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# A DYNAMIC SCHOOL CHOICE MODEL

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# Abstract

We present a dynamic school choice problem, which consists in assigning positions to overlapping generation of teachers, taking into account a unique priority order. From one period to another, agents are allowed either to retain their current position, either to choose a preferred one (when available). We introduce a fairness concept that takes into account this individually rational condition. We show that it always exists a fair assignment, which is unique and can be reached by a modified version of the Deferred Acceptance Algorithm by Gale and Shapley. Moreover, we show that the mechanism is dynamic strategy-proof and respects improvements.

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# 1 Introduction

This work deals with a dynamic school choice model inspired by a recent problem, which consists in assigning positions to overlapping generations of teachers. In May 2008 the Mexican Federal Government, through the Ministry of Public Education (SEP), signed an agreement with the National Education Workers Union (SNTE), called the Alliance for the Quality of Education (ACE)[1]. Included in the agreement, was the National Contest for the Allocation of Teacher Positions, an assignment mechanism for teachers to teaching positions. Through the settlement, teachers who sought a position in the public education system are required to sit an exam which by ranking allocates each teacher a 'priority order' to which they are assigned a position within the school. However, any teacher that had been previously assigned a position would have the ability to choose between their current position and any open positions. Under the test score assignment mechanism used by the central authority, positions are offered to teachers based on their priority order. Therefore, the assignment mechanism for teachers to teaching positions raises a selection bias problem, which has some special features not modeled yet in the literature.

Due to some special features of the process, our model cannot be directly applied, but it casts some light on the resource allocation problem faced by the SEP and SNTE. In the simplified framework that we present, it is shown that the mechanism actually used has some important flaws. In particular, as it is presented in the next Sections, the mechanism is neither efficient nor dynamic strategy-proof (it can be manipulated by teachers) and does not respect improvements done by teachers (a teacher may increase her 'priority order' and however be punished with a worth position)[2].

The principal subject of this work is to present a dynamic school choice problem, which consists in assigning positions to overlapping generation of teachers (agents). In each period, the central authority must assign positions to teachers taking into account a unique priority order (the same order for each position) and the previous assignment; hence, the central authority faces a dynamic allocation problem.

We introduce a new framework to model this dynamic problem, and in such context a new concept of fairness is introduced. Since we cannot assign a teacher to a worst position than the one she already has, a fair assignment must verify the individually rational condition. Next, we consider the claims that could exist in an assignment. A teacher has a claim over a position if it exists a position preferred by a teacher to her assignment upon which she has priority over the teacher assigned to that preferred position. But in our model two kinds of claims can happen. If teacher that holds the preferred position was not assigned to that position in the previous period, we say that it is a justified claim. On the contrary, if the teacher was assigned in the previous period to that position, we have an inappropriate claim. Note that the last type of claim takes into account the individually rational restriction. Finally we define a fair assignment as the one

[1] More information can be obtained in http://www.concursonacionalalianza.org

[2] More details about this issue can be found in Cantala [5].

which is individually rational, does not have justified claims and if it has inappropriate claims, they cannot be solved by another assignment that verifies the first two properties.

In the context of our model we show that it always exists a unique solution to our problem. The uniqueness is an interesting property of our model and, despite some papers (see, for example, Eeckhout [6] and Legros and Newman [10]), it is not a very frequent property in the literature[3]. In order to find the unique solution, a modified version of the Deferred Acceptance Algorithm by Gale-Shapley is introduced. Finally, the important properties of strategy-proofness and respecting improvements (both in a dynamic version) are established.

The literature on matching is devoted in its majority to static assignment problems (see for example Roth and Sotomayor [13] and Sönmez and Ünver [14]). Related to this literature, our model joints two of the main frameworks in matching theory. On the one hand, our concept of fairness relies heavily on the concept of eliminating justified envy, as it is defined by Abdulkadiroglu and Sönmez [2], and in this sense is related to the school choice problem literature. On the other hand, we take from house allocation theory with existing tenants the concept of individual rationality defined by Abdulkadiroglu and Sönmez [1].

Recently, some articles have presented assignment problems in dynamic context. In Ünver [15] a dynamic mechanism is presented with an application to kidney exchange for patients. More related to the model presented here are the articles of Bloch and Cantala [4] and Kurino [9]. Although both papers study a similar problem to the one presented in this work, their modeling is quite different.

The rest of the work is organized as follows. An illustrative example is presented in the next section. In Section 3 we introduce the ingredients of our model and the concept of fairness. Section 4 is devoted to the existence of a solution to our problem and its uniqueness. Section 5 considers the efficiency of the solution and in Section 6 the proposed mechanism is introduced. In Sections 7 and 8 the dynamic strategy-proofness and respecting improvements properties are considered. In Section 9, we present the conclusions and directions for future research of the work. Finally, in the Appendix, we present the omitted proofs.

## 2 An illustrative example: a three period economy

With this example we introduce the main concepts of our model. The example, despite being informal, attempts to motivate the formal definitions of the following Section. Suppose four positions to assign and denote the set of such positions as: $P = \{p_1, p_2, p_3, p_4\}$. Consider three periods: $t = 1, 2, 3$. At $t = 1$ we have the initial assignment of the problem, described as a function $\nu_1$ from the set of teachers that had been previously assigned to hold a position, denoted by $I_E^1$ ([4]), to the set of positions $P$. In this example we suppose that teachers (agents) $a_1$ and $a_2$ are initially assigned to positions $p_1$ and $p_2$, respectively. Then

[3] The difference between those papers and this work is that the uniqueness in our model is not derived by restricting the agents preference.

[4] The subscript $E$ is motivated by the fact that teachers in this group play the role of what is known in the literature as existing tenants.

the initial assignment is the function $\nu_1 : I_E^1 \to P$ such that $\nu_1(a_1) = p_1$ and $\nu_1(a_2) = p_2$. We also have the set of teachers in period 1: $I^1 = \{a_1, a_2, a_3, a_4, a_5\}$, that also includes new teachers that do not hold position and are competing to hold one. Another element is the preference profile $\succ\ = (\succ_i)_{a_i \in I^1}$ consisting of each teacher preferences defined over the set $P \cup \{p_0\}$, where $p_0$ denotes the null position that means not be assigned to any position. In this example we assume the following preferences:

$$\begin{bmatrix} \succ_1 & \succ_2 & \succ_3 & \succ_4 & \succ_5 \\ p_2 & p_1 & p_3 & p_4 & p_2 \\ p_1 & p_3 & p_2 & p_3 & p_1 \\ p_3 & p_2 & p_1 & p_0 & p_0 \\ p_4 & p_0 & p_0 & p_1 & p_3 \\ p_0 & p_4 & p_4 & p_4 & p_4 \end{bmatrix}$$

Finally we have a strict priority order of all teachers: $>^1 = (a_3, a_1, a_2, a_4, a_5)$. Note that this order is the same for all positions.

The elements $(P, I^1, \nu_1, \succ, >^1)$ define the market at $t = 1$, denoted as $M^1$. With these ingredients an outcome of this problem is an assignment of teachers to positions such that every agent is assigned to one position, and only the null position $p_0$ (that will be used to assign no position to agent) can be assigned to more than one agent. We will refer to that assignment as a matching, that is, a function $\mu_1 : I^1 \to P \cup \{p_0\}$.

To define the market at the next period, a transition rule must be specified. The element that links one period with the following is the matching of the period, because the assignment of one period defines the initial assignment of the next period. Then to complete the description of this economy one must specified a mechanism, that is, a systematic procedure that assigns a matching for each market. Assume that the used mechanism in this economy is the one used by SEP, mentioned in the introduction, which we will call the **test score assignment mechanism**. Hence, teacher $a_3$ who is first in the priority order can choose between the vacancies: $p_3$ and $p_4$. According to her preference she chooses $p_3$. The following teacher is $a_1$ that can choose between her position $p_1$ and the vacancy $p_4$; then she does not change and remains assigned to $p_1$. Continuing with the process we have: (the position under each teacher is the assignment of that teacher):

$$\mu_1 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ p_1 & p_2 & p_3 & p_4 & p_0 \end{pmatrix}$$

Note that the mechanism does not eliminate the justified envy: teacher $a_1$ prefers position $p_2$ to the one that she had been assigned, and is before teacher $a_2$ in the priority order $>^1$. In this sense, as we claim in the introduction, the mechanism is not fair. Observe also that matching $\mu_1$ is not efficient in the sense that is dominated by the following matching (teachers $a_3, a_4, a_5$ are weakly better off and teachers $a_1, a_2$ are strictly better off):

$$\hat{\mu}_1 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ p_2 & p_1 & p_3 & p_4 & p_0 \end{pmatrix}$$

Hence, as we affirm in the introduction, the test score assignment mechanism is not efficient.

In period $t = 2$ we have the set $I^2$ of teachers that are active (eligible to hold to a position) in this period. We also have the set $I_E^2$ defined as $I_E^2 \equiv \mu_1^{-1}(P) \cap I^2$, that is, the set of teachers that were assigned a position at $t = 1$ and do not have to retire in $t = 2$. Then the initial assignment of period 2 is $\nu_2$, defined as the restriction of $\mu_1$ to the set $I_E^2$. As in the previous period, the other elements of period 2 are $\succ, >^2$ . For this example we suppose $I^2 = \{a_2, a_3, a_4, a_5, a_6\}$ and $>^2 = (a_6, a_3, a_2, a_4, a_5)$ . Then we have: $I_E^2 = \{a_2, a_3, a_4\}$ , $\nu_2(a_2) = p_2, \nu_2(a_3) = p_3$ and $\nu_2(a_4) = p_4$.

Concerning to teachers preferences, we assume that they are revealed by each teacher in the period in which she enters the market and they cannot be changed in the next periods. Then we only have to define the preferences of teacher $a_6$, suppose $\succ_6 = (p_4, p_2, p_3, p_1, p_0)$ . The market at $t = 2$ is $(P, I^2, \mu_1, \nu_2, \succ, >^2)$ and the outcome of the assumed mechanism is:

$$\mu_2 = \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & a_6 \\ p_2 & p_3 & p_4 & p_0 & p_1 \end{pmatrix}$$

Finally, for the last period, we assume $I^3 = \{a_4, a_5, a_6, a_7, a_8\}$ and $>^3 = (a_7, a_6, a_4, a_8, a_5)$ . Therefore $I_E^2 = \{a_4, a_6\}$ , $\nu_3(a_4) = p_4$ and $\nu_3(a_6) = p_1$. Concerning to preferences we have to ad: $\succ_7 = (p_1, p_4, p_2, p_3, p_0)$ and $\succ_8 = (p_2, p_1, p_3, p_0, p_4)$ . Then, the last matching for this economy is:

$$\mu_3 = \begin{pmatrix} a_4 & a_5 & a_6 & a_7 & a_8 \\ p_4 & p_0 & p_3 & p_2 & p_1 \end{pmatrix}$$

Observe that if teacher $a_7$ wouldn't have worked so hard in the exam so that the order would have been $\tilde{>}^3 = (a_6, a_7, a_4, a_8, a_5)$ , the final matching would have been:

$$\tilde{\mu}_3 = \begin{pmatrix} a_4 & a_5 & a_6 & a_7 & a_8 \\ p_4 & p_0 & p_2 & p_1 & p_3 \end{pmatrix}$$

But in this case, teacher $a_7$ obtains a better position. Then, as we claim in the introduction, the mechanism does not respect improvements done by teachers: a teacher may increase her 'priority order' and however is punished with a worth position.

From this example it is clear the elements that define our model. To specify an economy we must define: the set of positions $P$, the initial assignment $\nu_1$, the sequence of sets of teachers that are active in each period $\{I^t\}_t$ , the preference profile $\succ \ = (\succ_i)_{a_i \in I^t}$, the strict priority order of all teacher $\{>^t\}_t$ and finally the mechanism or the systematic procedure that assigns a matching for each market, denoted by $\varphi$. All these ingredients define the state of the market in each period: $M^1 = \left\langle P, I^1, \nu_1, >^1, \succ \right\rangle$ and for $t \geq 2$, $M^t = \left\langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \right\rangle$ with $I_E^t = \mu_{t-1}^{-1}(P) \cap I^t$ and $\nu_t = \mu_{t-1} \diagup I_E^t = \varphi(P, I^{t-1}, \nu_{t-1}, >^{t-1}, \succ) \diagup I_E^t$, where $\mu_{t-1} \diagup I_E^t$ means the restriction of function $\mu_{t-1}$ to the set $I_E^t$. In the next Section we present the formal definitions.

# 3 The Model

## 3.1 Premises

We consider the allocation of positions to overlapping generations of teachers. Time is discrete, starts at $t = 1$ and lasts forever. Denote by $P = \{p_1, p_2, ..., p_M\}$ the set of positions. We assume that in all periods we have the same set of positions although our results extend to the case where $P$ varies over time. The null position is $p_0$ and will be used to assign no position to agent.

We assume that teachers (agents) live $T$ periods, $T \geq 2$. Denote by $I^t$ the set of all "active" teachers, that is, teachers eligible to hold a position in period $t$. Note that we do not suppose any relation between $|P|$ and $|I^t|$.

Each agent $a_i \in I^t$ has a preference relation over $P \cup \{p_0\}$, denoted by $\succsim_i$ and let $\succ_i$ be the induced strict preference relation over $P \cup \{p_0\}$ [5]. As usual, if the null position is preferred to another position $p_i$ for some teacher $a_j$ ($p_0 \succ_j p_i$), then $p_i$ is not acceptable to $a_j$. We work under the assumption that a teacher reveals her preference in the period in which she enters the market. In the following periods the teacher cannot modify the preferences that announced before. We will discuss the consequences of relaxing this assumption. Let $\Lambda_i$ be the set of strict preference relations of agent $a_i$. A preference profile is an element of the Cartesian product of the set of preferences of all agents: $\Lambda = \prod\limits_{a_i \in I^t} \Lambda_i$ , we will denote by $\succ \ = (\succ_i)_{a_i \in I^t}$ a preference profile.[6]

Another ingredient of the model is a strict priority order of all teachers: $>^t$, when teacher $a_i$ has priority over $a_j$ for any position at $t$, we write $a_i >^t a_j$. We will suppose that the relative order of teachers does not change over time, that is, if $a_i >^t a_j$ in some $t$ then $a_i >^\tau a_j$ $\forall \tau$ such that $a_i, a_j \in I^\tau$.

[5] By a preference relation we mean a bynary relation over $P \cup \{p_0\}$ that is reflexive, transitive, complete and antisymmetric. The induced strict preference relation is defined as: $p_i \succ p_j$ iff $p_i \succsim p_j$ and $\neg(p_j \succsim p_i)$, and is irreflexive, transitive and connected ($\forall p_i, p_j \in P \cup \{p_0\}$ , if $p_i \neq p_j$ then $p_i \succ p_j$ or $p_j \succ p_i$).

[6] Although the formal notation would be $\succ_t$, to simplify the notation we will not use the subindex $t$. Then with $\succ$ we will refer to teacher preferences in the period under study.

## 3.2 Assignments

A **matching** is an assignment of teachers to positions such that every agent is assigned one position, and only the null position $p_0$ can be assigned to more than one agent., i.e., a function $\mu_t : I^t \to P \cup \{p_0\}$. To refer that agent $a_i$ is assigned to position $p_j$ in period $t$, we write $\mu_t(a_i) = p_j$. Let $\mathcal{M}_t$ be the set of all matchings in period $t$. A **submatching** is a matching with restricted domain, i.e., a function $\nu_t : J \subset I^t \to P \cup \{p_0\}$. Let $\mathcal{S}_t$ be the set of all submatchings in period $t$.

At the initial period of the model we have a set of teachers (denoted by $I_E^1$) that had been previously assigned to hold a position, and the set of positions that they hold. We will refer to that subset of teachers and positions as the initial assignment of period 1, and we denote this initial assignment as a function $\nu_1 : I_E^1 \to P$. For any period $t \geq 2$ the initial assignment, denoted by $\nu_t$, is defined by the matching of the previous period; that is, given the assignment of the previous period $\mu_{t-1}$ and the sets $P$, $I^t$, we have that $\nu_t \equiv \mu_{t-1} \diagup I_E^t$ with $I_E^t \equiv \mu_{t-1}^{-1}(P) \cap I^t$. One can think the initial assignment of each period as a submatching of teachers in set $I_E^t$ to her positions. Note that set $I^t \setminus I_E^t$, is formed by teachers that do not hold position and are competing to hold one.

Given a matching $\mu_{t-1}$, the sets $P, I^t$, the strict order $>^t$ and the preference profile $\succ$, **an overlapping teacher placement problem** is represented by the market $M^t = \left\langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \right\rangle$, with $\nu_t \equiv \mu_{t-1} \diagup I_E^t$ and $I_E^t \equiv \mu_{t-1}^{-1}(P) \cap I^t$ if $t \geq 2$. When $t = 1$ we have $\mu_0 \equiv \nu_1$. An outcome of an overlapping teacher placement problem is a matching.

We define a **mechanism** as a systematic procedure that assigns a matching for each problem, that is, a function $\varphi$ such that $\varphi\left(\left\langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \right\rangle\right) \in \mathcal{M}_t$, for any problem $\left\langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \right\rangle$.

An **economy** is defined by a set of positions $P$, an initial assignment $\nu_1$, a sequence of sets $\{I^t\}_t$, $\succ \; = (\succ_i)_{a_i \in I^t}$, a strict priority order of all teachers $\{>^t\}_t$ for each period and finally, the mechanism or the systematic procedure that assigns a matching for each problem, denoted by $\varphi$. All these ingredients define the state of the market in each $t$ : $M^t = \left\langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \right\rangle$ since $\nu_t \equiv \mu_{t-1} \diagup I_E^t = \varphi(P, I^{t-1}, \nu_{t-1}, >^{t-1}, \succ) \diagup I_E^t$, $I_E^t \equiv \mu_{t-1}^{-1}(P) \cap I^t$ if $t \geq 2$ and $\mu_0 \equiv \nu_1$. Our main concern consist of finding a mechanism that, given the rest of the elements, defines an economy which verifies the properties that are enunciated in the following Sections.

## 3.3 Fairness

The remainder of the Section is devoted to the definition of our concept of fairness. In our problem we must combine two classic concepts of the literature. On the one hand, since we have existing tenants in our model, we cannot assign a teacher to a position worst that the one she already has. Then a fair matching must verify the individually rational condition, as it is defined in Abdulkadiroglu and Sönmez [1]. On the other hand, we must respect the strict priority order of all teachers: $>^t$. Hence, a fair matching must eliminate the justified envy, as it is defined by Abdulkadiroglu and Sönmez [2].

For the following definitions we suppose as given a matching $\mu_{t-1}$, the sets $P, I^t$, the preference profile $\succ$ and the strict order $>^t$ . All these elements specify a market $M^t = \langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \rangle$. Our aim is to define a fair matching $\mu_t$ for the market $M^t$. In the first place, we define the classic concept of individual rationality. Then, we consider the claims that could exist in a matching. We have a claim of a teacher over a position if it exist a position preferred by a teacher to her assignment upon which she has priority over the teacher assigned to that preferred position. But in our model two kinds of claims can happen. If teacher that holds the preferred position was not assigned to that position in the previous period, we say that it is a justified claim. On the contrary, if teacher was assigned in the previous period to that position, we have an inappropriate claim. The formal definitions are the following.

**Definition 1** *A matching* $\mu_t$ *is* individually rational *if:*

*i)* $\mu_t(a_i) \succsim_i p_0 \; \forall a_i \in I^t$ ,

*ii)* $\mu_t(a_i) \succsim_i \nu_t(a_i) \; \forall a_i \in I^t_E$.

**Definition 2** *Given a matching* $\mu_t$ *, teacher* $a_i$ *has a* ***justified claim*** *over* $p_j$ *if:*

*i)* $a_i$ prefers $p_j$ to her assignment $\mu_t(a_i)$: $p_j \succ_i \mu_t(a_i)$,

*ii)* $a_i$ has priority over teacher assigned to $p_j$ : $a_i >^t \mu_t^{-1}(p_j)$, or $p_j$ is not assigned to any teacher: $p_j \notin \mu_t(I^t)$,

*iii)* if $p_j \in \mu_t(I^t)$, teacher $\mu_t^{-1}(p_j)$ was not assign to $p_j$ in the last period: $\nu_t^{-1}(p_j) \neq \mu_t^{-1}(p_j)$ or $p_j \notin \nu_t(I^t_E)$.

We will say that a matching eliminates the *justified claims* if it does not exist a justified claim in that matching.

**Definition 3** *Given a matching* $\mu_t$ *, teacher* $a_i$ *has an* ***inappropriate claim*** *over* $p_j$ $(p_j \in \nu_t(I^t_E) \cap \mu_t(I^t))$ *if:*

*i)* $a_i$ prefers $p_j$ to her assignment $\mu_t(a_i)$: $p_j \succ_i \mu_t(a_i)$,

*ii)* $a_i$ has priority over teacher assigned to $p_j$ : $a_i >^t \mu_t^{-1}(p_j)$,

*iii)* teacher $\mu_t^{-1}(p_j)$ was assign to $p_j$ in the last period: $\nu_t^{-1}(p_j) = \mu_t^{-1}(p_j)$.

We will say that a matching eliminates the *inappropriate claims* if it does not exist an inappropriate claim in that matching.

**Definition 4** *Let* $\Gamma(\mu_t)$ *the* set of all inappropriate claims *in matching* $\mu_t$, *that is:*

$\Gamma(\mu_t) = \{(a_i, p_j) \in I^t \times P$ such that $a_i$ has an inappropriate claim over $p_j$ in $\mu_t\}$ .

**Definition 5** *Given a matching $\mu_t$, we say that $\mu_t$ is* acceptable *if it:*

*i)* is individually rational,

*ii)* eliminates the justified claims

Let denote by $\mathcal{C}_t \subset \mathcal{M}_t$ the set of all acceptable matchings.

We have the elements to define our concept of fairness, but before, we motivate it with the following example.

**Example 1** *Consider the market $M^1 = \left\langle P, I^1, \nu_1, \succ, >^1 \right\rangle$ and matchings $\mu_1$, $\hat{\mu}_1$ defined in the example of Section 2. Although both matchings are acceptable since they are individually rational and eliminate the justified claims, they present an important difference. In $\mu_1$ we have that teacher $a_1$ has an inappropriate claim over $p_2$, and then $\Gamma(\mu_1) = \{(a_1, p_2)\}$. But in the case of $\hat{\mu}_1$ we have that $\Gamma(\hat{\mu}_1) = \varnothing$. Then, in this example, it is clear that we cannot define matching $\mu_1$ as fair, because as we saw, it exists another assignment that solves the inappropriate claim and does not create a new one. Our concept of fairness captures this idea: we will say that a matching is fair if it is acceptable and, in the case that it has inappropriate claims, it does not exist another acceptable matching that solves at least one of these claims without creating a new one.*

**Definition 6** *We say that matching $\mu_t$ is* fair *if:*

*i)* it is acceptable

*ii)* in the case that $\Gamma(\mu_t) \neq \phi$ then $\nexists\ \mu_t' \in \mathcal{C}_t$ such that $\Gamma(\mu_t') \subsetneq \Gamma(\mu_t)$.

Note that if it exists an individually rational matching without any kind of claim, by the previous definition, it is fair. Note also that the concept of fairness adopted here, does not imply a utilitarian perspective. We can have two fair matchings $\mu_t, \mu_t'$ even if $|\Gamma(\mu_t)| < |\Gamma(\mu_t')|$.

**Definition 7** *A matching $\mu_t$ is* Pareto efficient or Pareto superior (or simply efficient) *if there is no other matching $\mu_t'$ that makes all agents weakly better off and at least one agent strictly better off.*

In the following example we present a problem in which it does not exist a fair matching with $\Gamma(\mu_t) = \phi$.

**Example 2** *Consider the following problem:*

$I_E^t = \{a_2, a_3, a_4\} \subset I^t = \{a_1, a_2, a_3, a_4\}$
$P = \{p_1, p_2, p_3, p_4\}$, $\nu_t = \{(a_2, p_2), (a_3, p_3), (a_4, p_4)\}$
$>^t = (a_1, a_2, a_3, a_4)$ and the following teacher preferences:

$$\begin{bmatrix} \succ_1 & \succ_2 & \succ_3 & \succ_4 \\ p_3 & p_4 & p_3 & p_2 \\ p_1 & p_2 & p_1 & p_4 \end{bmatrix}$$

In this problem it is easy to prove that it does not exist an individually rational matching that eliminate both justified and inappropriate claims. These two matchings are acceptable since both are individually rational and eliminate the justified claims:

$$\mu_t = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ p_1 & p_2 & p_3 & p_4 \end{pmatrix} \text{ and } \mu_t' = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ p_1 & p_4 & p_3 & p_2 \end{pmatrix},$$

but only the second matching is fair because $\Gamma(\mu_t') = \{(a_1, p_3)\} \subsetneq \Gamma(\mu_t) = \{(a_1, p_3), (a_2, p_4)\}$ .■

The last example shows that we cannot guarantee in any problem the existence of a fair matching with $\Gamma(\mu_t) = \phi$. But, can we guarantee the existence of a fair matching in any overlapping teacher placement problem? The next Section is devoted to that question.

# 4 Existence and Uniqueness

To prove the existence of a fair matching we introduce the concept of what we will call a related market. Given an overlapping teacher placement problem represented by the market $M^t = \langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \rangle$, let $|P| = M$. We have a set of positions held by some teachers and another set that are not. Let denote as $p_1, ..., p_k$ the free positions and as $p_{k+1}, ...., p_M$ positions held by teachers $a_{k+1}, ...., a_M$, respectively. For each position $p_i$ with $i = k+1, ..., M$ we define the following strict priority order of all teachers for that position: $>_i^t$ is such that $a_i$ is the first teacher in the order $>_i^t$ and for the rest of teachers the order is the defined by $>^t$ . For $p_j$ with $j = 0, 1, ..., k$, we define $>_j^t = >^t$ . Let $O^t = \{>_i^t\}_{i=0}^M$ be the set of all orders indexed by the number of position. Finally, we have a strict order for each position. Then, given a market $M^t = \langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \rangle$ , the **related market** is $\langle P, I^t, \succ, O^t \rangle$ . The idea of this definition is very similar to the one used in Balinski and Sönmez [3]. Following Ergin [7] we present the following definition.

**Definition 8** *Given an overlapping teacher placement problem $\langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \rangle$ and the related market $\langle P, I^t, \succ, O^t \rangle$ , we say that matching $\mu_t$* ***violates the priority*** *of $a_i$ for $p_j$, if there is a teacher $a_j$ such that $\mu_t(a_j) = p_j$, $p_j \succ_i \mu_t(a_i)$ and $a_i >_j^t a_j$. The matching $\mu_t$* ***adapts*** *to $O^t$ if it does not violate any priorities.*

The relation between the concepts of a matching that adapts to $O^t$ and an acceptable matching is straightforward, as we prove in the next proposition.

**Proposition 1** ***Given an overlapping teacher placement problem*** $M^t = \langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \rangle$ ***and the related market*** $\langle P, I^t, \succ, O^t \rangle$ , ***a matching is acceptable (related to the market*** $M^t$***) if and only if it adapts to*** $O^t$ ***(respect to the related market)***.

**Proof.** ($\Rightarrow$) Suppose that $\mu_t$ is acceptable but violates the priority of $a_i$ for $p_j$. Then there is a teacher $a_j$ such that $\mu_t(a_j) = p_j$, $p_j \succ_i \mu_t(a_i)$ and $a_i >_j^t a_j$. We have two possibilities: $a_i >^t a_j$ or $a_i$ was originally assigned to position $p_j$. The last possibility violates the individually rational assumption. And the first implies, by definition of acceptable matching, that it must be the case that $a_j$ was originally assigned to $p_j$, but then we must have that $a_j >_j^t a_i$, and then a contradiction.

($\Leftarrow$) Suppose that $\mu_t$ adapts to $O^t$ but it is not acceptable. Then we have two cases: $\mu_t$ is not individually rational or it exists a justified claim in $\mu_t$. In the first case suppose that $a_i$ is such that $p_j = \nu_t(a_i) \succ_i \mu_t(a_i)$. Then we have that $a_i >_j^t \mu_t^{-1}(p_j)$ and then $\mu_t$ violates the priority of $a_i$ to $p_j$. In the case that it exists a justified claim in $\mu_t$, then we have a teacher $a_i$ and a position $p_j$ such that $p_j \succ_i \mu_t(a_i)$, $a_i >^t \mu_t^{-1}(p_j)$ and ${\nu_t}^{-1}(p_j) \neq \mu_t^{-1}(p_j)$. But then it must be $a_i >_j^t \mu_t^{-1}(p_j)$, and then $\mu_t$ does not adapts to $O^t$. ■

Then the problem of finding an acceptable matching in our original framework is equivalent to find a matching that adapts to $O^t$.

**Proposition 2** ***Given an overlapping teacher placement problem*** $\left\langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \right\rangle$, $\mathcal{C}_t$ ***is not empty.***

**Proof.** Given the related market $\langle P, I^t, \succ, O^t \rangle$ we can apply the Gale-Shapley deferred acceptance algorithm (Gale and Shapley [8]). It is well know, see Ergin [7], that the outcome of that algorithm is a matching that adapts to the orders $O^t$. Then by Proposition 1 we have an acceptable matching for our problem $\left\langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \right\rangle$. ■

**Corollary 1** ***Given an overlapping teacher placement problem*** $\left\langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \right\rangle$, ***it always exists a fair matching.***

**Proof.** We know that $\mathcal{C}_t$ is nonempty and finite. For each matching $\mu_t \in \mathcal{C}_t$, calculate $|\Gamma(\mu_t)|$. Then we have a finite set of real numbers, take $\mu_t \in \mathcal{C}_t$ such that $|\Gamma(\mu_t)| \leq |\Gamma(\mu_t')| \;\; \forall \mu_t' \in \mathcal{C}_t$. Clearly, $\mu_t$ is a fair matching. ■

The following question is about uniqueness and, as we will prove in the last part of this Section, it always exists a unique fair matching for each problem $M^t$. The key of the proof are the following two lemmas.

**Lemma 1** ***Given an overlapping teacher placement problem*** $\left\langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \right\rangle$, ***if*** $\mu_t$ ***is a fair matching such that*** $\Gamma(\mu_t) = \phi$, ***then*** $\mu_t$ ***is Pareto efficient.***

(See Appendix for a proof)

Observe that in the problem without existing tenants, that is, when no teacher had been previously assigned to hold a position, if we have a claim, it must be a justified claim. Then we have to following Corollary.

**Corollary 2** ***Given an overlapping teacher placement problem*** $\left\langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \right\rangle$, ***such*** $I_E^t = \mu_{t-1}^{-1}(P) \cap\ I^t = \varnothing$, ***then if*** $\mu_t$ ***is a fair matching,*** $\mu_t$ ***is Pareto efficient.***

**Lemma 2** ***Given an overlapping teacher placement problem*** $\langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \rangle$, ***suppose two different fair matchings*** $\mu_t$ ***and*** $\hat{\mu}_t$, ***then*** $\mu_t$ ***does not dominate*** $\hat{\mu}_t$ ***in the sense of Pareto (and vice versa).***

(See Appendix for a proof)

It is a well known result [7] that if preferences are strict, it only exists one acceptable matching (the outcome of the deferred acceptance algorithm) that is Pareto superior to any other acceptable matching. With this result we have the main proposition of the Section.

**Proposition 3** ***Given an overlapping teacher placement problem*** $\langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \rangle$***, it always exists a unique fair matching.***

**Proof.** The existence was proven by Proposition 2 and Lemma 1, where we show that we can apply the deferred acceptance algorithm of Gale and Shapley, and the resulting matching $\mu_t^G$ is an acceptable matching. It is straightforward that $\mu_t^G$ is fair: suppose the opposite, then it must exists an acceptable matching $\mu_t$ such that $\Gamma(\mu_t) \subsetneq \Gamma(\mu_t^G)$. But we also know that as $\mu_t^G$ is Pareto superior to any other acceptable matching, then by the last Lemma we have that $\Gamma(\mu_t^G) \subsetneq \Gamma(\mu_t)$, which is not possible

Now, suppose that it exists another fair matching $\mu_t$. By the cited result we have that $\mu_t^G$ dominates $\mu_t$ . But since the last Lemma we know that is not possible. Finally, we cannot have two different fair matchings in an overlapping teacher placement problem $\langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \rangle$ . ■

## 5 Efficiency

When in an overlapping teacher placement problem we have a fair matching which eliminates the inappropriate claims, we know by Lemma 1 that it is efficient. But what can be said if the fair matching is such that $\Gamma(\mu_t) \neq \varnothing$?, it will be always efficient? Unfortunately, as we present in the next example, the answer is negative.

**Example 3** *Consider the following problem:*

$I_E^t = \{a_2, a_3\} \subset I^t = \{a_1, a_2, a_3\}$
$P = \{p_1, p_2, p_3\}$ , $\nu_t = \{(a_2, p_2), (a_3, p_3)\}$
$>^t = (a_2, a_1, a_3)$ and the following teacher preferences:

$$\begin{bmatrix} \succ_1 & \succ_2 & \succ_3 \\ p_2 & p_3 & p_2 \\ p_1 & p_2 & p_3 \\ p_3 & p_1 & p_1 \end{bmatrix}.$$

[7] See Ergin [7], Proposition 1 and Balinski and Sönmez [3], Theorem 2.

Then the following matching:

$$\mu_t = \begin{pmatrix} a_1 & a_2 & a_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$$

is the unique fair matching of the problem but is dominated by this (unfair) matching

$$\mu_t' = \begin{pmatrix} a_1 & a_2 & a_3 \\ p_1 & p_3 & p_2 \end{pmatrix} \blacksquare$$

The problem of finding a fair and efficient matching in any overlapping teacher placement problem can be also solved using the result of Ergin [7]. Following this author a cycle for a given priority structure $O^t$ is constituted of distinct positions $p_n$, $p_m \in P$ and teachers $a_i, a_j, a_k \in I^t$ such that $a_i >_n^t a_j >_n^t a_k >_m^t a_i$. By Theorem 1 of Ergin [7] we know that the deferred acceptance algorithm is Pareto efficient (that is, always selects a Pareto efficient matching) if and only if the priority structure is acyclical (that is, the priority structure has no cycle). In our problem, under the assumption that $|I_E^t| \geq 3$, the priority structure $O^t$ always has at least one cycle: take $a_i, a_j, a_k \in I_E^t$ with $\nu_t(a_i) = p_i$ ,$\nu_t(a_j) = p_j$ and $\nu_t(a_k) = p_k$, then $a_i >_i^t a_j >_i^t a_k >_k^t a_i$ or $a_i >_i^t a_k >_i^t a_j >_j^t a_i$, but in both cases we have a cycle. Finally applying the mentioned theorem we know that the deferred acceptance algorithm is not Pareto efficient.

It must be noted that it exists other mechanisms that selects Pareto efficient matchings. Gale's top trading cycles mechanism (described in Abdulkadiroglu and Sonmez [1]) is one of them. But as we noted above, all these efficient mechanisms are not fair.

# 6 The Mechanism

The definition of an economy includes a mechanism, because the dynamic of our problem is defined by the relation between the matching of one period and the initial assignment of the following period. As we saw in Section 3 a mechanism is a systematic procedure that assigns a matching for each overlapping teacher placement problem. We proved that it exists a unique fair matching, the outcome of the deferred acceptance algorithm of Gale and Shapley. Then the searched mechanism is the deferred acceptance algorithm, but it must be applied to the related market defined in Section 4.

**Definition 9** *A mechanism is:* ***Pareto efficient*** *if it always selects a Pareto efficient matching and* ***fair*** *if it always selects a fair matching.*

**Definition 10** *Given an overlapping teacher placement problem $\langle P, I^t, \mu_{t-1}, \nu_t, \succ, >^t \rangle$, the* ***teacher proposing deferred acceptance mechanism*** *is the deferred acceptance algorithm of Gale and Shapley applied to the related market $\langle P, I^t, \succ, O^t \rangle$.*

In Section 4 we proved that in every problem we only have one fair matching. We say that two mechanisms are **essentially** the same, if they always yield the same result. Then we have the following Proposition.

**Proposition 4** *The teacher proposing deferred acceptance mechanism is essentially the only fair mechanism, i.e., other fair mechanism is essentially the same as the teacher proposing deferred acceptance mechanism.*

**Definition 11** ***The teacher proposing deferred acceptance economy*** *is an economy* $P, \nu_1, \{I_E^t\}_t, \{I^t\}_t, \succ, \{>^t\}_t, \varphi$, *with* $\varphi$ *the teacher proposing deferred acceptance mechanism.*

**Definition 12** *Given an economy* $P, \nu_1, \{I_E^t\}_t, \{I^t\}_t, \succ, \{>^t\}_t, \varphi$, *we said that it is a fair economy if the used mechanism is fair.*

**Proposition 5** ***An economy*** $P, \nu_1, \{I_E^t\}_t, \{I^t\}_t, \succ, \{>^t\}_t, \varphi$, ***is fair if and only if it is the teacher proposing deferred acceptance economy.***

# 7 Dynamic Strategy-Proof

Suppose that at time $t$ a new teacher enters the market to compete for a position. A natural question is if this new teacher can ever benefit by unilaterally misrepresenting her preferences. It is a well known result that she cannot benefit in period $t$ by manipulating her preferences (Roth [12]). But, what can be said about the following periods? Can teacher benefits, in the following periods, by sacrificing her position in period $t$? After some definitions, we study this issue.

**Notation 1** *We denote by* $\varphi\left[P, I^t, \mu_{t-1}, \nu_t, \succ, >^t\right](i)$ *the position assigned to teacher* $a_i$ *under the mechanism* $\varphi$, *i.e.,* $\varphi\left[P, I^t, \mu_{t-1}, \nu_t, \succ, >^t\right](i) = \mu_t(a_i)$.

**Definition 13** *Suppose an economy* $P, \nu_1, \{I^t\}_t, \succ, \{>^t\}_t, \varphi$ *and a teacher* $a_i$ *that enters the market at time* $t$. *We say that the mechanism* $\varphi$ *is* ***dynamic*** *strategy-proof if* $a_i$ *cannot ever benefits by unilaterally misrepresenting her preferences, that is:* $\varphi$ *is dynamic strategy-proof if*[8] $\varphi\left[P, I^t, \mu_{t-1}, \nu_t, >^t, \succ_{-i}, \succ_i\right](a_i) \succsim_i \varphi\left[P, I^t, \mu_{t-1}, \nu_t, >^t, \succ_{-i}, \succ_i'\right](a_i)$ $\forall a_i, \forall \succ_{-i}, \forall \succ_i'$ *and* $\forall t$ *such that* $a_i \in I^t$, *where* $\succ_{-i}$ *are the preferences of teachers in the set* $I^t \diagdown \{i\}$.

**Remark 1** *The classic notion of strategy-proofness in static matching problems only makes reference to the benefits in one period. In our framework the notion involves not only the period when teacher enters the market (and reveals her preference) but also all periods where teacher is in the market.*

In the next example we prove that, as we have claimed in the introduction, the test score assignment mechanism is not dynamic strategy proof.

[8] Remember that when teacher enters the market she reveals her preference and cannot change it in the following periods.

**Example 4** *Consider the following problem:*

$I_E^t = \{a_3, a_4\} \subset I^t = \{a_1, a_2, a_3, a_4\}$
$P = \{p_1, p_2, p_3, p_4\}$ , $\nu_t = \{(a_3, p_3), (a_4, p_4)\}$
$>^t = (a_1, a_3, a_4, a_2)$ and the following teacher preferences:

$$\begin{bmatrix} \succ_1 & \succ_2 & \succ_3 & \succ_4 \\ p_3 & p_4 & p_1 & p_2 \\ p_1 & p_2 & p_3 & p_3 \\ p_2 & p_3 & p_2 & p_4 \\ p_4 & p_1 & p_4 & p_1 \end{bmatrix}.$$

The outcome of the test score assignment mechanism is:

$$\mu_t = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ p_1 & p_4 & p_3 & p_2 \end{pmatrix}$$

Assume that in the next period we have:
$I^{t+1} = \{a_1, a_2, a_5, a_6\}$
$>^{t+1} = (a_5, a_1, a_6, a_2)$
$\succ_5 = (p_1, p_3, p_4, p_2)$ and $\succ_6 = (p_2, p_4, p_1, p_3)$.
Then, the outcome of the mechanism is:

$$\mu_{t+1} = \begin{pmatrix} a_1 & a_2 & a_5 & a_6 \\ p_1 & p_4 & p_3 & p_2 \end{pmatrix}$$

Now suppose that instead of her true preferences, teacher $a_1$ reveals the following preferences: $\succ_1' = (p_3, p_2, p_1, p_4)$. Hence, the outcomes of the mechanism are:

$$\mu_t' = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ p_2 & p_4 & p_1 & p_3 \end{pmatrix} \quad \text{and} \quad \mu_{t+1}' = \begin{pmatrix} a_1 & a_2 & a_5 & a_6 \\ p_3 & p_4 & p_1 & p_2 \end{pmatrix}$$

Consistent with the fact that this mechanism is strategy-proof (the proof is straightforward), we have: $\mu_t(a_1) \succ_1 \mu_t'(a_1)$. But note that $\mu_{t+1}'(a_1) \succ_1 \mu_{t+1}(a_1)$, and then, teacher $a_1$ benefits in period $t+1$ by manipulating her preference. Hence the test score assignment mechanism is not dynamic strategy proof.$\blacksquare$

**Definition 14** *An economy is* ***dynamic strategy-proof*** *if the mechanism used $\varphi$ is dynamic strategy-proof.*

**Proposition 6** ***The teacher proposing deferred acceptance economy is dynamic strategy-proof.***

(See Appendix for a proof)

A stricter property of a mechanism is the **group strategy-proofness**. Specifically, a mechanism is group strategy-proof if no subset of teachers can gain by jointly misrepresenting their preferences: if it does not exist $\varnothing \neq J \subset I^t$ and $\succ_J, \succ'_J$ (where $\succ_J$ is a preference profile of teachers in $J$) such that $\varphi\left[P, I^t, >^t, \succ_{-J}, \succ'_J\right](a_h) \succsim_h \varphi\left[P, I^t, >^t, \succ_{-J}, \succ_J\right](a_h)$ $\forall a_h \in J, \forall \succ_J, \forall \succ'_J, \forall \succ_{-J}$ and for some $a_k \in J$ we have: $\varphi\left[P, I^t, \mu_{t-1}, \nu_t, >^t, \succ_{-j}, \succ'_J\right](a_k) \succ_k \varphi\left[P, I^t, \mu_{t-1}, \nu_t, >^t, \succ_{-J}, \succ_J\right](a_k)$. Unfortunately, as we show in the next example, the teacher proposing deferred acceptance mechanism is not group strategy-proof.

**Example 5** *The teacher proposing deferred acceptance mechanism is not group strategy-proof.*

Consider the following problem:

$I^t_E = \{a_2, a_3\} \subset I^t = \{a_1, a_2, a_3\}$

$P = \{p_1, p_2, p_3\}$ , $\nu_t = \{(a_2, p_2), (a_3, p_3)\}$

$>^t = (a_2, a_1, a_3)$ and the following preferences of teachers:

$$\begin{bmatrix} \succ_1 & \succ_2 & \succ_3 \\ p_2 & p_3 & p_2 \\ p_3 & p_2 & p_3 \\ p_1 & p_1 & p_1 \end{bmatrix}.$$

The outcome of the teacher proposing deferred acceptance mechanism is the following matching:

$$\mu_t = \begin{pmatrix} a_1 & a_2 & a_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$$

If we define $J = \{a_1, a_2\}$ and $\succ'_J = ((p_3, p_1, p_2) \succ_2)$ the outcome of the mechanism is:

$$\mu'_t = \begin{pmatrix} a_1 & a_2 & a_3 \\ p_1 & p_3 & p_2 \end{pmatrix}$$

Then we have that teacher $a_1$ can maintain her assignment and teacher $a_2$ is assigned to a preferred position. ■

By Lemma 1 of Pápai [11] we know that the property of being group strategy-proof implies strategy-proof and nonbossy. Then, the last result is consistent with that Lemma since the teacher proposing deferred acceptance mechanism violates the nonbossy condition. Formally we have:

**Definition 15** *A mechanism is* ***nonbossy*** *if for any $a_i \in I^t$ and $\succ_i, \succ'_i$, if $\varphi\left[P, I^t, \mu_{t-1}, \nu_t, >^t, \succ_{-i}, \succ_i\right](a_i) = \varphi\left[P, I^t, \mu_{t-1}, \nu_t, >^t, \succ_{-i}, \succ'_i\right](a_i)$ then $\varphi\left[P, I^t, \mu_{t-1}, \nu_t, >^t, \succ_{-i}, \succ_i\right](a_j) = \varphi\left[P, I^t, \mu_{t-1}, \nu_t, >^t, \succ_{-i}, \succ'_i\right](a_j)$ $\forall a_j \in I^t$.*

Then, as we show in the last example, teacher $a_1$ can maintain her assignment and cause a change in other teacher 's assignment by reporting preferences $\succ'_1$ instead of $\succ_1$ .

# 8 Respecting Improvements

As it is illustrated in Section 2, an important flaw of the test score assignment mechanism used by SEP is that it does not respect improvements done by teachers. Then, a major question is if the teacher proposing deferred acceptance mechanism also has this failure. Fortunately, the answer is negative.

**Definition 16** *An overlapping teacher placement problem* $\left\langle P, I^t, \mu_{t-1}, \nu_t, \tilde{>}^t, \succ \right\rangle$ *is an improvement for teacher* $a_i$ *over another problem* $\langle P, I^t, \mu_{t-1}, \nu_t, >^t, \succ \rangle$ *, if* $a_i >^t a_j$ *implies that* $a_i \tilde{>}^t a_j$ *, and for all teachers* $a_k, a_h$ *different to* $a_i$ *we have that* $a_h \tilde{>}^t a_k \Leftrightarrow a_h >^t a_k$*. That is, an improvement for a teacher is basically the same problem with the only difference that teacher is possibly in a better place in the strict order of all teachers.*

**Definition 17** *A mechanism respects improvements if for any teacher* $a_i$ *and* $\left\langle P, I^t, \mu_{t-1}, \nu_t, \tilde{>}^t, \succ \right\rangle$ *an improvement for that teacher over another problem* $\langle P, I^t, \mu_{t-1}, \nu_t, >^t, \succ \rangle$ *, the position assigned to teacher* $a_i$ *by the mechanism in the path beginning with the improvement (that is, in all periods* $\tau \geq t$*) is at least as good as the position assigned in the path beginning with the problem* $\langle P, I^t, \mu_{t-1}, \nu_t, >^t, \succ \rangle$ *.*

**Remark 2** *The same comment that we did in Remark 1 applies to this definition. Our concept of respecting improvements involves not only the period when the teacher improves her place in the strict order (like the classic notion), but also all periods where the teacher is in the market.*

**Definition 18** *An economy respects improvements if the mechanism used* $\varphi$ *respects improvements.*

The following is the result of this Section.

**Proposition 7** *The teacher proposing deferred acceptance economy respects improvements.*

(See Appendix for a proof.)

# 9 Conclusions

In recent years some articles have studied assignment problems in dynamic context. In this work we develop a new framework to model a dynamic school choice problem faced by the Mexican Ministry of Public Education (SEP). Since 2008, teachers who sought a position in the public education system are required to sit an exam which by ranking allocates each teacher a 'priority order' to which they are assigned a position within the school. However, any teacher that had been previously assigned a position would have the ability to choose between their current position and any open positions. In this work we model the mechanism used by SEP (which is called test score assignment mechanism) and we have shown that it does not comply with desirable properties about incentives and fairness. In particular we have proven that the mechanism in use is not fair, dynamic strategy-proof and does not respect improvements done by teachers. The main contribution of this work is to present a mechanism that solves the problem faced by SEP, and has these interesting properties.

In the context of our model, we introduce a new fairness concept. With this new fairness version, that is very natural in our context, we proved that it always exists a unique fair matching. The fair assignment can be reached by a modified version of the Deferred Acceptance Algorithm by Gale and Shapley. With these ingredients we define a fair economy as the one that uses the Gale-Shapley deferred acceptance algorithm in the way explained in this work. Related to the properties of the fair economy, we have demonstrated that it is dynamic strategy-proof and respect improvements done by teachers. Hence, in the context of our model, the use of the proposed mechanism, would improve the assignment of teachers to positions.

Our results rely heavily on the assumption that teacher preferences do not change over time. A very interesting extension of the model presented in this work, would be a way to include the changing over time of teacher preferences and investigate which properties are still valid. But this is material for future research.

# 10 Appendix

### A.1 Proof of Lemma 1

**Proof.** Suppose that $\mu_t$ is fair with $\Gamma(\mu_t) = \phi$ but it is not an efficient matching. Then we must have another matching that makes all teachers weakly better off and at least one teacher strictly better off. Hence, we have a teacher $a_i$ who improves when she changes from $p_i$ to $p_k$. Observe that $p_k \neq p_0$, otherwise if $p_k = p_0$, then $p_0 \succ_i \mu_t(a_i)$,and this contradicts the fact that $\mu_t$ is individually rational. But, since $\mu_t$ is fair, it exists another teacher $a_k$ that before the change she had the position $p_k$ and after $p_l$, and also $a_k$ is not worst off with the change, instead, she must be better since $p_0 \neq p_l \neq p_k$. Since matching $\mu_t$ is fair and $\Gamma(\mu_t) = \phi$ we have that $a_k >^t a_i$. We claim that $p_i \neq p_l$. Suppose that it is not the case. Then, if $p_i = p_l$ we have that $a_i >^t a_k$, but this is a contradiction. Then $p_i \neq p_l$. Following the argument, we have a teacher $a_l$ such that $\mu_t(a_l) = p_l \neq p_0$, and changes to position $p_h$. As before we know $p_h \neq p_0$. Once again, we must have $a_l >^t a_k >^t a_i$. If we suppose that $p_h = p_k$, then the last argument applies and we have a contradiction. Then $p_h \neq p_k$. We claim that in some step of the process, a teacher changes to a position that had already appeared. If it is not the case, then it must exists a free position (respect to $\mu_t$) such as it is preferred by some teacher to her assignment; but, since $\mu_t$ is fair, that it is not possible. Be $a_u$ teacher that had position $p_u$ and changes to $p_j$. By the construction of the process we have that $a_u >^t ... >^t a_j >^t ... >^t a_i$. But also, as $p_j \succ_u p_u$, it must be that $a_j >^t a_u$, and then we have a contradiction. ■

### A.2 Proof of Lemma 2

**Proof.** Suppose $\Gamma(\mu_t) \neq \varnothing$, and then we must have that $\Gamma(\hat{\mu}_t) \neq \varnothing$. We will prove that if $\hat{\mu}_t$ dominates $\mu_t$, then $\Gamma(\hat{\mu}_t) \subsetneq \Gamma(\mu_t)$, but this is a contradiction since $\mu_t$ is fair. Then suppose that $\hat{\mu}_t$ dominates $\mu_t$ and that it exists a pair $(a_i, p_j) \in \Gamma(\hat{\mu}_t)$ but $(a_i, p_j) \notin \Gamma(\mu_t)$. In particular, note that $p_j$ can not be the null position. Denote by $a_j$ teacher such that $\hat{\mu}_t(a_j) = p_j$. We have that $p_j \succ_i \hat{\mu}_t(a_i)$, $a_i >^t \hat{\mu}_t^{-1}(p_j) = a_j$ and $\hat{\mu}_t(a_j) = \nu_t(a_j)$. Since $(a_i, p_j) \notin \Gamma(\mu_t)$ we have two possibilities: $\mu_t(a_i) \succsim_i p_j$ or $\mu_t^{-1}(p_j) >^t a_i$. In the first case we have that $\mu_t(a_i) \succsim_i p_j \succ_i \hat{\mu}_t(a_i)$, but this contradicts the fact that $\hat{\mu}_t$ dominates $\mu_t$. In the second case $\mu_t(a_j) \neq p_j$, but since $\hat{\mu}_t$ dominates $\mu_t$ it must be that $\hat{\mu}_t\ (a_j) \succ_j \mu_t(a_j)$ and then $\nu_t(a_j) = \hat{\mu}_t(a_j) \succ_j \mu_t(a_i)$. Then $\mu_t$ is not individually rational, but this contradicts that $\mu_t$ is fair.

It remains to prove that it exists a pair $(a_i, p_j) \in \Gamma(\mu_t)$ such that $(a_i, p_j) \notin \Gamma(\hat{\mu}_t)$, that is, the inclusion proved above is strict. Suppose the opposite, then we have that for all $(a_i, p_j) \in \Gamma(\mu_t)$, $(a_i, p_j) \in \Gamma(\hat{\mu}_t)$, but this implies $\Gamma(\mu_t) = \Gamma(\hat{\mu}_t)$. We know that $\hat{\mu}_t$ dominates $\mu_t$, and then there must be a teacher $a_i$ such that $\hat{\mu}_t(a_i) = p_i \succ_i \mu_t(a_i) = p_j$ (note $p_i \neq p_0$ since $\mu_t$ is fair and then individually rational). But then it must exists another teacher $a_h$ (since $\mu_t$ is fair) such that $\hat{\mu}_t(a_h) = p_h \succ_h \mu_t(a_h) = p_i$. We have two possibilities: $a_h >^t a_i$ or $a_i >^t a_h$ . In the last case, since $\mu_t$ is fair it must be that $\mu_t(a_h) = \nu_t(a_h)$, and $(a_i, \mu_t(a_h)) = (a_i, p_i) \in \Gamma(\mu_t) \Rightarrow (a_i, p_i) \in \Gamma(\hat{\mu}_t)$, but as $\hat{\mu}_t(a_i) = p_i$ this is a contradiction. Then we have $a_h >^t a_i$. Following with the argument we have a teacher $a_k$ such that $\hat{\mu}_t(a_k) = p_k \succ_k \mu_t(a_k) = p_h$. If $a_h >^t a_k$ then $\mu_t(a_k) = \nu_t(a_k) \Rightarrow (a_h, p_h) \in \Gamma(\mu_t) \Rightarrow (a_h, p_h) \in \Gamma(\hat{\mu}_t)$, but as $\hat{\mu}_t(a_h) = p_h$, this is a contradiction. Then it must be $a_k >^t a_h \Rightarrow a_k >^t a_h >^t a_i$. Continuing this process we construct a list of teachers: $a_l >^t a_r >^t ... >^t a_k >^t a_h >^t a_i$. We claim that $p_j$ is not the null position and in some step of the process we must have a teacher that $\hat{\mu}_t(a_j) = p_j \succ_j \mu_t(a_j) = p_u$. To prove the claim note first that if a teacher $a_s$ is such that $\mu_t(a_s) \neq p_0$ and $\hat{\mu}_t(a_s) \neq \mu_t(a_s)$ then also $\hat{\mu}_t(a_s) \neq p_0$, because in the case that $\hat{\mu}_t(a_s) = p_0$ we have $p_0 \succ_s \mu_t(a_s)$. Then if in the previous process $p_j$ does not appear again or $p_j = p_0$, it must exist a position $p_w \neq p_0$ and a teacher $a_z$ such that $\hat{\mu}_t(a_z) = p_w \succ_z \mu_t(a_w)$ and $p_w$ is not assigned to any teacher in matching $\mu_t$, but this contradicts that $\mu_t$ is a fair matching.

Then we have a teacher $a_j$ such that $\hat{\mu}_t(a_j) = p_j \succ_j \mu_t(a_j) = p_u$. Since the construction of the process, we know that $p_j \neq p_0$. Then, as before, we must have that $a_j >^t ... >^t a_k >^t a_h >^t a_i$. But as $\mu_t(a_i) = p_j$, it must be that $\mu_t(a_i) = \nu_t(a_i) \Rightarrow (a_j, p_j) \in \Gamma(\mu_t) \Rightarrow (a_j, p_j) \in \Gamma(\hat{\mu}_t)$, but as $\hat{\mu}_t(a_j) = p_j$, once again, we have a contradiction.

Finally, we proven that if $\hat{\mu}_t$ dominates $\mu_t$, then $\Gamma(\hat{\mu}_t)$ is strictly included in $\Gamma(\mu_t)$. But then $\mu_t$ can not be fair.

In the case that $\Gamma(\mu_t) = \varnothing$ we must have that $\Gamma(\hat{\mu}_t) = \varnothing$ (because if $\Gamma(\hat{\mu}_t) \neq \varnothing$ then $\Gamma(\mu_t) \subsetneq \Gamma(\hat{\mu}_t)$ and then $\hat{\mu}_t$ can not be fair). But by Lemma 1 we know that $\mu_t$ and $\hat{\mu}_t$ are efficient, and then $\mu_t$ does not dominate $\hat{\mu}_t$ in the sense of Pareto (and vice versa). ■

### A.3 Proof of Proposition 5

**Proof.** For the **first period**, observe that for the strategy-proofness only matters the agent preferences. Then, since in the first step of our mechanism we do not modify teacher preferences, strategy-proofness for the period when teacher enters, is a direct corollary of Theorem 5 of Roth [12]. We only have to prove that teacher neither benefits by unilaterally misrepresenting her preferences in the next periods in which she is active.

For the **second period,** suppose a teacher $a_i$ with true preferences $\succ_i$ . Then we have that

$\varphi\left[P, I^t, \mu_{t-1}, \nu_t, >^t, \succ_{-i}, \succ_i\right](a_i) = \mu_t(a_i)$, and when $a_i$ misrepresents her preferences stating $\succ_i'$, she obtains $\varphi\left[P, I^t, \mu_{t-1}, \nu_t, >^t, \succ_{-i}, \succ_i'\right](a_i) = \mu_t'(a_i)$. We know that $\mu_t(a_i) \succsim_i \mu_t'(a_i)$ and, in particular, if $\mu_t(a_i) \neq \mu_t'(a_i)$ then $\mu_t(a_i) \succ_i \mu_t'(a_i)$. Each matching in $t$ generates a different initial assignment for the next period. Denote by $\nu_{t+1}$ and $\nu_{t+1}'$ the initial assignment generated by $\mu_t$ and $\mu_t'$, respectively. When the mechanism $\varphi$ is applied to the markets $P, I^{t+1}, \mu_t, \nu_{t+1}, >^{t+1}, \succ_{-i}, \succ_i$ and $P, I^{t+1}, \mu_t', \nu_{t+1}', >^{t+1}, \succ_{-i}, \succ_i'$, matchings $\mu_{t+1}$ and $\mu_{t+1}'$ are generated. We have to prove that $\mu_{t+1}(a_i) \succsim_i \mu_{t+1}'(a_i)$.

Note first that if $\nu_{t+1} = \nu_{t+1}'$, the argument used in the first period can be applied to prove $\mu_{t+1}(a_i) \succsim_i \mu_{t+1}'(a_i)$. Then suppose $\nu_{t+1} \neq \nu_{t+1}'$.

Since teacher $a_i$ can not manipulate the order $>^t$, the difference between problems in period $t+1$ (beyond the preference of teacher $a_i$) is the strict priority order of all teacher for each position, but each order $>_i^{t+1}$ [9] can only differ in the first place, the other relative orders between teachers are identical. That is: as $>^{t+1}$ is the same in both problems, and $\nu_{t+1} \neq \nu_{t+1}'$, the order for each position in both problems is the same **except** for the first place that can be held by other teacher, and the teacher that held the first place, return to her position defined by $>^{t+1}$ .

Suppose, to the contrary, $\mu_{t+1}'(a_i) \succ_i \mu_{t+1}(a_i)$, and denote $\mu_{t+1}'(a_i) = p_j$. Observe that $p_j \neq p_0$. Then we have that $a_i$ is proposed in some step of the mechanism when it is applied to the problem with $\nu_{t+1}$, to $p_j$ and is rejected. Therefore we have another teacher $a_k$ with $\mu_{t+1}(a_k) = p_j$ and $a_k >_j^{t+1} a_i$. Since in the problem $\nu_{t+1}'$, $\mu_{t+1}'(a_k) \neq p_j$, we only have two possibilities: $\mu_{t+1}'(a_k) \succ_k p_j = \mu_{t+1}(a_k)$ (and then $a_k$ is not proposed to $p_j$ in the problem with $\nu_{t+1}'$), or $p_j \succ_k \mu_{t+1}'(a_k)$ and then $a_k$ is proposed to $p_j$ in the problem with $\nu_{t+1}'$ but $a_i \tilde{>}_j^{t+1} a_k$[10]. In the second case we know that: $\nu_{t+1}(a_k) = p_j$ (because teacher $a_k$ is the first in the order $>_j^{t+1}$), $p_j \succ_k \mu_{t+1}'(a_k)$ and in the order $>^{t+1}, a_i >^{t+1} a_k$ (and also $a_i >^t a_k$).

We claim that this last case is not possible. First, observe that $\nu_{t+1}(a_k) \succ_k \nu_{t+1}'(a_k)$, because in the contrary case $p_j \succ_k \mu_{t+1}'(a_k) \succsim_k \nu_{t+1}'(a_k) \succsim_k \nu_{t+1}(a_k) = p_j$. Note also that $\nu_t(a_k) \neq p_j$ since in the opposite case if $\nu_t(a_k) = p_j$, then $\nu_t(a_k) = p_j = \nu_{t+1}(a_k) \succ_k \nu_{t+1}'(a_k) = \mu_t'(a_k)$, and then $\mu_t'$ is not individually rational (remember that in both problems $a_k$ states the same preferences). Finally we have that $a_i$ has a justified claim over $\mu_t(a_k)$ because: $p_j = \nu_{t+1}(a_k) = \mu_t(a_k) = \mu_{t+1}'(a_i) \succ_i \mu_{t+1}(a_i) \succsim_i \nu_{t+1}(a_i) = \mu_t(a_i)$, $a_i >^t a_k$ and $\nu_t(a_k) \neq p_j$, but this is a contradiction since $\mu_t$ is a fair matching. Finally we have: $\mu_{t+1}'(a_k) \equiv p_k \succ_k \mu_{t+1}(a_k)$.

For $a_k$ we have the same situation as for $a_i$ : $a_k$ is proposed to $p_k$ in the problem with $\nu_{t+1}$ and is

[9] Remember that $>_i^{t+1}$ is the strict priority order of all teacher for position $p_i$.

[10] Here $\tilde{>}_j^{t+1}$denote the strict priority order of all teacher for position $p_j$ in the problem with $\nu_{t+1}'$.

rejected. Therefore we have another teacher $a_l$ with $\mu_{t+1}(a_l) = p_k$ and $a_l >_k^{t+1} a_k$. Since in the problem $\nu'_{t+1}$, $\mu'_{t+1}(a_l) \neq p_k$, we only have two possibilities: $a_l$ is not proposed to $p_k$ in the problem with $\nu'_{t+1}$, or $a_l$ is proposed to $p_k$ in the problem with $\nu'_{t+1}$ but $a_k \tilde{>}_k^{t+1} a_l$. In the first case we have that $\mu'_{t+1}(a_l) \succ_l p_k = \mu_{t+1}(a_l)$. In the second case we know that: $\nu_{t+1}(a_l) = p_k$, $p_k \succ_l \mu'_{t+1}(a_l)$ and in the order $>^{t+1}, a_k >^{t+1} a_l$ (and also $a_k >^t a_l$). A similar argument as before shows that the second case is not possible, and then $\mu'_{t+1}(a_l) \succ_l \mu_{t+1}(a_l)$. Following this argument we have that $\mu'_{t+1}(a) \succ \mu_{t+1}(a)$ for all teachers and then, the matching $\mu'_{t+1}$ is preferred for all teachers to $\mu_{t+1}$. We also have proved that $\mu'_{t+1}(a) \neq p_0$ and $\mu_{t+1}(a) \neq p_0$ $\forall a \neq a_i$.

Suppose that $\mu_{t+1}(a_i) \neq p_0$ Then in both matchings $\mu_t$ and $\mu'_t$ neither teacher is assigned to the null position. Observe that if a teacher $a_l$ makes a match at the last state of the mechanism (that is $a_l$ is proposed to her ultimate position $\mu_{t+1}(a_l) \neq p_0$ at the last step of the process) then $a_l$ was the only teacher who was proposed to that position. If that is not the case, then at least one more step would have occurred. Then take the teacher that makes the match at the last stage in $\langle P, I^{t+1}, \mu_t, \nu_{t+1}, >^{t+1}, \succ_{-i}, \succ_i \rangle$, let $a_k$ be this teacher and $\mu_{t+1}(a_k)$ her assignment. Since the last claim we have that $\mu'_{t+1}(a_k) \succ_k \mu_{t+1}(a_k)$. Note also that it must exist another teacher $a_h$ that $\mu'_{t+1}(a_h) = \mu_{t+1}(a_k)$, because, otherwise, we have a position $p_n$ not assigned to any teacher in $\mu_{t+1}$ but assigned in $\mu'_{t+1}$; but that is not possible since $\mu'_{t+1}(m) \succ \mu_{t+1}(m)$ and $\mu_{t+1}$ is fair. But since $a_h$ was not proposed to position $\mu_{t+1}(a_k)$ in the problem $\langle P, I^{t+1}, \mu_t, \nu_{t+1}, >^{t+1}, \succ_{-i}, \succ_i \rangle$, we have that $\mu_{t+1}(a_h) \succsim_h \mu'_{t+1}(a_h)$. But this contradicts the last claim.

If $\mu_{t+1}(a) = p_0$ then **only** teacher $a_i$ is assigned to the null position. As it was made above, it is easy to verify in this case the existence a position $p_n$ such that is not assigned under the matching $\mu_{t+1}$ but it is assigned to a teacher in the other matching. Clearly, since all teachers are better under $\mu'_{t+1}$ than $\mu_{t+1}$, that is a contradiction.

For the next period we have that $\mu_{t+1}(a_i) \succsim_i \mu'_{t+1}(a_i)$ and the same argument applies to prove that $a_i$ can ever benefit in the following periods. ∎

### A.4 Proof of Proposition 6

**Proof.** For the **first period**, we have a problem $\langle P, I^t, \mu_{t-1}, \nu_t, >^t, \succ \rangle$ and another problem $\left\langle P, I^t, \mu_{t-1}, \nu_t, \tilde{>}^t, \succ \right\rangle$ that represents an improvement for one teacher, say $a_i$. Let denote by $\mu_t$ the matching selected by the teacher proposing deferred acceptance mechanism in problem $\langle P, I^t, \mu_{t-1}, \nu_t, >^t, \succ \rangle$ and by $\tilde{\mu}_t$ the one selected in problem $\left\langle P, I^t, \mu_{t-1}, \nu_t, \tilde{>}^t, \succ \right\rangle$. It must be shown that $\tilde{\mu}_t(a_i) \succsim_i \mu_t(a_i)$. Suppose that $\mu_t(a_i) \succ_i \tilde{\mu}_t(a_i)$. We claim that in that case $\mu_t(a) \succ \tilde{\mu}_t(a)$ for all teachers. Let $p_i = \mu_t(a_i) \succ_i \tilde{\mu}_t(a_i) = p_j$. First note that this implies $p_i \neq p_0$, since $\tilde{\mu}_t$ is individually rational. Then, as $\tilde{\mu}_t$ is fair, we must have another teacher $a_k$ such that $\tilde{\mu}_t(a_k) = p_i$ and $\mu_t(a_k) = p_k$. If $a_i \tilde{>}^t a_k$ then as $p_i \succ_i p_j$ it must be the case that $\nu_t(a_k) = \tilde{\mu}_t(a_k)$. But since the matching $\mu_t$ is individually rational, $\mu_t(a_k) \succ_k \tilde{\mu}_t(a_k)$. Suppose $a_k \tilde{>}^t a_i$ and then we have $a_k >^t a_i$. If $\tilde{\mu}_t(a_k) \succ_k \mu_t(a_k)$ we must have that $\nu_t(a_i) = \mu_t(a_i)$ and then $\nu_t(a_i) \succ_i \tilde{\mu}_t(a_i)$, but this contradicts that $\tilde{\mu}_t$ is individually rational. Then we have $\mu_t(a_k) \succ_k \tilde{\mu}_t(a_k)$. Also note that $\mu_t(a_k) = p_k \neq p_0$ and $\tilde{\mu}_t(a_k) = p_i \neq p_0$. Following with this process, it must exists another teacher $a_l$ such that $\tilde{\mu}_t(a_l) = p_k$. If $a_k \tilde{>}^t a_l$ as $\tilde{\mu}_t(a_l) = \mu_t(a_k) \succ_k \tilde{\mu}_t(a_k)$ then $\nu_t(a_l) = \tilde{\mu}_t(a_l)$ and $\mu_t(a_l) \succ_l \tilde{\mu}_t(a_l)$. In the case that $a_l \tilde{>}^t a_k$, then $a_l >^t a_k$, and if $\tilde{\mu}_t(a_l) = \mu_t(a_k) \succ_l \mu_t(a_l)$, it must be $\nu_t(a_k) = \mu_t(a_k) \succ_k \tilde{\mu}_t(a_k)$, but this is a contradiction since $\tilde{\mu}_t$ is individually rational. Then we have $p_0 \neq \mu_t(a_l) \succ_l \tilde{\mu}_t(a_l) \neq p_0$. Using induction in the number of teachers, we prove the claim. Also note that we have proved that the matching $\mu_t$ verifies that $\mu_t(a) \neq p_0$ $\forall a$ and that $\tilde{\mu}_t(a) \neq p_0$ $\forall a \neq a_i$.

Now observe that the same argument as the applied in the proof of the last proposition can be used here to obtain a contradiction.

For the **second period**, denote by $\langle P, I^{t+1}, \mu_t, \nu_{t+1}, >^{t+1}, \succ \rangle$ and $\left\langle P, I^{t+1}, \tilde{\mu}_t, \tilde{\nu}_{t+1}, \tilde{>}^{t+1}, \succ \right\rangle$ the markets of the following period in the case that we have in period $t$ $\langle P, I^t, \mu_{t-1}, \nu_t, >^t, \succ \rangle$ and $\left\langle P, I^t, \mu_{t-1}, \nu_t, \tilde{>}^t, \succ \right\rangle$, and by $\mu_{t+1}$ and $\tilde{\mu}_{t+1}$ the outcome of the mechanism in each market in period $t+1$. We define the following market at $t+1$ : $\left\langle P, I^{t+1}, \mu_t, \nu_{t+1}, \tilde{>}^{t+1}, \succ \right\rangle$ and be $\mu''_{t+1}$ the matching obtained by the mechanism in this market. We know that $\tilde{\nu}_{t+1}(a_i) \succsim_i \nu_{t+1}(a_i)$. Then by the argument used in strategy-proofness we have $\tilde{\mu}_{t+1}(a_i) \succsim_i \mu''_{t+1}(a_i)$, and by the property of respecting improvement for one period, $\mu''_{t+1}(a_i) \succsim_i \mu_{t+1}(a_i)$[11]. Repeating the argument for the following periods we have $\tilde{\mu}_\tau(a_i) \succsim_i \mu_\tau(a_i)$ $\forall \tau \geq t$. ■

# References


[1] Abdulkadiroglu, A. and Sönmez, T. (1999) "House allocation with existing tenants", *Journal of Economic Theory* **88**, 233-260.

[2] Abdulkadiroglu, A. and Sönmez, T. (2003) "School choice: a mechanism design Approach", *The American Economic Review* **93 (3)**, 729-747.

[3] Balinski, M. and Sönmez,T. (1999) "A Tale of Two Mechanisms: Student Placement", *Journal of Economic Theory*, **84**, 73-94.

[4] Bloch, F. and Cantala, D. (2008) "Markovian assignment rules", *Working Paper.*

[5] Cantala, D. (2010) "Eficiencia y justicia en la administración publica: la educación en México", Mimeo, El Colegio de México.

[6] Eeckhout, J. (2000) "On the uniqueness of stable marriage matchings", *Economics Letters*, **69**, 1-8.

[7] Ergin, H (2002) "Efficient resources allocation on the basis of priorities", *Econometrica* **70 (6),** 2489-2497.


[11] Remember the assumption that teacher reveals her preference in the period when she enters the market. In the following periods the teacher cannot modify the preference that announced before

[8] Gale, D. and Shapley, L (1962) "College Admissions and the Stability of Marriage", *American Mathematical Monthly*, **69**, 9-15.

[9] Kurino, M. (2009) "House Allocation with Overlapping Agents: A Dynamic Mechanism Approach", *Jena Economic Research Papers*, **2009-075**.

[10] Legros, P. and Newman, A. (2010) "Co-ranking mates: Assortative matching in marriage markets", *Economics Letters*, **106**, 177-179.

[11] Pápai, S.(2000) "Strategyproof Assignment by Hierarchical Exchange", *Econometrica* , **68**, 1403-1433

[12] Roth,.E (1982) "The economics of matching: stability and incentives", *Mathematics of Operations Research*, **7**, 617-628.

[13] Roth, E. and Sotomayor, M (1990) "Two-sided matching: a study in game-theoretic modeling and analysis", Cambridge, Cambridge University Press.

[14] Sönmez, T. and Ünver, U. (2009) "Matching, Allocation, and Exchange of Discrete Resources ", Jess Benhabib, Alberto Bisin, and Matthew Jackson (eds.), Handbook of Social Economics, Elsevier, forthcoming.

[15] Ünver, M. (2009) "Dynamic Kidney Exchange". Forthcoming in *Review of Economic Studies.*