of students with measure not exceeding its capacity; (3) is a consistency condition, requiring that a college is matched to a student if and only if the student is matched to the college. The technical Condition (4) is novel. It states that given a sequence of student types $\theta^k = (\succ, e^k)$, which are decreasingly desirable, with scores $e^k \longrightarrow e$, then for large enough k all student types $(\succ, e^k)$ in the sequence receive the same allocation, and the students whose score is the limit $(\succ, e)$ receive this allocation too. The last condition does not affect the set of stable matchings, it only implies that a stable matching always allows a set of extra students of measure 0 into a college when this can be done without compromising stability.

A student-college pair (θ, c) **blocks** a matching μ at economy E if the student θ prefers c to her match and either (i) college c does not fill its quota, or (ii) college c is matched to another student that has a strictly lower score than θ. Formally, (θ, c) blocks μ if $c \succ^\theta \mu(\theta)$ and either (i) $\eta(\mu(c)) < s_c$, or (ii) $\exists\, \theta' \in \mu(c)$ with $e_c^{\theta'} < e_c^\theta$.

Definition 2. **(*Azevedo and Leshno* [6])** A matching μ for a continuum economy E is **stable** if it is not blocked by any student-college pair.

As stated before, the target of this paper is to address the problem of investing in school quality by colleges. In doing so, the characterization of a matching in terms of supply and demand will be used. Recall that a cutoff is a minimal score $P_c \in [0, 1]$ required for admission at a college c. A student's **demand** given a vector of cutoffs is her favorite college among those she can afford. That is,

$$D^{\theta}(P) = \arg\max_{\succ^{\theta}}\{c|P_c \leq e_c^{\theta}\} \cup \{\theta\}.$$

The **aggregate demand** for college c is the mass of students that demand it,

$$D_c(P) = \eta(\{D^{\theta}(P) = c\}).$$

The aggregate vector $\{D_c(P)\}_{c\in C}$ is denoted by $D(P)$. A market clearing cutoff is a vector of cutoffs that clears supply and demand for colleges.

Definition 3. (*Azevedo and Leshno* [6]) A vector of cutoffs P is a market clearing cutoff if it satisfies the following market clearing equations

$$D_c(P) \leq s_c$$

for all c, and $D_c(P) = s_c$ if $P_c > 0$.

There is a one-to-one correspondence between stable matchings and market clearing cutoffs. To define this correspondence, two operators are defined. Given a market clearing cutoff P, we define the associated matching $\mu = \mathcal{M}P$ using the demand function:

$$\mu(\theta) = D^{\theta}(P).$$

Conversely, given a stable matching μ, we define the associated cutoff $P = \mathcal{P}\mu$ by the score of the marginal students matched to each college:

$$P_c = \inf_{\theta\in\mu(c)} e_c^{\theta}.$$

The operators $\mathcal{M}$ and $\mathcal{P}$ form a bijection between stable matchings and market clearing cutoffs.

Lemma 1. (*Supply and Demand Lemma - Azevedo and Leshno* [6]) *If μ is a stable matching, then $\mathcal{P}\mu$ is a market clearing cutoff. If P is a market clearing cutoff, then $\mathcal{M}P$ is a stable matching. In addition, the operators $\mathcal{P}$ and $\mathcal{M}$ are inverses of each other.*

This lemma implies that, given E, computing stable matchings is equivalent to finding market clearing cutoffs, as $\mathcal{P}$ and $\mathcal{M}$ are a one-to-one correspondence between the two sets.

2.2 Investment In School Quality

Now we are able to say that, given δ, there exists a unique stable matching μ_δ. Let $P^*(\delta)$ be the unique associated market clearing cutoffs. More specifically, Azevedo and Leshno (2013) define the aggregate quality of a college's entering class as

$$Q_c(\delta) = \int_{\mu_\delta(c)} e_c^\theta d\eta_\delta(\theta) \tag{1}$$

That is, the integral of scores e_c^θ over all students matched to the college c. Why do we need this term? Consider how a college's quality δ_c affects the quality of its entering class Q_c. Once we have a direct link between δ_c and Q_c, we are able to say something about the incentives to improve quality, it could be to improve the quality of the entering class as a whole or to attract a specific kind of students.

Taking into account that students' utility is increasing in δ_c, we need to define some relevant quantities:

- The number N_c of additional students attracted to college c by a marginal increase in quality:

$$N_c \equiv \frac{d}{d\delta_c} D_c(P) \mid_{P=P^*(\delta)} = \int_{\{\theta : D^\theta(P^*(\delta))=c\}} \frac{d}{d\delta_c} f_\delta(\theta) d\theta.$$

- The average quality of the attracted students:

$$\bar{e}_c \equiv \int_{\{\theta : D^\theta(P^*(\delta))=c\}} e_c^\theta \cdot \frac{d}{d\delta_c} f_\delta(\theta) d\theta / N_c.$$

- The set of students who are marginally accepted to college c' and would go to college c otherwise:

$$\tilde{M}_{c'c} = \{\theta : c' \succ^\theta c, P_{c'} = e_{c'}^\theta, P_c \leq e_c^\theta, P_{c''} > e_{c''}^\theta \forall c'' \neq c' : c'' \succ^\theta c\}.$$

- The number of students in this margin, and their average scores:

$$M_{c'c} = \int_{\tilde{M}_{c'c}} f_\delta(\theta) d\theta,$$

$$\bar{P}_{c'c} = E[e_c^\theta \mid \theta \in \tilde{M}_{c'c}].$$

At this point, notice that a cutoff P_c^* is the quality of a marginal accepted (or rejected) student at college c. The effect of school quality δ_c on the quality of the entering class Q_c is as follows.

Proposition 1. (***Azevedo and Leshno*** **[6]**) *Assume that $P^*(\delta) > 0$, and that P is differentiable in δ_c. Then the quality of the entering class Q_c is differentiable in school quality δ_c, and its derivative can be decomposed as*

$$\frac{dQ_c}{d\delta_c} = \underbrace{[\bar{e}_c - P_c^*] \cdot N_c}_{\textit{Direct Effect}} - \underbrace{\sum_{c' \neq c} [\bar{P}_{c'c} - P_c^*] \cdot M_{c'c} \cdot (-\frac{dP_{c'}^*}{d\delta_c})}_{\textit{Market Power Effect}}. \tag{2}$$

The direct effect term is weakly positive, always giving incentives to invest in quality. The market power terms increase (decrease) the incentives to invest in quality if an increase (decrease) in the quality of college c increases the market clearing cutoff of college c', that is $dP_{c'}^*(\delta)/d\delta_c > 0 (< 0)$.

2.3 Income Restrictions

So far, as mentioned before, we only have replicated the Azevedo and Leshno's model (the remaining is ours) in order to extend it in the form of a constrained maximization problem. Suppose that, in the context of the model described above, there is a budget constraint for colleges R_c, i.e. investment in quality depends both on the behavior of the other colleges in the economy, and on an amount available to the college c to invest in quality.

Investing in quality is costly, we denote $h(\cdot)$ the cost function for investing in quality (i.e. depends on δ) for college c, $h : [0, \infty) \longrightarrow \mathbb{R}$. We assume the following:

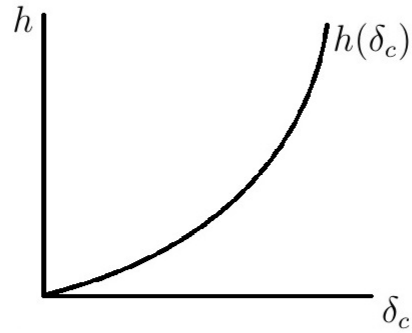

Figure 1: The strictly convex form of the cost function.

$$h'(\delta) > 0$$

and also

$$h''(\delta) > 0.$$

The intuition of an increasing and strictly convex $h(\cdot)$ is the following (see Figure 1): suppose that college c wants to increase its quality of, say, Δ_δ. What is the cost associated with this increase? It stands to reason that c can achieve this target at a lower cost when the quality is low than when it is high, and that the greater δ_c, the greater the amount of money required to improve (e.g. c could hire a recent graduate of the Master in Economics of Colmex to teach Microeconomics when δ_c is low, while it would have to hire a SNI[2] III researcher to obtain the same change when δ_c is high).

Having defined above, we can address the investment in quality for c as a maximization problem subject to a budget constraint as follows:

$$\max_{\delta_c} Q_c(\delta)$$

subject to

$$h(\delta_c) \leq R_c.$$

Using the Lagrange method, the problem becomes

[2]Sistema Nacional de Investigadores.

$$L = Q_c(\delta_c) - \lambda[h(\delta_c) - R_c], \tag{3}$$

from which we get the following first order conditions:

$$\frac{\partial L}{\partial \delta_c} = \frac{dQ_c(\delta_c)}{d\delta_c} - \lambda \frac{dh(\delta_c)}{d\delta_c} = 0 \tag{4}$$

$$\frac{\partial L}{\partial \lambda} = h(\delta_c) - R_c = 0. \tag{5}$$

Recalling that the vector of cutoffs that clear the market P^* is determined by the condition $D(P) = S$, for a given $\delta_{c'}$, $\forall c' \neq c$, we can substitute (2) into (4) to obtain

$$\delta_c = \delta_c(R_c, \delta_{c' \in \{C \backslash c\}}). \tag{6}$$

Since this expression depends on the amount R_c and the quality of the other colleges, it clearly captures the two effects mentioned above: the effect of the behavior of the other colleges in the economy, and the effect of the budget constraint for college c to invest in quality. It is a reaction function.

At this point, assuming that all colleges choose their quality simultaneously, we can model this problem as a game:

$$\mathcal{G} = \langle C, \{\Delta_c\}, \{Q_c(\cdot)\} \rangle, \quad \text{for all } c \in C,$$

where C is the set of players (colleges), $\{\Delta_c\}$ is the set of strategies for each player c (with $\delta_c \in \Delta_c$), and $\{Q_c(\delta_{c_1}, ..., \delta_{c_C})\}$ the payoff function giving the von Neumann-Morgenstern utility levels associated with the outcome arising from strategies $(\delta_{c_1}, ..., \delta_{c_C})$. Given the reaction functions in (6), we can say that:

Proposition 2. *For a given function $Q_c(\delta)$, quasi-concave in δ_c, a pure-strategy Nash equilibrium, $\delta^* = (\delta^*_{c_1}, ..., \delta^*_{c_C})$, exists in the game $\mathcal{G}$.*

Proof. We begin with **Lemma 2**, which provides a key technical result.

Lemma 2. *If the sets $\Delta_{c_1}, ..., \Delta_{c_C}$ are nonempty, Δ_c is compact and convex, and $Q_c(\cdot)$ is continuous in $(\delta_{c_1}, ..., \delta_{c_C})$ and quasi-concave in δ_c, then player c's best-response correspondence $b_c(\cdot)$ is nonempty, convex-valued, and upper hemi-continuous.*

Proof. Note first that $b_c(\delta_{-c})$ is the set of maximizers of the continuous function $Q_c(\cdot, \delta_{-c})$ on the compact set Δ_c. Hence, it is nonempty[]. The convexity of $b_c(\delta_{-c})$ follows because the set of maximizers of a quasi-concave function [in this case, the function $Q_c(\cdot, \delta_{-c})$] on a convex set [Δ_c, in this case] is convex. Finally, for upper hemi-continuity, we need to show that for any sequence $(\delta_c^n, \delta_{-c}^n) \longrightarrow (\delta_c, \delta_{-c})$ such that $\delta_c^n \in b_c(\delta_{-c}^n)$ for all n, we have $\delta_c \in b_c(\delta_{-c})$. To see this, note that for all n, $Q_c(\delta_c^n, \delta_{-c}^n) \geq Q_c(\delta_c^{'}, \delta_{-c}^n)$ for all $\delta_c^{'} \in \Delta_c$. Therefore, by the continuity of $Q_c(\cdot)$, we have $Q_c(\delta_c, \delta_{-c}) \geq Q_c(\delta_c^{'}, \delta_{-c})$. □

Now, in order to prove *Proposition 2*, define the correspondence $b : \Delta \rightarrow \Delta$ by

$$b(\delta_{c_1}, ..., \delta_{c_C}) = b_{c_1}(\delta_{-c_1}) \times \cdots \times b_{c_C}(\delta_{-c_C})$$

Note that $b(\cdot)$ is a correspondence from the nonempty, convex, and compact set $\Delta = \Delta_{c_1} \times \cdots \times \Delta_{c_C}$ to itself. In addition, by Lemma 2, $b(\cdot)$ is a nonempty, convex-valued, and upper hemi-continuous correspondence. Thus, all the conditions of the Kakutani fixed point theorem are satisfied. Hence, there exists a fixed point for this correspondence, a strategy profile $\delta \in \Delta$ such that $\delta \in b(\delta)$. The strategies at this fixed point constitute a Nash equilibrium because by construction $\delta_c \in b_c(\delta_{-c})$ for all $c = c_1, ..., c_C$. □

3 Application: Duopoly

We now restrict our attention to the duopoly case. This section attempts to show the main results of the previous sections. First of all, we need to make the allocation mechanism clear. Consider the case when there are only two colleges,

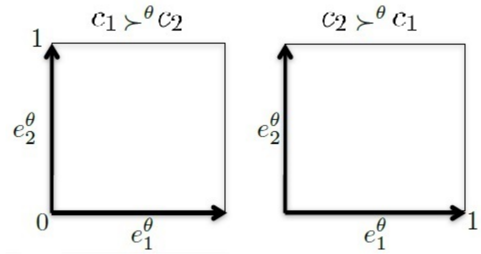

Figure 2: These squares represent the set of student types. The left square represents students that prefer college c_1, and the right square students who prefer college c_2. Scores at each college are represented by the (x, y) coordinates.

c_1 and c_2. The distribution of the students η is a truncated distribution to the interval $[0, 1]$. That is, we assume that there is a mass $m \in [0, 1]$ of students with preference list c_1, c_2, and a mass $(1 - m)$ of students with preference list c_2, c_1, where we assume that m is the mean of the truncated distribution.

Each mass of students has scores distributed uniformly over $[0, 1]^2$ as it is shown in Figure 2. Note the following: if both colleges had capacity 1/2, the unique stable matching would have each student matched to her favorite college. To make the model interesting, assume that the capacity of college c_1 is s_{c_1}, and the capacity of college c_2 is s_{c_2}, and $s_{c_1} + s_{c_2} < 1$. That is, colleges do not necessarily have the capacity to accept all students who prefer them.

As Azevedo and Leshno [6] have shown, there are two ways of finding stable matchings: the continuous version of the student-proposing deferred acceptance algorithm and the application of *Lemma 1.* As they show, we may simply look for cutoffs that equate supply and demand. This is illustrated in Figure 3, where it is easy to see that the market clearing equations are:

$$s_{c_1} = (1 - p_1)(1 + p_2)(m)$$

$$s_{c_2} = (1 - p_2)(1 + p_1)(1 - m)$$

Solving this system, we get $p_1 = p_1(m, s_{c_1}, s_{c_2})$ and $p_2 = p_2(m, s_{c_1}, s_{c_2})$, which are the market clearing cutoffs in terms of the capacities and the mean of the distribution η.

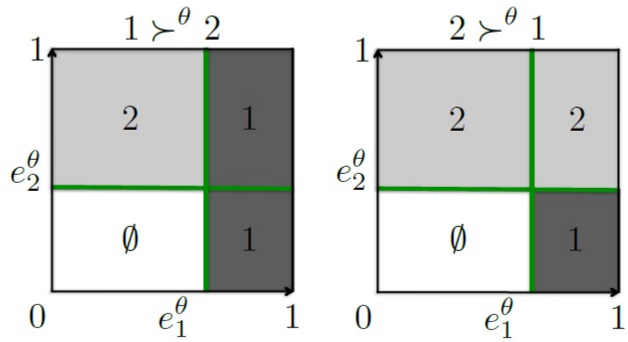

Figure 3: These squares are the same as in Figure 2, but here we can see the mass of students assigned to each college. The area with $\emptyset$ represents the mass of unmatched students.

As we said in section 2.3, this is the starting point in order to analyze the investment in quality, since the mechanism assigns students to colleges once the quality of the latter has been observed. Now, we need an expression to capture the effect of the qualities of both colleges on the students' demand. The following expression for m, the mean of the distribution, is proposed

$$m(\delta_{c_1}, \delta_{c_2}) = 1 - \frac{1}{1 + \frac{\delta_{c_1}}{\delta_{c_2}}}. \tag{7}$$

This way of expressing the mean has two advantages. First, we can take the following limits in equation (7):

$$\lim_{\left(\frac{\delta_{c_1}}{\delta_{c_2}}\right) \to 0} m = 0$$

$$\lim_{\left(\frac{\delta_{c_1}}{\delta_{c_2}}\right) \to \infty} m = 1$$

This tells us that for all possible values of the qualities, captured by the ratio between them, the mean is always in $[0, 1]$. Remember that we assumed that the mean determines the demand for each college, and that we truncated the distribution to $[0, 1]$ (so that, a mean outside this range would be a big problem!). Second, this expression is consistent with our assumptions because it implies that an increase in the quality of college c_1, increases the demand for it $(\partial m(\cdot)/\partial\delta_{c_1} > 0)$, and an increase in the quality of college c_2 decreases the demand for college c_1 $(\partial m(\cdot)/\partial\delta_{c_2} < 0)$, which is very intuitive.

Now, we have to be sure that our market clearing cutoffs are indeed positive. Solving the equations of p_1 and p_2 above, we obtain

$$p_1 = \frac{s_{c_1}(1-m) - s_{c_2}m}{4m(m-1)} - \frac{\sqrt{(s_{c_1}(m-1)+s_{c_2}m)^2 + 8m(m-1)[s_{c_1}(1-m)+m(2(m-1)+s_{c_2})]}}{4m(m-1)},$$

and

$$p_2 = \frac{s_{c_1}(m-1) + s_{c_2}m}{4m(m-1)} - \frac{\sqrt{(s_{c_1}(m-1)+s_{c_2}m)^2 + 8m(m-1)[s_{c_1}(1-m)+m(2(m-1)+s_{c_2})]}}{4m(m-1)}.$$

Suppose that $s_{c_1} = 1/4$ and $s_{c_2} = 1/2$. Replacing this values in p_1 and p_2, and computing them using *Mathematica* for m in $[0,1]$, we get the result shown in Figure 4, where we can see not only that our results make sense, but the range in which p_1 and p_2 are positive. Remember that we have, basically, two kinds of results. The first one is theoretical, in which we have shown that there exists a Nash equilibrium in the game where colleges choose quality simultaneously. It is well known that even in the basic Cournot model, the sufficient conditions for equilibrium to exist are quite restrictive (Roberts and Sonnenschein [15]), so we had to assume a quasi-concave function of the quality of the entering class in order to use the Kakutani's fixed point theorem to prove existence. The second result is related to comparative statics, where, by assuming a truncated distribution function, we show that the market clearing cutoffs of the colleges, are indeed positive in some interval. This can be extended to several cases when the sum of the capacities colleges is less than 1 (see Figure 7 in the Appendix). At this point, we can replace (7) into p_1 and p_2 to obtain $p_1(\delta_{c_1}, \delta_{c_2})$ and $p_2(\delta_{c_1}, \delta_{c_2})$.

This expressions are useful to analyze the incentives for colleges to invest in school quality. Since the cutoffs depend on the qualities, we can compute the

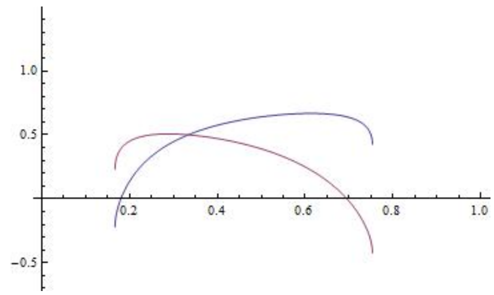

Figure 4: We can see that for a range between 0.2 and 0.7 of the mean, our clearing market cutoffs are positive. Here, the cutoffs are plotted on the y axis and the mean on the x axis.

sign of $\partial P_c/\partial\delta_c$ and $\partial P_c/\partial\delta_{c'}$, $\forall c, c' = \{c_1, c_2\}$, in order to find an interval for equation (2) to be positive, giving incentives to invest in quality.

Having defined above, we set up the maximization problem for colleges in the case of duopoly:

$$\max_{\delta_c} Q_c(\delta)$$

subject to

$$h(\delta_c) \leq R_c$$

for all $c = \{c_1, c_2\}$. At this point, we note that our assumption of the quasi-concavity and the strategic behavior of the colleges derive in two cases for the solution of this problem, both related to the constraint. Let $\tilde{\delta_c}$ be the value of the quality of college c for which the constraint is satisfied with equality. The cases are the following:

1. When the constraint is not active. This is the case when $\lambda = 0$ in the maximization problem of equation (3). A non active constraint implies that the function $Q_c(\cdot)$ reaches an interior maximum $\delta^* \in (0, \tilde{\delta_c})$ (see Figure 5). We assume that this is so because of the strategic behavior of the colleges, and suppose that the problem can be posed as follows

$$\max_{\delta_c} Q_c(\delta_c, \delta_{c'}(\delta_c)),$$

which is a standard problem of Industrial Organization (for example, the Stack-

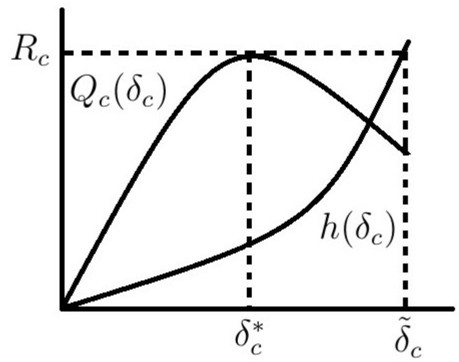

Figure 5: When the constraint is not active, we have an interior maximum of $Q_c(\cdot)$.

elberg model). We can obtain the following first order condition:

$$\frac{dQ_c}{d\delta_c} = \frac{\partial Q_c}{\partial \delta_c} + \frac{\partial Q_c}{\partial \delta_{c'}} \cdot \frac{d\delta_{c'}}{d\delta_c} = 0,$$

from which we get

$$\frac{d\delta_{c'}}{d\delta_c} = -\frac{\frac{\partial Q_c}{\partial \delta_c}}{\frac{\partial Q_c}{\partial \delta_{c'}}}, \qquad \text{for all } c, c' \in \{c_1, c_2\}. \tag{8}$$

It is clear that (8) is a reaction function. Assuming that $\partial Q_c / \partial \delta_c \geq 0$ and $\partial Q_c / \partial \delta_{c'} \leq 0$ for all $c, c' \in \{c_1, c_2\}$, we can say that colleges are strategic complements.

2. When the constraint is active. This is the case when we do not have to take into account the strategic behavior of colleges, that is, when $\lambda > 0$. Here we are in the case of equation (3), from which we obtain the same first order conditions:

$$\frac{\partial L}{\partial \delta_c} = \frac{dQ_c(\delta_c)}{d\delta_c} - \lambda \frac{dh(\delta_c)}{d\delta_c} = 0,$$

and

$$\frac{\partial L}{\partial \lambda} = h(\delta_c) - R_c = 0.$$

Given that $Q_c(\cdot)$ is a continuous and quasi-concave function of δ_c, by the Bolzano-Weierstrass theorem, we can say that, in this case, $Q_c(\cdot)$ reaches its maximum in δ_c^*, which is the supreme of the compact interval $[0, \delta_c^*]$ (see Figure 6).

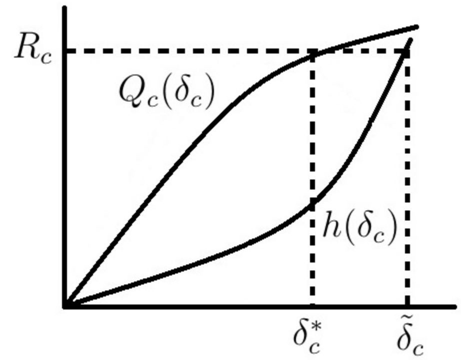

Figure 6: When the constraint is active, $Q_c(\cdot)$ has a maximum in the interior point δ_c^*.

3.1 Comparative Statics

Finally, we want to derive comparative statics in the constrained maximization problem. Once we know how the mechanism works, we just have to note that, in the second case discussed before (when the constraint is active), δ_c^* is given by the second F.O.C., so we can assume a specific functional form of $h(\cdot)$ to obtain it. Then, we replace δ_c^* and $\delta_{c'}^*$ in equation (7) to obtain the value of the mean of our distribution and, finally, by replacing them in equation (1) the quality of the entering class to college c is obtained. It should be clear that we only have to know the score of the students assigned to each college (see Figure 3), which depends on the distribution of e^θ.

First of all, we get the market clearing cutoffs P_c^* and $P_{c'}^*$ in terms of R_c and $R_{c'}$ as follows

$$P_c^* = \frac{2R_c + \sqrt{R_c}\sqrt{R_{c'}} - R_{c'} - \sqrt{4R_c^2 - 60R_c^{3/2}\sqrt{R_{c'}} + 157R_cR_{c'} - 34\sqrt{R_c}R_{c'}^{3/2} + R_{c'}^2}}{16\sqrt{R_c}\sqrt{R_{c'}}}$$

and

$$\begin{aligned} P_{c'}^* = & \frac{1}{16R_cR_{c'}}\left(-2R_c^{3/2}\sqrt{R_{c'}} - R_cR_{c'} + \sqrt{R_c}R_{c'}^{3/2}\right) \\ & - \frac{1}{16R_cR_{c'}}\left(\sqrt{R_c}\sqrt{R_{c'}}\sqrt{4R_c^2 - 60R_c^{3/2}\sqrt{R_{c'}} + 157R_cR_{c'} - 34\sqrt{R_c}R_{c'}^{3/2} + R_{c'}^2}\right) \end{aligned}$$

Now, having obtained $P_c^*(R_c, R_{c'})$ and $P_{c'}^*(R_c, R_{c'})$ we need to calculate the integral of equation (1). We use *Mathematica* to compute the integral in order to obtain $Q_c(R_c, R_{c'})$ and $Q_{c'}(R_c, R_{c'})$ This calculation is a little difficult, and the expression is very large (we omit the result here). Assuming a nice distribution of the student types, one can see the following

$$\frac{\partial Q_c(R_c, R_{c'})}{\partial R_c} \geq 0,$$

which means that the quality of the entering class to college c is increasing in the budget assigned to college c by the State. It is easy to see that, by equation (7),

$$\frac{\partial Q_c(R_c, R_{c'})}{\partial R_{c'}} \leq 0,$$

which tells us that the quality of the entering class to college c is decreasing in the budget assigned to college c'. This is so because c' is attracting the best students. Considering polar cases, for example $R_c > 0$ and $R_{c'} = 0$, it should be noted that the quality of students depends on the capacities of colleges, so that, assuming $s_c = s_{c'}$ we will find that $Q_c \geq Q_{c'}$, for all $c, c' = \{c_1, c_2\}$. This means that, when colleges have the same capacities, the quality of the students is higher for the one with the largest budget.

4 Conclusion

This paper proposes an extension of the Azevedo and Leshno's work [6] in the choice of the quality offered by colleges in a model of matching markets with a continuum of students on one side, and a discrete set of colleges on the other. This model admits complex heterogeneous preferences, so that we assume that colleges are not necessarily perfect substitutes even if they have the same quality.

We show that there exists a Nash equilibrium in the constrained maximization problem, i.e. when colleges choose quality simultaneously subject to a budget constraint. Moreover, we analyze the case of the duopoly in order to make both the mechanism and our results clear. The distribution for the students used in the model was a truncated one to the interval $[0, 1]$, and a specific functional form (in terms of the qualities of both colleges) of the mean of it was proposed. Additionally, we obtain the cutoffs in terms of the qualities by solving the supply and demand equations of Azevedo and Leshno [6], and we found that there is a range in the values of them for which there are incentives for colleges to invest in quality.

We consider two cases of the maximum quality. First, when the constraint is not active, where we obtained conditions under which the colleges are strategic complements (as in a standard problem of Industrial Organization). Second, when the constraint is active, where we solved the standard problem by the Lagrange method; this case let us finding some comparative statics. Once we obtained the maximum quality of colleges in this case in terms of the budget assigned by the State to each one, we calculated the quality of the entering class and considered different cases for the distribution of that budget.

A contrastation can be made between the paradigm of assortative matching and our results, more related to the distribution of education by using two decision variables of the State, the capacities of colleges, and the budget assigned to each one to invest in school quality. It was shown that the amount assigned to each college has a positive impact on the quality of the students, so that, the greater the amount assigned to one college, the higher the quality of its entering class. It is only an analysis tool to address the issue of education and the incentives to invest in school quality for colleges.

5 Appendix

Proof. (***Proposition 1. Azevedo and Leshno, 2013***) We need to define some additional relevant quantities in order to prove this proposition:

- The set of students who are marginally accepted to college c and would go to college c' otherwise:

$$\tilde{M}_{cc'} = \{\theta : c \succ^{\theta} c', P_c = e_c^{\theta}, P_{c'} \leq e_{c'}^{\theta}, P_{c''} > e_{c''}^{\theta} \forall c'' \neq c : c'' \succ^{\theta} c'\}.$$

- The number of students in this margin:

$$M_{cc'} = \int_{\tilde{M}_{cc'}} f_{\delta}(\theta) d\theta.$$

- The set of students who are marginally accepted to college c and would be matched to herself otherwise:

$$\tilde{M}_{c\emptyset} = \{\theta : c \succ^{\theta} \emptyset, P_c = e_c^{\theta}, P_{c'} > e_{c'}^{\theta} \forall c' \neq c : c' \succ^{\theta} c\}.$$

- The number of students in this margin:

$$M_{c\emptyset} = \int_{\tilde{M}_{c\emptyset}} f_{\delta}(\theta) d\theta,$$

  it is intuitive that, since these students are "marginally accepted", their average score is precisely the cutoff of college c.

Aggregate quality is defined as

$$\begin{aligned} Q_c(\delta) &= \int_{\mu_{\delta}(c)} {e_c}^{\theta} d\eta_{\delta}(\theta) \\ &= \int_{\{\theta : D^{\theta}(P^*(\delta)) = c\}} {e_c}^{\theta} \cdot f_{\delta}(\theta) d\theta \end{aligned}$$

By Leibniz's rule, Q_c is differentiable in δ_c, and the derivative is given by

$$\begin{aligned} \frac{dQ_c(\delta)}{d\delta_c} &= \int_{\{\theta : D^{\theta}(P^*(\delta)) = c\}} {e_c}^{\theta} \cdot \frac{d}{d\delta_c} f_{\delta}(\theta) d\theta \\ &+ \sum_{c' \neq c} \frac{dP_{c'}^*}{d\delta_c} \cdot M_{c'c} \cdot \bar{P}_{c'c} \\ &- \frac{dP_c^*}{d\delta_c} \cdot [M_{c\emptyset} + \sum_{c' \neq c} M_{cc'}] \cdot P_c^* \end{aligned}$$

The first term is the integral of the derivative of the integrand, and the last two terms the change in the integral due to the integration region $\{\theta : D^\theta(P^*(\delta)) = c\}$ changing with δ_c.

The terms in the second line are the changes due to changes in the cutoffs $P^*_{c'}$, the students that college c gains (or loses) because college c' becomes more (less) selective. The quantity of these students is $\frac{dP^*_{c'}}{d\delta_c} \cdot M_{c'c}$, and their average quality $\bar{P}_{c'c}$. The last line is the term representing the students lost due to college c raising its cutoff P_c. These students number $[M_{c\emptyset} + \sum_{c' \neq c} M_{cc'}]$, and have average quality P^*_c. Note that, since the total number of students admitted at college c is constant and equal to s_c, we have

$$\begin{aligned} 0 = &\int_{\{\theta : D^\theta(P^*(\delta)) = c\}} \frac{d}{d\delta_c} f_\delta(\theta) d\theta \\ &+ \sum_{c' \neq c} \frac{dP^*_{c'}}{d\delta_c} \cdot M_{c'c} \\ &- \frac{dP^*_c}{d\delta_c} \cdot [M_{c\emptyset} + \sum_{c' \neq c} M_{cc'}] \end{aligned}$$

Therefore, if we substitute $\frac{dP^*_c}{d\delta_c} \cdot [M_{c\emptyset} + \sum_{c' \neq c} M_{cc'}]$ in the above equation we have

$$\begin{aligned} \frac{dQ_c(\delta)}{d\delta_c} = &\int_{\{\theta : D^\theta(P^*(\delta)) = c\}} [e^\theta_c - P^*_c] \cdot \frac{d}{d\delta_c} f_\delta(\theta) d\theta \\ &+ \sum_{c' \neq c} \frac{dP^*_{c'}}{d\delta_c} \cdot M_{c'c} \cdot [\bar{P}_{c'c} - P^*_c] \end{aligned}$$

The term in the second line is the market power effect as defined. That the term in the first line equals the expression in *Proposition 1* follows from the definition of N_c and $\bar{e}_c$. To see that the direct effect is positive, note that by definition $\bar{e}_c \geq P^*(\delta)$, and since $u^i_c(\delta)$ is increasing in δ_c we have $N_c \geq 0$. □

In Figure 7, we can see the extension of the market clearing cutoffs to several cases when the sum of the capacities colleges is less than 1:

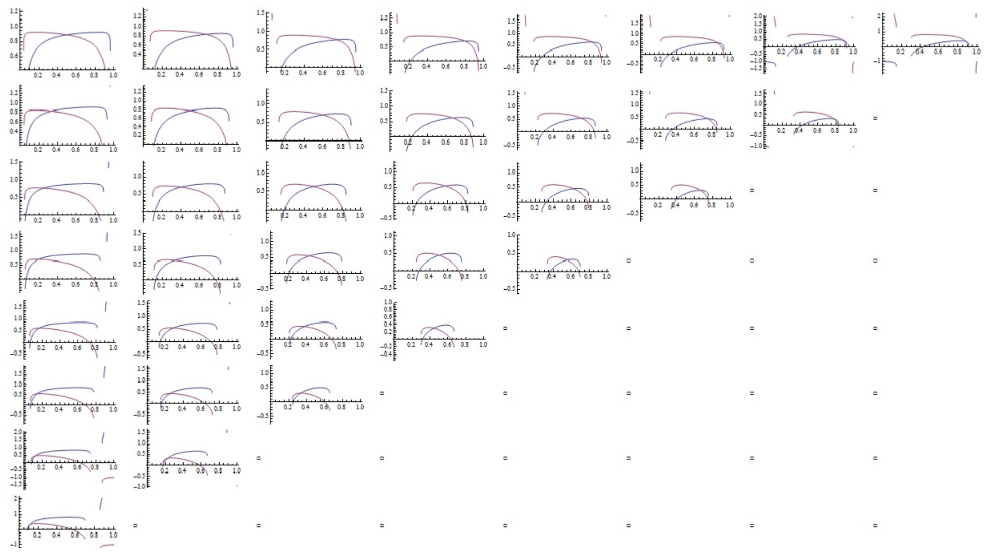

Figure 7: The cutoffs are positive in a an interval of the mean for different values of capacities of the colleges. Here the cutoffs are plotted on the y axis and the mean on the x axis, and we consider only the cases in which $s_{c_1} + s_{c_2} < 1$ for several combinations of values of them.

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