EL COLEGIO DE MÉXICO, A.C.
CENTRO DE ESTUDIOS ECONÓMICOS

"Three Essays on the Allocation of Indivisible Goods"

Tesis presentada por:

DAMIÁN EMILIO GIBAJA ROMERO

Promoción 2011-2014

México D.F. Agosto de 2014

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Para optar por el grado de

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Contents

Chapter 1

Introduction

This thesis is a collection of three essays on assignment problems in matching markets. Each essay is self-contained. However, the first two essays are closely related, specifically, some of the results of the first essay are applied in the second one.

We study the allocation of indivisible goods (e.g., apartments, school seats, scholarships) to different agents (e.g., households, students) in the context of three different assignment problems. The first essay focuses on a school choice problem where all students have the same priority for all schools and students can be indifferent between them. In the second essay we establish the existence and uniqueness of the price vector of equilibrium in the assignment game when markets are large. The last essay deals with the subsidized housing problem in Paris.

The first essay analyses the School Choice problem in Toluca to allocate seats in elementary schools to six years old children. Parents submit a preference list of up to five elementary schools to the assignment procedure used by the local Ministry of Education, the Sistema Anticipado de Inscripción y Distribución (SAID). This mechanism presents efficiency problems because indifferences are pervasive in the market. Well-known school choice mechanisms, like the Deferred Acceptance and Top Trading Cycles algorithms, have a loss of efficiency due to the presence of indifferences. We introduce the Top Trading Cover (TTCo) mechanism to deal with them. This new mechanism is characterized by Pareto efficiency and the fact that it recursively respects top ranking. Nevertheless, this assignment procedure is manipulable, that is to say students have incentives to misreport their preferences. To support the use of this mechanism, we analyze its incentives properties when the market is large. In a Bayesian game where student only know their own ranking of elementary schools, we prove that truth-telling is the unique epsilon Nash equilibrium when the markets satisfy a "thickness" condition.

In the second essay we re-examine the heterogeneity feature of real estate markets by modelling them as large auction markets. More specifically, considering large enough markets, we study the equilibrium prices in the Assignment Game when buyers and sellers do not know the valuation of other. We model the problem as a three-stage game. First, nature draws the valuation of each agent, namely over the good they own for sellers, or the goods on sale for buyers. At stage two sellers simultaneously set prices, which are observed by buyers before stage three begins, when buyers report their valuation to the assignment mechanism. The payoffs are determined using the Top Trading Cover (TTCo) algorithm. We prove that the price vector at equilibrium is unique when markets satisfy the thickness condition and agents preferences are independent and follow an exponential distribution with the same parameter. This result is robust when we assume that preferences are independent but not identically distributed. Concretely, we extend our result to different parameters and overlapping valuations, assuming an exponential distribution.

Finally, in the last essay we study the subsidized housing problem in Paris. The French tradition in subsidized housing aims at "mixité sociale", namely promoting social diversity in all districts. A high population percentage in Paris is eligible to one of the many existing programs, promoting by a wide quantity of institutions, which is financially profitable to the programs. The allocation of a common pool of subsidized housing is solved by the institutions in Committees. The assignment process, thus, is not transparent, which raises criticisms over its discretionality. Our work focuses on designing a fair and efficient mechanism that incorporates different scoring schemes. We introduce the Nested Deferred Acceptance algorithm to get an assignment for this three sided market. However, it does not satisfy the "mixité sociale" condition nor fairness for the same type. We encompass these deficiencies by detecting institution acting as interrupters, and deleting them accordingly in the Efficiently Adjusted Deferred Acceptance Mechanism.

Chapter 2

An equilibrium analysis for the Top Trading Cover algorithm in large markets

2.1 Introduction

In Toluca, seats in public elementary schools are assigned through the Sistema Anticipado de Inscripción y Distribución (Anticipated System of Inscription and Distribution, SAID). The parents of six years old children submit a preference list of up to five elementary schools to the local Ministry of education; schools do not distinguish between kids and only can accept a limited number of kids determined by a quota. Schools run two rounds of classes; the quota is the number of available seats in the morning turn plus the available seats in the afternoon turn, which are not necessarily equal. Since schools are indifferent between students, this assignment problem differs from the School Choice problem because all students have the same priority over schools. The authority runs the SAID to determine the final allocation between students and schools. This mechanism has the following deficiencies[1]:

1. It does not deal with indifferences. This situation can be presented in the case of "twin schools", which are two public schools in the same block. Given their proximity, parents do not distinguish between them. The existence of these schools is due to the fact that years ago male and

[1] http://saidedomex.wordpress.com/

female children were separated in two different schools in the same block. Now, they are two mixed gender schools[2].

2. It is wasteful. A common situation after the publication of the final allocation is that parents find an available place in a preferred school to the school they have been assigned to.

3. It is not always Pareto efficient. Parents exchange seats in order to get a better school.

The school choice problem in Toluca consists of a finite set of students; a finite set of elementary schools, each student has a non-strict ranking of elementary schools and elementary school with a finite capacity.

We introduce the Top Trading Cover (TTCo) algorithm to deal with indifferences. The mechanism relies on a graph representation of the market, where a node is associated to all students and schools, and directed edges represent the top choices of students. A cover of the graph is selected by a tie breaking rule; each student is assigned to her partner in the cover. The procedure is iterated with the remaining students, analogically to David Gale's Top Trading Cycle; indeed TTCo also encompasses Hierarchical Exchange rules by Papai (2000).

We show that TTCo mechanism is characterized by: 1. Pareto efficiency, there is no other assignment that improves a student's allocation without harming other; and 2. the fact that it recursively respects top rankings: if a student is not assigned to her top ranked school, then this school is assigned to a student that also top ranks it, and this is true when one removes iteratively students assigned to one of their top ranked schools. Both properties are satisfied when we consider tie breaking rules that remove a top trading cover of maximum cardinality at each step. However, this tie breaking rules do not find all the Pareto efficient assignments.

Previous properties are useful to support the TTCo mechanism. For example, the existence of indifferences induces a loss of efficiency in the Student Optimal Stable Mechanism (Erdil and Ergin, 2008). Moreover, queue allocation rules (Svensson, 1994) do not recursively respect top rankings despite the fact that they deal with indifferences and they are Pareto efficient.

[2] http://portal2.edomex.gob.mx/edomex/temas/educacion/index.htm

Our characterization of the TTCo algorithm is related with the characterization of others mechanisms. The closest study to ours is Abdulkadiroglu and Che (2010), who provide a characterization of Top Trading Cycle mechanism through Pareto efficiency, strategy-proof and the fact that it recursively respects top ranking. Kojima and Manea (2010) characterize the Deferred Acceptance algorithm using non-wastefulness, population monotonicity and weak Maskin monotonicity. Kojima and Unver (2010) show that a mechanism coincides with the Boston mechanism if and only if the mechanism respects preference profile, resource monotonicity, and consistency. On the other hand, Papai (2000) study and characterize the hierarchical exchange rules, connecting the Serial Dictatorship and the TTC mechanism when agents have strict preferences.

Assuming non-strict preferences, Svensson (1994) characterizes the queue allocation rules with Pareto efficiency and strategy-proofness, when there are no property rights. Ehlers (2002) shows the existence of a unique maximal domain on which efficiency and coalitional strategy-proofness are compatible, moreover, mixed dictator-pairwise-exchange rules are the only rules that satisfy both properties on this domain. Later, Bogomolnaia, Deb and Ehlers (2005) extend the queue allocation rules to the case of private information introducing Bi-polar Serially Dictatorial Rules. They characterized these rules and discuss the consequences on equity when information is private.

Under the existence of property rights and indifferences, Alcalde and Molis (2011) and Jaramillo and Manjunath (2012) extend the Top Trading Cycles mechanism through the Top Trading Absorbing Sets and the Top Cycle Rules mechanism, respectively. Both mechanisms are Pareto efficient and strategy-proof.

Truth-telling, however, is not a dominant strategy for the TTCo. We extend the analysis on large matching markets, developed by Immorlica and Mahdian (2005) and Kojima and Pathak (2009), to the TTCo. In a simultaneous game where students report their preference list to the TTCo and only know their own schools valuations, we prove that truth-telling is an Epsilon Bayesian Nash equilibrium when market is sufficiently thick. There are crucial differences between our analysis and previous ones:

- Unlike in the Deferred Acceptance Algorithm (DA), the side of the market which manipulates the mechanism is the one that points to the favorite option.

- Fuhito and Pathak (2009) uses the rural hospital theorem to prove that dropping strategies are exhaustive is a general property of stable mechanisms. TTCo, however, is not stable; the argument used to establish exhaustiveness relies on the fact that TTCo is Pareto Efficient for all tie breaking rules.

- Students with more than one effective school have incentives to misreport their preference list. While previous works count schools using an algorithm based on reaction chains, we construct an algorithm based on dropping strategies that erase schools from the top, one-by-one, to analyze whether the most preferred school in this ranking can be an effective school when other students do not change their ranking.

- We use the thickness condition of Kojima and Pathak (2009), which specifies that for each school the number of students is balanced; the condition which implies that TTCo ends at the first iteration of the mechanism, when DA does not.

The paper is organized as follows. The model and the TTCo are presented in Sections 2 and 3. The characterization of this assignment procedure is carried out in Section 4. Finally, the equilibrium analysis under the thickness condition is done in Section 5.

2.2 The Model

2.2.1 Preliminary Definitions

We consider an economy with a set of students and a set of schools. Let $A = \{1, 2, \dots, n\}$ be the set of students, a generic student is denoted by i; and a set of elementary schools $O = \{\omega_1, \omega_2, \dots, \omega_m\}$, where ω denotes a generic elementary school, or simply school. Each school has a capacity of $q_\omega > 0$. First, we assume that each school has capacity $q_\omega = 1$. We consider that n and m are not necessarily equal, i.e. n can be less than, greater than or equal to m. A generic element in $A \cup O$ is represented by r.

We suppose that each student i has a preference relation represented by an utility $u_i(\cdot)$ over $O_i = O \cup \{i\}$. We assume the following quasi-linear utility function

$$u_i(r; \hat{v}_i) = \begin{cases} v_{ji} + \theta((v_{\tau i})_{\tau \neq j}) & \text{if } r = \omega_j \in O \\ 0 & \text{if } r = i. \end{cases}$$

Each student i has a valuation $v_{ji} \in \mathbb{R}$ of school ω_j for all $j \in \{1, 2, \ldots, m\}$. The type of student i is the vector $\hat{v}_i = (v_{1i}, \ldots, v_{mi})$. The set of all possible types of student i is denoted by $\hat{V}_i \subseteq \mathbb{R}^m$. The state of the market is the vector $v = (\hat{v}_1, \ldots, \hat{v}_n) \in \mathbb{R}^n$. The set of all possible states of the market is the Cartesian product between all sets $\hat{V}_i$; let $V \equiv \prod_{i=1}^{n} \hat{V}_i$. We suppose that the state of the market $v \in V$ is drawn according to a probability function f from V to $\mathbb{R}$, of common knowledge.

An school ω is *individually rational (IR)* for student i if $u_i(\omega) \geq u_i(i)$. From now on, we omit not *IR* schools from the representation of R_i, for all $i \in A$.

An *assignment* is a function γ from $A \cup O$ to $A \cup O$ such that

1. $\gamma(i) \in O_i = O \cup \{i\}$, for all $i \in A$,

2. $\gamma(\omega) \in A_\omega = A \cup \{\omega\}$, and

3. $\gamma(i) = \omega$ if and only if $\gamma(\omega) = i$.

In words, each student i is assigned exactly one school in O_i and each school ω is assigned exactly one student in A_ω, under an assignment γ. If $\gamma(r) = r$, r remains unassigned. The set of all assignments is denoted by Γ.

An assignment γ is *individually rational (IR)* if $u_i(\gamma(i)) > u_i(i)$ for all $i \in A$. In words, every student is assigned an *IR* school at γ.

We say that an assignment γ is *blocked* by a pair $(i, \omega) \in A \times O$ with respect to R if and only if

1. $\gamma(\omega) = \omega$, and

2. $u_i(\omega) > u_i(\gamma(i))$.

If γ is blocked by the pair (i,ω), we say that it is a *blocking pair* at R. An assignment γ is *non-wasteful* if and only if it is individually rational and it is not blocked by any pair. Non-wastefulness establishes than an student cannot be allocated with an school strictly preferred to her allocation because it has been assigned to some other student.

2.2.2 The two-step Game

We consider a two-step game. Nature moves first, determining the type of each student according to the probability distribution f. All students of the market observe their type, but do not observe the type of the others. At stage 2, students report a preference list of schools. Each student observes the schools in the market and sets her preference relation $R_i(\hat{v}_i)$, or R_i, over the set O_i. Naturally, students can be indifferent between different schools. This preference relation is complete, reflexive and transitive. Let P_i and $\sim_i$ represent the asymmetric and symmetric parts of R_i. For all $\omega,\omega' \in O_i$, we write $\omega P_i \omega'$ when student i strictly prefers ω to ω', or $u_i(\omega) > u_i(\omega')$; and $\omega \sim_i \omega'$ if i is indifferent between them, $u_i(\omega) = u_i(\omega')$. The *indifference class of* ω at R_i is the set $[\omega]_i = \{\omega' \in \ \mid \omega \sim_i \omega'\}$. If ω is the unique element in its indifference class at R_i, we consider that $[\omega]_i = \omega$. The preference list of student i is represented by

$$R_i : [\omega_1]_i, [\omega_2]_i, \ldots, [\omega_k]_i, [i]_i, [\omega_{k+1}]_i \ldots [\omega_K]_i.$$

A *preference profile* is a $n-$tuple of preferences, denoted by R. As in the usual way, let $R_{-i} = (R_j)_{j\neq i}$, $R_{A'} = (R_i)_{i\in A'}$ and $R_{-A'} = (R_i)_{i\in A-A'}$, for all $A' \subset A$. We denote by $\Re_i$ the set of all preference lists of student i, $\Re = \prod_{i\in A} \Re_i$ is the set of all possible preferences profile. Let $R_i(\omega)$ be the rank of school ω in R_i, i.e. $R_i(\omega) = k$ if and only if ω belongs to the $k-$th most preferred indifference class.

Given the profile R of reported preference profile, the final allocation of each student is determined by a fix mechanism (assignment procedure). A *mechanism* is a systematic procedure ϕ from $\Re$ to Γ that maps a preference profile into an assignment. The assignment generated by the mechanism ϕ at a preference profile R' is represented by $\phi[R']$. Let $r \in A \cup O$, the allocation of r under the assignment $\phi[R']$ is denoted by $\phi[R'](r)$. A mechanism ϕ is *IR* and *non-wasteful* if $\phi[R']$ is IR and non-wasteful for all $R' \in \Re$. To determine the allocation of each student, we introduce the *Top Trading Cover* algorithm to deal with indifferences. For all preference profile $R \in \mathfrak{R}$, this mechanism generates the assignment $TTCo[R]$.

2.2.3 Solution Concept

We need some extra concepts and notations. Given the realization of her type $\hat{v}_i$, an action of student i is a preference list R_i over the set O_i. A *decision rule/pure strategy* $\beta_i(\hat{v}_i)$ for student i is a function that maps types into preference lists.

Definition 1. Let $\beta^* = (\beta_1^*, \ldots, \beta_n^*)$ be a profile of strategies for students, and $\varepsilon > 0$. An $\varepsilon-$**Bayesian Nash equilibrium** is a profile of pure strategies $(\beta_1^*, \ldots, \beta_n^*)$ such that

$$E[u_i(TTCo[\beta_i^*, \beta_{-i}^*](i)] + \varepsilon \geq E[u_i(TTCo[\beta_i', \beta_{-i}^*](i)],$$

for all $i \in A$ and β_i' decision rule.

2.3 The Assignment Procedure

At the end of the game, the students payoffs are induced by the TTCo algorithm. In order to describe it, we must first introduce some concepts from Graph Theory.

2.3.1 Graph Theory Preliminaries

Consider a set of schools, $O' \subseteq O$; a set of students, $A' \subseteq A$; and a profile of preferences $R' = R_{A'}$. We define the **bipartite directed graph** $\overline{G}(A', O', R)$ as a pair $(\overline{V}(A', O', R), \overline{E}(A', O', R))$, where $\overline{V}(A', O', R) = A' \cup O'$ is the set of nodes; and $\overline{E}(A', O', R)$ is the set of all directed edges $(i, \omega) \in A' \times O'$, such that $(i, \omega) \in \overline{E}$ if and only if student ω is the most preferred school of i at R_i'. Since we consider non-strict preference lists, the most preferred school of student i is not necessarily unique. Thus, the fact that there is more than one edge from i to the set of schools represents indifference. We refer us to this bipartite graph only by $\overline{G} = (V(\overline{G}), E(\overline{G}))$ whenever there is no confusion.

Given that not all schools are *IR* for all students, it is possible that an student prefers to remain unassigned to be assigned with some school $\omega \in O$, i.e. $R_i(i) = 1$ and $R_i(\omega) > 1$ for all $\omega \in O$. In such situation, we use *loops*, which are pairs $(i,i) \in A \times A$. In words, a loop is an edge from the set of students to the set of students. Consequently, loops are not admitted in bipartite graphs, that is why our assignment procedure makes use of quasi-bipartite graphs. We define a **quasi-bipartite directed graph** $G(A',O',R)$ as a pair $(V(A',O',R),E(A',O',R))$, where $V(A',O',R) = A' \cup O'$ is the set of nodes; and $E(A',O',R)$ is the set of all directed edges $(i,\omega) \in A' \times O'$ and loops $(j,j) \in A' \times A'$, such that:

1. student i prefers school ω to any other school, and

2. student j prefers to remain unassigned to being assigned any school $\omega \in O$.

An arbitrary element of $E(A' \cup O',R)$ is denoted by $\vec{e}$. So, the quasi-bipartite graph G is the graph on $A' \cup O'$ which results when each student i in A' points to her most preferred element in A'_i. Whenever there is no confusion, a quasi-bipartite graph is denoted by $G = (V(G),E(G))$. In the following example we show the construction of a quasi-bipartite graph.

Example 2.3.1. Consider $O = \{\omega_1,\omega_2,\omega_3\}$, $A = \{1,2,3\}$. The preference relations are:

$$\begin{aligned} R_1 &: [\omega_1,\omega_3],\omega_2,1, \\ R_2 &: [\omega_2,2], \\ R_3 &: \omega_3,3. \end{aligned}$$

Figure 2.1 shows the quasi-bipartite graph $G(A,O,R)$.

2.3.2 Top Trading Covers and Tie Breaking Rules

Given a quasi-bipartite graph G, a **top trading cover (ttco)** is a subset $T = \{\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_k\}$ of $E[G]$, such that there are no two edges in T with a node in common. For example, the empty set and the set with a unique edge are top trading covers. Consequently, there always exist at least one ttco for all quasi-bipartite graph. A **maximal top trading cover** is a ttco of G that is no longer a ttco when an edge, not in it, is added to it. In other words, a maximal ttco is not a proper subset of any other ttco of the quasi-bipartite graph G. A **maximum top trading cover** is a ttco that covers the largest

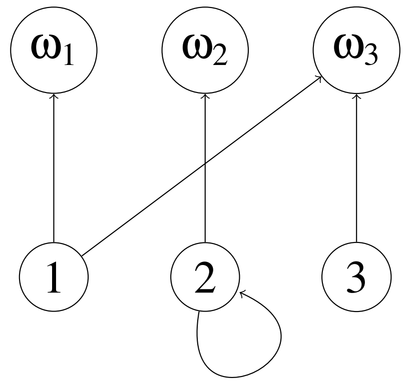
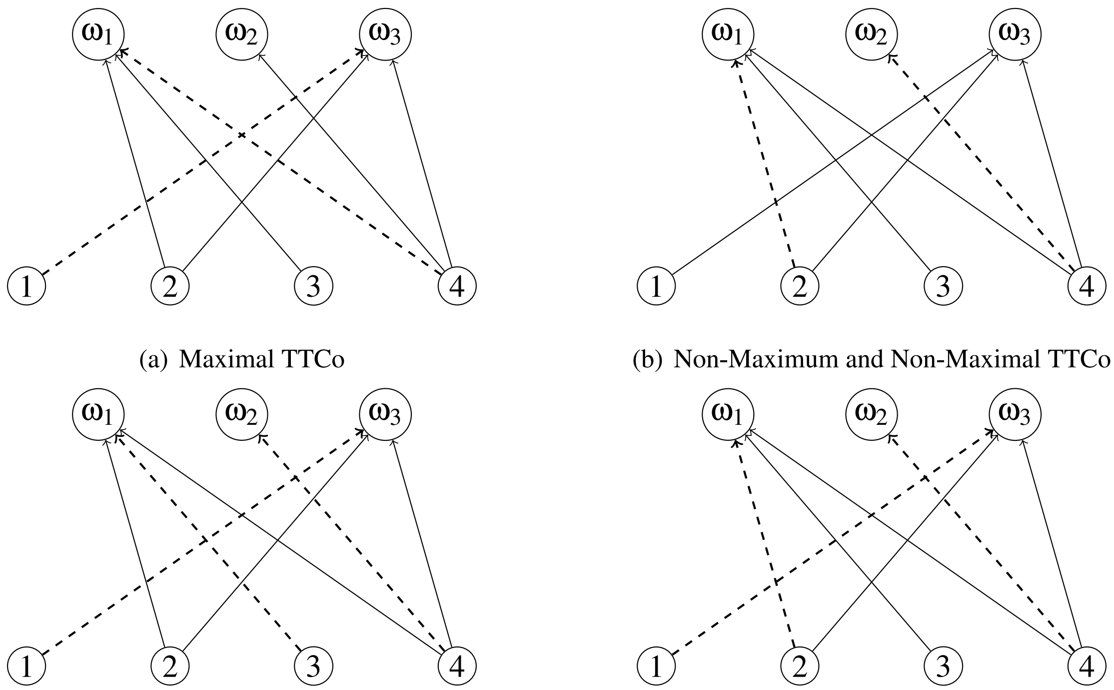


Figure 2.1: A Quasi-bipartite Graph



(a) Maximal TTCo

(b) Non-Maximum and Non-Maximal TTCo

(c) Maximum TTCo 1

(d) Maximum TTCo 2

Figure 2.2: Top Trading Covers in dotted lines for the same Quasi-Bipartite Graph

possible number of nodes. It is clear that all maximum top trading covers are also maximal; however, a maximal ttco is not always maximum.

Figure 2.2 shows four different top trading covers (in dotted lines) for the same quasi-bipartite graph. Note that all edges not in the ttco of Figure 2.2(a) have a node in common with it. Then, this ttco is a maximal ttco because adding an edge, not in it, the resulting subset of edges is not a ttco. Now, the ttco illustrated in 2.2(a) covers 4 nodes and the ttco in 2.2(c) covers 6 nodes, the largest possible of nodes than can be covered. Therefore, 2.2(c) is a maximum top trading cover and 2.2(a) is not. Finally, if we add the edge $(\omega_3, 1)$ to the ttco in 2.2(b), we get a larger top trading cover. Therefore, this ttco is non-maximum and non-maximal.

The existence of at least one maximum top trading cover is a well-known result from Graph Theory.

Proposition 2.3.1. *Let $G(A', O', R)$ a quasi-bipartite graph, a maximum $TTCo$ always exists.*

Proof. See Appendix A1. □

However, the maximum top trading cover is not unique. Figure 2.2(d) shows a ttco that covers the same number of nodes than the ttco in 2.2(c), but they differ in the edges $(\omega_1, 3)$ and $(\omega_1, 2)$. That is, we have two different maximum top trading covers for the same quasi-bipartite graph.

The non-uniqueness of the maximum ttco represents a problem for the Top Trading Cover algorithm. In each iteration of it, every student i points to her most preferred school ω, between the remaining schools in the market. After that, we remove a maximum top trading cover and each student is assigned its corresponding partner in the ttco removed. To choose only one maximum ttco we use *tie breaking rules*. Formally, consider a quasi-bipartite graph G and let $\Upsilon[G] = \{T \subseteq E[G] \mid T \text{ is a ttco of G}\}$ be the set of all top trading covers of G. The set of elements which are subsets of $\Upsilon[G]$ is denoted by $2^{\Upsilon[G]}$. A *tie breaking rule* is a function ς from $2^{\Upsilon[G]}$ to $\Upsilon[G]$ that chooses maximum top trading covers, for all quasi-bipartite graph G.

2.3.3 The Top Trading Cover Algorithm

Consider an economy (A, O, R) and a tie breaking rule ς. Initialize the algorithm with $A^0 = A$ and $O^0 = O$. The assignment procedure proceed as follows:

Step t: Every student in A^{t-1} points to her most preferred school between the schools in O^{t-1}. According to the tie breaking rule ς, we remove a maximum top trading cover of G^t, every student in the cover is assigned to its partner in the cover.

Let A^t and O^t be the sets of students and schools remaining in the economy after removing $\varsigma(\Upsilon[G^t])$. If both are non-empty, we continue to the next step. Otherwise, the algorithm stops.

The final allocation produced by the TTCo algorithm depends on the economy (A, O, R) and the tie breaking rule ς that we use. We sometimes write the final allocation as $TTCo[A, O, R\varsigma]$, and we denote by $TTCo[A, O, R, \varsigma](r)$ the assignment given to $r \in A \cup O$ under this assignment procedure. If there is no confusion we refer to the assignment only by $TTCo$.

To illustrate how the TTCo algorithm works, we use the "lexicographic tie-breaking rule". This rule is based on a lexicographic order defined over the edges in $E(G) \cap A' \times O'$ of the quasi-bipartite graph G.

Definition 2. Let $G = (V(G), E(G))$ be a quasi-bipartite graph, and $(i, \omega_j, i), (i', \omega_{j'}) \in E(G) \cap (A' \times O')$. The **lexicographic order** $\preccurlyeq_L$ over $E(G) \cap (A' \times O')$ is defined as follows

$$(i, \omega_j) \preccurlyeq_L (i', \omega_{j'}) \text{ if and only } \begin{cases} i < i' \text{ or} \\ i = i' \text{ and } j \leq j'. \end{cases}$$

We recall that students and schools are indexed by the set of natural numbers, a well-ordered set, which implies the veracity of the next observation.

Observation 2.3.1. Given a quasi-bipartite graph, let $\tilde{E}$ be a subset of $E(G) \cap (A \times O)$, then there exists an edge $(i, \omega) \in A \times O$ such that $(i, \omega) \preccurlyeq_L (i', \omega')$ for all $(i', \omega') \in \tilde{E}$. In other words, the minimum element of $\tilde{E}$ exists for all $\tilde{E} \subseteq E(G) \cap (A \times O)$. We write $(i, \omega) = \min \tilde{E}$.

By Proposition 2.3.1 we can ensure the existence of at least one maximum top trading cover regardless the quasi-bipartite graph. Thus, we can proceed to define the lexicographic tie-breaking rule over $\Upsilon[G]$ using Definition 2 and Observation 2.3.1.

Definition 3. Consider a quasi-bipartite graph G, and let $T = \{T_\eta \mid T_\eta \in \Upsilon[G]$ and is maximum for all $1 \leq \eta \leq K\}$ be a finite subset of maximum top trading covers. Consider $T'_\eta = T_\eta \cap (A' \times O')$ for all $\eta \in \{1, 2, \ldots, K\}$. The **lexicographic tie breaking rule** $\varsigma_{\mathbf{L}}$ is the function $\varsigma_L : 2^{\Upsilon[G]} \to \Upsilon[G]$ such that $\varsigma_L(T) = T_\kappa$ if and only if

$$\min \left[T'_\kappa \setminus \bigcap_{\eta=1}^{K} T'_\eta \right] \preccurlyeq_L \min \left[T'_t \setminus \bigcap_{\eta=1}^{K} T'_\eta \right],$$

for all $t \neq \kappa$.

We provide an example to show how the Top Trading Cover algorithm and the lexicographic tie breaking rule algorithm work.

Example 2.3.2. Consider the set of schools $O = \{\omega_1, \omega_2, \omega_3\}$, and the set of students $A = \{1, 2, 3, 4\}$. The preference list of each student is

$$\begin{aligned} R_1 &: [\omega_1, \omega_3], \omega_2, 1, \\ R_2 &: \omega_1, \omega_2, 2, \\ R_3 &: [\omega_1, 3], \\ R_4 &: \omega_1, \omega_2, \omega_3, 4. \end{aligned}$$

Figure 2.3 illustrates the step 1 of the TTCo algorithm. The maximum ttco that satisfies the lexicographic is shown in dotted lines. The other maximum ttco, not chosen, includes the edge $(4, \omega_1)$ instead of $(2, \omega_1)$. Therefore, the set $\{(2, \omega_1), (1, \omega_3), (3, 3)\}$ is removed from the market.

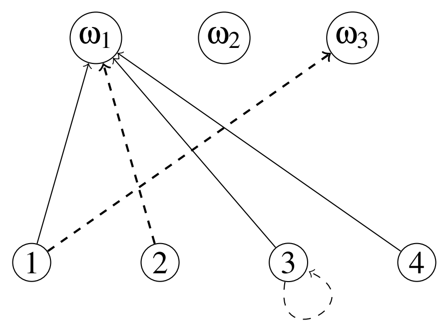


Figure 2.3: First step of the TTCo algorithm

Figure 2.4 shows the quasi-bipartite generated by the schools and students remaining in the market. Consequently, the maximum ttco chosen by ς_L is the unique edge $(4, \omega_2)$.

The algorithm finishes at step 2 because all schools and students are removed from the market at the end of this stage. Therefore, the final allocation is

$$TTCo = \begin{pmatrix} \omega_3 & \omega_1 & 3 & \omega_2 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Unless we specify some other tie breaking rule, in each example we will show the assignment produced by the the lexicographic tie breaking rule. However, it is important to remark that non-wastefulness is a property that satisfies the TTCo algorithm for all tie breaking rule ς. This is because

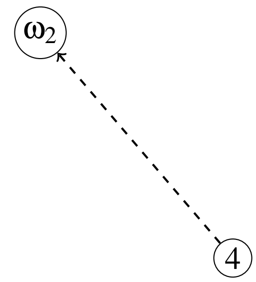


Figure 2.4: Second and last step of TTCo algorithm

the existence of a blocking pair contradicts the definition of maximum top trading cover, as we formalize in the following proposition.

Proposition 2.3.2. *Consider a market* (A,O,R) *and* ς *a tie breaking rule. The assignment*

$$TTCo[A,O,\varsigma]$$

is non-wasteful for all tie-breaking rule ς*.*

Proof. By construction, it is clear that $TTCo[R,\varsigma]$ is individually rational.
To prove that $TTCo[R,\varsigma]$ is not blocked by any pair, we proceed by contradiction. Suppose that this assignment is blocked by a pair $(i,\omega) \in A \times O$. Then, $TTCo[R,\varsigma](\omega) = \omega$ and $\omega \, P_i \, TTCo[R,\varsigma]$. We have the following cases.
Case I. student i remains unassigned under $TTCo[R,\varsigma]$, i.e. $TTCo[R,\varsigma](i) = i$. By assumption, student i strictly prefers ω to its allocation under this assignment. Then, there exists a step $t_i \in \mathbb{N}$ such that the pair (i,ω) is an edge of the graph G^{t_i}. Moreover, we have that

$$(i,\omega) \notin \varsigma(\Upsilon[G^{t_i}])$$

because we assume that $TTCo[R,\varsigma](\omega) = \omega$. Consequently, (i,ω) does not have nodes in common with the maximum ttco removed at step t_i. So, the set $\varsigma(\Upsilon[G^{t_i}]) \cup \{(i,\omega)\}$ is a ttco greater than $\varsigma(\Upsilon[G^{t_i}])$. This is a contradiction with the definition of maximum ttco.
Case II. Suppose that $TTCo[R,\varsigma](i) = \omega'$, then, the edge (i,ω') was removed from the market in some step t' of the TTCo algorithm, i.e. (i,ω') belongs to the maximum ttco $\varsigma[\Upsilon(G^{t'})]$. Also, we suppose that $\omega P_i \omega'$, this implies that (i,ω) must be an edge of G^t, for some $t \leq t'$. Moreover, (i,ω)

was not included in the maximum ttco $\varsigma(\Upsilon[G^t])$ because ω was not assigned to any student during the algorithm. Now, if $t = t'$, we have that (i,ω) and (i,ω') belong to the same graph G^t. By construction of G^t we conclude that $\omega \sim_i \omega'$, contradicting that (i,ω) is a blocking pair. Otherwise, $t < t'$, we have that $(i,\omega) \in G^t$ and $(i,\omega) \notin \varsigma(\Upsilon[G^t])$. Even more, the edge (i,ω) does not have nodes in common with the edges in the removed ttco. Consequently, the set $\varsigma(\Upsilon[G^t]) \cup \{(i,\omega)\}$ is a ttco greater than the ttco $\varsigma(\Upsilon[G^t])$, contradicting the definition of maximum ttco.
Therefore, the assignment $TTCo$ is not blocked by any pair regardless the tie breaking rule that we use. $\square$

2.3.4 Picking rules and Other mechanisms

The tie breaking rules are quite restrictive since they remove maximum top trading covers. Picking rules are more general functions than tie breaking rules. Given a quasi-bipartite graph G, a **picking rules** is a function $\hat{\varsigma}_G$ that chooses one and only one non-empty top trading cover of G.

In the following lemma we show that every Pareto efficient assignment γ can be gotten using the TTCo algorithm and a picking rule $\hat{\varsigma}_\gamma$ that never chooses an empty ttco.

Lemma 2.3.1. *If $\gamma[R]$ is a Pareto efficient assignment, there is some picking rule $\hat{\varsigma}_\gamma$ such that $\hat{\varsigma}_\gamma(T) \neq \emptyset$ for all $T \in 2^{\Upsilon[G]} \backslash \{\emptyset\}$, subset of top trading covers, and $\gamma[R] = TTCo[R, \hat{\varsigma}_\gamma]$.*

Proof. Let $T^\gamma = \{(i,r) \in A \times (A \cup O) \mid \gamma(r) = i\} \cup \{(i,i) \mid \gamma(i) = i\}$. The pairs in the set T^γ include all members in the market and do not have elements in common because γ is also a feasible assignment, i.e. each element in the economy is assigned one and only one element in $A \cup O$.
We partition T^γ in the following way:

- $T_1^\Gamma = \{(i,r) \in T^\Gamma \mid i \in A$ and r is the most preferred element of i in $R_i\}$.

- For all $k \geq 2$, define

$$T_k^\Gamma = \{(i,r) \in T^\gamma \mid i \in A \text{ and } r \text{ is the top ranked in } R_i \text{ after removing } \bigcup_{l=1}^{k-1} T_l^\gamma\},$$

  and

- Let $K \equiv \max\{k \in \mathbb{N} | T_k^{\gamma} \neq \emptyset\}$, the set of remaining students is $T_{K+1}^{\gamma} = T \setminus \left(\bigcup_{k=1}^{K} T_k \right)$.

Note that, since γ is individually rational, T_{K+1} is the set of schools that are assigned to themselves at the end of the TTCo algorithm.

We claim that each T_k^{γ} is a non-empty top trading cover of the graph G^k for all $k \geq 1$.

Affirmation 2.3.1. All sets T_k^{γ} are non-empty for all $1 \leq k \leq K$.

Proof. We proceed by induction.

Induction Base. By contradiction. Suppose that $T_1^{\gamma} = \emptyset$, then, any buyer is assigned her most preferred basket. We have the following cases:

Case I. There are i and i' such that $\gamma(i')$ is the most preferred school of i and $\gamma(i)$ is the most preferred school of i', consequently

$$\gamma(i') P_i \gamma(i) \text{ and } \gamma(i) P_{i'} \gamma(i'),$$

i.e. i strictly prefers the allocation of i' than her allocation and vice versa. Then, defining an assignment γ' such that

- $\gamma'(i) = \gamma(i')$ and $\gamma'(\gamma(i')) = i$,
- $\gamma'(i') = \gamma(i)$ and $\gamma'(\gamma(i)) = i'$,
- $\gamma'(r) = \gamma(r)$ for all $r \in (A \cup O) \setminus \{i, i', \gamma(i), \gamma(i')\}$,

we contradict the fact that γ is Pareto efficient.

Case II. There is an school ω such that $\gamma(\omega) = \omega$ and it is the most preferred school of some student i. Then, $\omega P_i \gamma(i)$. Consequently (i, ω) is a blocking pair. However, this is not possible because γ is Pareto efficient.

Therefore $T_1^{\Gamma} \neq \emptyset$.

Induction Hypothesis. Assume that $T_k^{\Gamma} \neq \emptyset$, for all $1 \leq k < K$.

Induction Step. We have to prove that $T_{k+1} = \emptyset$, for $k+1 < K+1$. We proceed by contradiction.

Suppose that $T^{\Gamma}_{k+1} = \emptyset$, then, applying the hypothesis of induction we have that

$$\begin{aligned} T^{\gamma}_{k+2} &= \left\{ (i,r) \in T^{\gamma} \mid i \in A \text{ and } r \text{ is the top ranked in } R_i \text{ after remove } \bigcup_{l=1}^{k+1} T^{\gamma}_{l} \right\} \\ &= \left\{ (i,r) \in T^{\gamma} \mid i \in A \text{ and } r \text{ is the top ranked in } R_i \text{ after remove } \bigcup_{l=1}^{k} T^{\gamma}_{l} \right\}. \end{aligned}$$

So, $T^{\gamma}_{k+2} = T^{\gamma}_{k+1} = \emptyset$. Analogously, we conclude that $T^{\gamma}_{k'} = \emptyset$ for all $k' > k+1$. Then $\max\{t \in \mathbb{N} \mid T^{\gamma}_{t} \neq \emptyset\} = k < K = \max\{t \in \mathbb{N} \mid T^{\gamma}_{t} \neq \emptyset$, which is not possible. Therefore $T^{\gamma}_{k+1} \neq \emptyset$. □

Now, define the picking function $\hat{\varsigma}_{\gamma} : 2^{\Upsilon[G]} \to \upsilon[G]$ as

$$\hat{\varsigma}(\Upsilon[G']) = \begin{cases} T^{\gamma}_{k} & \text{if } G' = G^{k}[A^{k}, O^{k}, R^{k}] \text{ for all } 1 \leq k < K-1, \\ T^{\gamma}_{K} \cup T^{\gamma}_{K+1} & \text{if } G' = G^{K}[A^{K}, O^{K}, R^{K}]. \end{cases}$$

Affirmation 2.3.1 guarantees that $\hat{\varsigma}_{\gamma}$ is a picking function that chooses the non-empty top trading cover T^{γ}_{k} in every step k of the algorithm. Moreover, it is clear that $\gamma(r) = TTCo[R, \hat{\varsigma}_{\gamma}](r)$ for all $r \in A \cup O$. □

The TTCo algorithm and other mechanisms

Assuming strict preferences, the top trading cover algorithm encompasses with hierarchical exchange rules. We describe the corresponding picking rules for the polar cases: the top trading cycles and the serial dictatorship mechanisms. Consider P a profile of strict preferences and G the quasi-bipartite graph induced by these preferences.

Top Trading Cycles. Assume that the initial endowment of each agent is described by a bijection $\mu : O \to A$. Note that a **cycle** is a ttco $C \subseteq E[G]$ if and only if for all $(i,\omega) \in C$ there exists $\omega' \in O$ such that $(\mu(\omega), \omega') \in C$. So, a **cycle picking rule** is a picking rule ς_C such that $\varsigma_C(\Upsilon[G])$ is a cycle.

Serial Dictatorship. Given an exogenous priority π of agents in A, the **serial picking rule** ς_S is defined as follows: $\varsigma_S(\Upsilon[G^t]) = \{(i,\omega) \in E[G^t] \mid i = \pi(t)\}$, for all $t = 1, \ldots, |A|$.

As we know, top trading cycle and serial dictatorship mechanism belong to a more general family of mechanisms: the hierarchical exchange rules. The TTCo algorithm encompasses with this family of mechanisms. To show it, we first describe the inheritance trees.

An **inheritance tree** $\Gamma_a = (V,\tilde{A})$ for object a is a **rooted tree** where V is the set of nodes, and $\tilde{A} \subseteq V \times V$ is the set of arcs (or edges) such that there is a path from v_i to v_j for all v_i, $v_j \in V$. The **root** of an inheritance tree is a node such that there is no incident arcs on it. An inheritance tree has the following properties

1. All nodes are students, $V \subseteq A$,
2. Every vertex of a path represent a different individual.
3. All arcs in $\tilde{A}$ are labelled by schools, that is to say, for all $(v_i, v_j) \in \tilde{A}$, this arc corresponds to a school in $O \setminus \{a\}$.
4. Every arc in a path represents a different school.
5. Arcs from the same node represent different schools.

Consider G_a the set of inheritance tree, $G = \times_{a \in O} G_a$, where $\Gamma \in G$ is a list of inheritance trees. When $A = O = \varnothing$, the *initial endowment* ξ_i^Γ of each student is given by

$$\xi_i^\Gamma = \{\omega \in O | i \text{ is the root in } \Gamma_a\}.$$

Let $\sigma_{T,P} : T \to P$ a bijection between T and P. Now, consider $T \subseteq A$, $P \subseteq O$ and a student $i \in A \setminus T$, the corresponding *non-initial endowment* $\xi_i^\Gamma(T, P, \sigma_{T,P})$ is given by

$$\xi_i^\Gamma = \{\omega \in O \setminus P | i \text{ is the root in } \Gamma_a \text{ or there exists a } v_0 - v_r \text{ path}$$

$$\text{such that } v_0 = i, (v_s, v_{s+1}) \in P, \phi(v_s) = (v_s, v_{s+1})\}.$$

The initial endowment is denoted by $E_1(i,R) = \xi_i^\Gamma(\varnothing)$, and a cycle at the first step is defined as follows:

$$S_1(i,R) = \begin{cases} \{j_1, \ldots, j_g\} & \text{if there exist } j_1, \ldots, j_g \in A \text{such that } R_{j_a}^{-1}(1) \in E_1(i,R), \\ \varnothing & \text{otherwise.} \end{cases} \tag{2.1}$$

Consequently, the set of *assigned students* is $W_1(R) = \{i : S_1(i,R) \notin \varnothing\}$, and the set of *assigned schools* is $F_1(R) = \{R_j^{-1}(1) : j \in W_1(R)\}$. Consequently, the non-endowments are defined as follows

$$E_t(i,R) = \xi_i^\Gamma(W_t(R), F^t(R), \phi_{W_t(R), F^t(R)}).$$

We define the hierarchical exchange rules as a picking rule ς_H such that $(i,\omega) \in \varsigma_H[G^t]$ if and only there exists in $(i',\omega') \in \varsigma_H[G^t]$ such that $\omega \in E_t(i',R)$. In other words, in a hierarchical picking rule me remove a top trading cover if and only the owner of each object is also in the ttco, according to the endowment $E_t(i',R)$.

Now, consider R a profile of non-strict preference list, Alcalde and Molis (2011) propose the Top Trading Absorbing Sets mechanisms as a generalization of the Top Trading Cycles mechanism. To describe this mechanism through picking rules, we need the following concepts.

Let $i,\omega_j \in V[G]$, we say that there is a **path** from the student i to the school ω_j if there is a sequence of nodes $v_1 = i,\dots,v_{2m} = \omega_j$ that satisfies two conditions: 1. $v_{2k-1} \in A$ for all $k \in \{1,\dots,m\}$, $v_{2k} \in O$ for all $k \in \{1,\dots,m\}$; and 2. $(v_{2k-1},v_{2k}) \in E[G]$ and $\mu(v_{2k}) = v_{2k+1}$ for all $k \in \{1,\dots,m-1\}$. Analogously, we define a path between a school ω_j and a student i. The pair i,ω_j is **symmetric** if $(i,\omega_j) \in E[G]$ and $\mu(\omega_j) = i$.

An **absorbing set** is a set of nodes $A \subseteq\in V[G]$ such that

1. for any two nodes $i,\omega_j \in A$ there is a path from one to the other,

2. there is no path from any node $r \in A$ to any node $r' \notin A$.

An observing set is **paired symmetric** if each of its nodes belongs to a symmetric pair.

The **absorbing rule** is a picking rule ς_{AS} that chooses a collection of paired symmetric absorbing sets. In the corresponding $TTC[\varsigma_{AS}]$ algorithm, the quasi-bipartite graph G^{t+1} actualizes the endowments of each agent not in a paired-symmetric absorbing set. Every agent in a non paired-symmetric absorbing set point to the maximal object with the highest priority, different from her initial endowment. Each agent in a cycle changes her endowment for the object that she is pointing to.

2.4 Characterization

The objective of this section is to present the axioms that characterize the Top Trading Cover algorithm. We get a characterization through Pareto efficiency and the Recursively Respect Top Preference axiom, introduced by Abdulkadiroglu and Che in [1] as the fact that an assignment Recursively Respects Top Rankings.

2.4.1 Pareto Efficiency

An assignment γ is *Pareto efficient* if there is no other assignment γ' such that $\gamma(i) \sim_i \gamma(i)$ for almost $i \in A$, and there exists at least one i such that $\gamma'(i) P_i \gamma(i)$.

Proposition 2.4.1. *Let γ be a Pareto efficient assignment, then γ is IR and non-wasteful.*

Proof. Let γ be a Pareto efficient assignment. To prove that γ is IR, we proceed by contradiction, so we assume that γ is not IR. Consider $A^- = \{i \in A \mid iP_i\gamma(i)\}$, the set of students assigned to some not *IR* school. Since γ is not IR, then $A^- \neq \varnothing$.

Now, define the assignment γ' in the following way:

$$\gamma'(r) = \begin{cases} \gamma(r) & \text{if } r \in A \setminus A^- \text{ or } \gamma(r) \in A \setminus A^-, \\ r & \text{if } r \in A^- \text{ or } \gamma(r) \in A^-, \\ r & \text{if } r \in O \text{ and } \gamma(r) = r. \end{cases}$$

It is clear that $\gamma'(i) R_i i$ for all $i \in A$. Then, γ is *IR*.

Consequently, by definition of γ' we have that

$$\gamma'(i) = \gamma(i) \text{ for all } i \in A \setminus A^-,$$

and, by individual rationality, we conclude that

$$\gamma'(i) P_i \gamma(i) \text{ for all } i \in A^-.$$

This is a contradiction because γ is Pareto efficient. Therefore, γ is IR.

Also, to prove that γ is wasteful, then there exists a blocking pair (i, ω) such that

$$\gamma(s) = s \text{ and } sP_i\gamma(i).$$

Define the assignment γ' as follows

$$\gamma'(r) = \begin{cases} \gamma(r) & \text{if } r \in (A \cup O) \setminus \{i, s\}, \\ s & \text{if } r = i, \\ i & \text{if } r = s. \end{cases}$$

Since (i,s) is a blocking pair, the definition of γ' implies that

$$\gamma'(j) = \gamma(j) \text{ for all } j \in A \setminus \{i\}, \text{ and } \gamma'(i) P_i \gamma(i).$$

This is a contradiction because γ is Pareto efficient. Therefore γ is non-wasteful. $\square$

Now, we prove that every tie breaking induces a Pareto efficient assignment.

Proposition 2.4.2. *Let ς be a tie breaking rule. The assignment $TTCo[R,\varsigma]$ is Pareto efficient.*

Proof. We proceed by contradiction. Suppose that there exists an assignment γ such that $\gamma(i) \sim_i TTCo[R,\varsigma](i)$ for almost $i \in A$, and there exists at least one j such that $\gamma(j) P_j TTCo[R,\varsigma](i)$. Consider that $TTCo[R,\varsigma] = \cup_{k=1}^{K} TTCo^k$, where $TTCo^k$ is the maximum ttco removed at step t of the TTCo algorithm. Since $\gamma(j) P_j TTCo[R,\varsigma]$, for at least one $j \in A$, then $(j, \gamma(j)) \notin TTCo^k$ for some $k \in \{1,2,\ldots,K\}$. Let k^* be the minimum k such that some j improves under γ. We know that other students in $TTCo^{k^*}$ are indifferent between γ and $TTCo[R,\varsigma]$. This implies that $TTCo^{k^*}$ is not a maximum ttco, which is not possible by the definition of ς. Therefore $TTCo[R,\varsigma]$ is Pareto efficient for all tie breaking rule ς. $\square$

2.4.2 Recursively Respects Top Preferences

An assignment γ *respects top preferences* if for each $\omega \in O$ and $i \in A$ such that $R_i(\omega) = 1$ and $\omega P_i \gamma(i)$ implies that

$$R_{\gamma(\omega)}(\omega) = 1.$$

In words, if an student is not assigned some of her most preferred schools, this implies that her most preferred schools are assigned to students who also put these schools as their most preferred schools.

All tie breaking rules generate an assignment that respects top preferences.

Proposition 2.4.3. *$TTCo[\varsigma]$ respects top preferences for all tie breaking rule ς.*

Proof. Consider $(i,\omega) \in A \times O$ such that $R_i(\omega) = 1$ and $\omega P_i \gamma(i)$. Then

$$(i,\omega) \in E[G^1] \text{ and } (i,\omega) \notin \varsigma[\Upsilon(G^1)].$$

Then, by definition of maximum top trading, there is $i' \in A$ such that

$$(i', \omega) \in E[G^1] \text{ and } (i', \omega) \in \varsigma[\Upsilon(G^1],$$

otherwise ς did not remove a maximum ttco which contradicts the fact that it is a tie breaking rule. Consequently, there is i' such that $i' = \gamma(\omega)$ and $R_{i'}(\omega) = 1$. Therefore $TTCo[\varsigma]$ respects top preferences for all tie breaking rule. $\square$

The TTCo mechanism is related with the *Gales Top Trading Cycles (TCC)* mechanism. Both mechanisms respect top preferences, and remove a set of pairs (students-schools) at the end of its iteration. Both properties are crucial to describe the TTCo algorithm when we use a tie breaking rule.

If γ respects top preferences, then $\varsigma_\gamma(\Upsilon[G^1])$ is a maximum ttco. However, this axiom says nothing about other iterations.

Abdulkadiroglu and Che (2010) made a characterization of the Top Trading Cycle mechanism for the assignment of indivisible schools between students. In this work, they introduced the axiom *recursively respect top rankings* (top preferences in our case) to explain how the TTC mechanism removes a set of students in each iteration. Before introduce this axiom, note that respecting top preferences axiom leads to the following concept.

Definition 4. A *top preference group* is a subset $A_0 = \{a_1, a_2, \ldots, a_k\} \subseteq A$ such that there is $\omega_i \in O$ such that

$$\gamma(a_i) = \omega_i \text{ and } R_{a_i}(\omega_i) = 1 \text{ for all } 1 \leq i \leq k.$$

A top preference group is a set of students assigned to its most preferred school.

Let γ be an assignment and $A_0 \subseteq A$ a top ranked group of the initial economy $E = (A, O, R)$. The *sub-economy induced* by A_0 is the economy $(A \setminus A_0, O^{\gamma}_{A \setminus A_0}, R_{A\setminus})$, to simplify, we sometimes represent the induced economy by $E \setminus A_0$. In words, the induced economy $E \setminus A_0$ is the economy where all the students in A_0 left the initial economy with her allocations under the assignment γ. As in [1], we use this elements to generalize the idea of respecting top preferences in a recursive form.

Let $E^0 = (A^0, O^0, R^0) = (A, O, R)$ be the original economy and consider an assignment γ. We denote by E^t the sub-economy induced by a top preference group A^t_0 of the economy E^{t-1}, for all $t \geq 1$.

Definition 5. We say that the assignment γ **recursively respects top preferences** if $\gamma[E^t]$ respects top preferences for all $t \geq 0$, regardless the sequence of top ranked groups $\{A_0^t\}_{t\geq 0}$.

In words, an assignment recursively respects top preferences if it respects top preferences in any sub-economy induced by a top preference group and this goes recursively in each sub-economy. Moreover, it is important to note that this must be true regardless the top preference group initially chosen.

We say that $R_i(\omega; E^t) = k$ if and only if $(i, \omega) \in A^t \times O^t$ and ω is the $k-$th most preferred student in O^t with respect to R_i.

We have already proved that every Pareto efficient assignment can be gotten using a picking rule that never chooses the empty top trading cover, see Lemma 2.3.1. Before to establish our characterization of the TTCo algorithm and tie breaking rules, we require an extra lemma.

The following lemma shows a sufficient condition to choose a maximum top trading cover in the first step of the TTCo algorithm. That is to say, if an IR assignment respects top preferences, then the corresponding picking rule chooses a maximum ttco.

Lemma 2.4.1. *Let γ be an IR assignment that respects top preferences, then there is a picking rule $\hat{\varsigma}$ such that $\gamma = TTCo[\hat{\varsigma}]$ and $\hat{\varsigma}[\Upsilon(G^1)]$ is a maximum ttco.*

Proof. By Lemma 2.3.1, there exists $\hat{\varsigma}_\gamma$ such that $\gamma = TTCo[\hat{\varsigma}_\gamma]$. We have to prove that

$$\hat{\varsigma}_\gamma[\Upsilon(G^1)]$$

is a maximum ttco.

By construction, $\hat{\varsigma}_\gamma[\Upsilon(G^1)] = \{(i, \gamma(i)) \mid R_i(\gamma(i)) = 1\}$.

We proceed by contradiction. Suppose that it is not a maximum ttco, then there is $(i, r) \in E[G^1]$ such that

$$(i, r) \notin \hat{\varsigma}_\gamma[\Upsilon(G^1)],$$

and (i, r) does not have nodes in common with $\hat{\varsigma}_\gamma[\Upsilon(G^1)]$.

Case I. If $r = i$, then $\hat{\varsigma}_{\gamma}[\Upsilon(G^1)] \cup \{(i,i)\}$ is a larger ttco than $\hat{\varsigma}_{\gamma}[\Upsilon(G^1)]$, which is not possible by construction of ς_{γ}.

Case II. If $r = \omega \in O$, we know that $R_i(\omega) = 1$ and $\gamma(i) \neq \omega$. Then there is $r' \in (A \backslash \{i\}) \cup \{\omega\}$ such that $\gamma(\omega) = r'$.

II.A If $r' = \omega$, this means that ω remains unassigned under γ and $\omega P_i \gamma(i)$ because i is not removed in the first step (remember that (i,r) does not have nodes in common with the ttco removed). This is a contradiction because γ respects top preferences.

II.B If $r' = j \in A \backslash \{i\}$, then $R_j(\omega) \geq 1$. If $R_j(\omega) = 1$, the fact that γ respects top preferences implies that (j,ω) belongs to $\hat{\varsigma}_{\gamma}(\Upsilon[G^1])$ which contradicts that (i,ω) does not have nodes in common with the ttco. If $R_j(\omega) > 1$ we contradict that γ respects top preferences.

In any case, we get a contradiction. Therefore $\hat{\varsigma}_{\gamma}(\Upsilon[G^1])$ is a maximum top trading cover. □

The Theorem establishes a characterization of the TTCo algorithm when it uses a tie breaking rule.

Theorem 2.4.1. *Given an economy (A, O, R), a mechanism ϕ is Pareto efficient and recursively respects top preferences if and only there is a tie breaking rule ς_{ϕ} such that $\phi[R] = TTCo[R, \varsigma]$.*

Proof. We have proven that $TTCo[\varsigma]$ is Pareto efficient and respects top preferences in propositions 2.4.2 and 2.4.3 for all tie breaking rule ς, respectively. We have to prove that $TTCo[\varsigma]$ recursively respects top preferences, i.e. we have to prove that $TTCo[E^t]$ respects top preferences for all $t \geq 0$. We proceed by induction.

Base of Induction. Consider A_0^1 a top ranked group of E^0. We have to prove that $TTCo[E^1, \varsigma]$ respects top preferences. Let $(i,\omega) \in A^1 \times O^1$ such that

$$R_i(\omega; E^1) = 1 \text{ and } \omega P_i TTCo[E^1, \varsigma](i).$$

Then, we have that

$$(i,\omega) \in G^1[E^1] \text{ and } (i,\omega) \notin \varsigma(\Upsilon[G^1[E^1]]).$$

Consequently, there exists $j \in A^1 \setminus \{i\}$ such that $(j,\omega) \in \varsigma(\Upsilon[G^1[E^1]])$. Otherwise, $(i,\omega) \cup \varsigma(\Upsilon[G^1[E^1]])$ is a top trading cover greater than $\varsigma(\Upsilon[G^1[E^1]])$, which is not possible because ς is a tie breaking rule. Since $j \in A^1$ and $G^1[E^1]$ is the quasi-bipartite graph where all students in A^1 point to her most preferred school in E^1, we conclude that $R_j(\omega; E^1) = 1$.

Hypothesis of Induction. Let $k \in \mathbb{N}$. Consider A_0^t a top ranked group of E^{t-1} for all $t = 0, 1, \ldots, k$. Suppose that $TTCo[E^k, \varsigma]$ respects top preferences.

Induction Step. Let A_0^{k+1} a top ranked group of E^k. We have to prove that $TTCo[E^{k+1}, \varsigma]$ respects top preferences. The proof is equal to the done in the Base of Induction.

Therefore, $TTCo[\varsigma]$ recursively respects top preferences for all tie breaking rule ς.

Now, let γ be a Pareto efficient assignment that recursively respects top preferences. We have to prove that $\gamma = TTCo[\varsigma]$ for some tie breaking rule ς.

In Lemma 2.3.1 we proved the existence of a picking rule ς_γ such that $\gamma = TTCo[\varsigma_\gamma]$. Even more, we know that $\varsigma(\Upsilon[G^k]) = T_k^\gamma$ is a non-empty top trading cover for all $k = 1, 2, \ldots, K$. Then, we have to prove that every ttco T_k^γ is a maximum ttco. We proceed by induction.

Base of Induction. Since γ recursively respects top preferences, particularly, γ respects top preferences. Therefore, $\varsigma_\gamma(\Upsilon[G^1]) = T_1^\gamma$ is a maximum top trading cover by Proposition 2.4.1.

Hypothesis of Induction. For $1 \leq k < K$, suppose that T_k^γ is a maximum ttco of G^k.

Induction Step. We have to prove that T_{k+1}^γ is a maximum ttco of G^{k+1}. By hypothesis of induction, we know that each ttco T_k^γ is a maximum ttco.

Note that T_1^γ is also a top preference group of the economy E^0 with maximum cardinality. If $E^1 = E^0 \setminus T_1^\gamma$, the construction of ς_γ implies that T_2^γ is a top preference group of the economy E^1 because T_1^γ is a maximum ttco. So, consider the sequence of economies E^t such that $E^t = E^{t-1} \setminus T_t^\gamma$.

Since γ recursively respects top preferences, we have that γ respects top preferences at economy E^{K-1}. By construction, T_K^γ is a top preference group of the economy E^{K-1}. Then, applying Lemma 2.4.1, we conclude that T_K^γ is a maximum ttco of $G[E^{K-1}]$.

So, any Pareto efficient assignment that recursively respects top preferences is generated by the TTCo mechanism through a tie breaking rule. $\square$

2.5 Equilibrium Analysis

The objective of this section is to show the existence of a unique $\varepsilon-$Bayesian Nash equilibrium, where all students are truth-telling, in large markets that satisfy the thickness condition. Remember, an $\varepsilon-$Bayesian Nash equilibrium of the game described above is a profile of decision rules such that in expectation no student has large incentives to deviate from it. In other words, no single student i can improve significantly her final allocation $TTCo[\beta^*, \varsigma](i)$ at equilibrium.

We search the best response correspondence of each student following the methodology developed by Immorlica and Mahdian (2005), and extended by Kojima and Pathak (2009). Since the TTCo algorithm is manipulable, we simplify the searching of the best responses analysing *dropping strategies*, and show that students have incentives to deviate whenever they have more than one effective school. In large markets, the proportion of students with more than one effective school tends to zero. Finally, the *thickness condition* ensures the existence of a unique $\varepsilon-$Bayesian Nash equilibrium where all students are truth-telling.

2.5.1 Dominant Strategies in the TTCo algorithm

A preference list R_i^* is a **dominant strategy** for student i if and only if

$$\phi[A, O, (R_i^*, \overline{R}_{-i})](i) R_i \phi[A, O, (R_i', \overline{R}_{-i})](i),$$

for all $R_i' \in \Re_i$ and for all $\overline{R}_{-i} \in \prod_{j \neq i} \Re_j$. An student i is **truth-telling** if her true preference list is a dominant strategy. We use R_i to denote the true ranking of i; if i is not truth-telling, she manipulates the TTCo algorithm.

Definition 6. An student i **manipulates the** ϕ at $\overline{R}$ through R_i' if there exists R_i' in $\Re_i$ such that

$$\phi[A, O, (R_i', \overline{R}_{-i})](i) P_i \phi[A, O, (R_i, \overline{R}_{-i})](i).$$

If an student manipulates the mechanism, the mechanism is said to be manipulable.

Recall that the $TTCo$ assignment depends on: a set of students, A, a set of schools, O, and the reported profile of preference lists $\overline{R}$. We define a **market** as a tuple $\aleph = (A, O, \overline{R})$.

Observation 2.5.1. The TTCo mechanism is manipulable.

The above observation is illustrated in the example below.

Example 2.5.1. Consider the market given in Example 2.3.2, students true preferences are

$$\begin{aligned} R_1 &: \omega_1, \omega_2, 1 \\ R_2 &: \omega_1, \omega_2, 2 \\ R_3 &: \omega_1, \omega_2, \omega_3, 3 \\ R_4 &: \omega_1, \omega_2, \omega_3, 4. \end{aligned}$$

Given the profile R of true preferences, the assignment produced by the TTCo algorithm under the market (S,B,R) is:

$$\begin{aligned} TTCo(1) &= \omega_1 \\ TTCo(2) &= \omega_2 \\ TTCo(3) &= \omega_3 \\ TTCo(4) &= 4. \end{aligned}$$

Now, suppose that student 4 reports

$$R'_4 : \omega_3,$$

while other students report their true preferences. Let $R' = (R_1, R_2, R_3, R'_4)$, the $TTCo$ algorithm under R' outputs the following assignment:

$$\begin{aligned} TTCo[R'](1) &= \omega_1 \\ TTCo[R'](2) &= \omega_2 \\ TTCo[R'](3) &= 3 \\ TTCo[R'](4) &= \omega_3. \end{aligned}$$

Since student 4 prefers ω_3 to remain unassigned, she manipulates the mechanism at (R_1, R_2, R_3) through R'_4. $\square$

Indeed, truth-telling is neither a dominant strategy for students nor a Nash equilibrium in *small markets*.

2.5.2 Dropping Strategies

To establish that truth-telling is an $\varepsilon-$Nash equilibrium in large markets, we show first that dropping strategies cover all the range of best responses, i.e. dropping strategies are exhaustive.

Definition 7. A preference list R'_b is said to be a *dropping strategy* for the TTCo at true preference list R_b if

1. $\omega R'_b \omega'$ then $\omega R_b \omega'$, and
2. $b R_b \omega$ then $b R'_b \omega$.

That is to say, a dropping strategy is a preference list that 1. drops some elements and 2. respects the order in R_b.

In [15], Kojima and Pathak showed the exhaustiveness of dropping strategies in their Lemma 1: "All manipulations of the Student Optimal Stable Mechanism (SOSM) can be gotten through a dropping strategy when others report true preferences". Even more, they proved that this is a general property of any stable mechanism. We establish a similar result for the family of non-wasteful assignments $TTCo[\varsigma]$ for all tie breaking rule ς. With respect to the proof of Kojima and Pathak, we restrict our attention to dropping strategies which drop all the schools preferred to the one assigned by the TTCo algorithm, denoted by $R_b^{TTCo(b)}$. That is to say, if student b reports a strategy

$$R'_b : [\omega^1]_b, [\omega^2]_b, \ldots, [TTCo(b) = \omega^k]_b, \ldots, [b]_b,$$

the corresponding dropping strategy $R_b^{TTCo(b)}$ is defined as follows

$$R_b^{TTCo(b)} : [TTCo(b) = \omega^k]_b, \ldots, [b]_b.$$

Lemma 2.5.1. *(Dropping strategies are exhaustive) Fix an student $b \in A$. Suppose that b reports $R'_b \in \Re_b$, and all the other students report any ranking R'_{-b}. Consider that the assignment γ is equal to $TTCo[R'_b, R'_{-b}; \varsigma]$, that is to say $TTCo[R'_b, R'_{-b}; \varsigma](i) = \gamma(i)$ for all $i \in A$. Then*

$$TTCo[R_b^{\gamma(b)}, R'_{-b}; \varsigma](b) = \gamma[R'_b, R'_{-b}],$$

for all tie breaking rules ς.

Proof. Specifically, we prove that $TTCo[R_b^{\gamma(b)}, \tilde{R}_{-b}; \varsigma](b) = \gamma(b)$ when $\gamma(b)$ is declared as the unique most preferred school of b at $R_b^{\gamma(b)}$, i.e. $R_b^{\gamma(b)} : [\gamma(b)]_b, \ldots, [b]_b$.

We have that $\gamma(b) = TTCo[R'_b, R'_{-b}](b) = \omega$. By construction of the TTCo algorithm, there is $\tau \in \mathbb{N}$, such that

$$(b, \omega) \in \varsigma_L(\Upsilon[G^\tau]).$$

Then, all student b' that points to ω are not removed in any step $t = 1, \ldots, \tau$. So, edges (b', ω) were never removed from the market.

Case I. If $\tau = 1$, then $TTCo[R_b^{\gamma(b)}, R'_{-b}] = TTCo[R'_b, R'_{-b}]$.

Case II. If $\tau \neq 1$. Running the TTCo algorithm with the profile $(R_{-b}^{\gamma(b)}, R'_{-b})$, in its first step we have that

$$b \in D^1_\omega(p) = \{i \in A \mid \omega \text{ is the most preferred school in } (R_b^{\gamma(b)}, R'_{-b})\}.$$

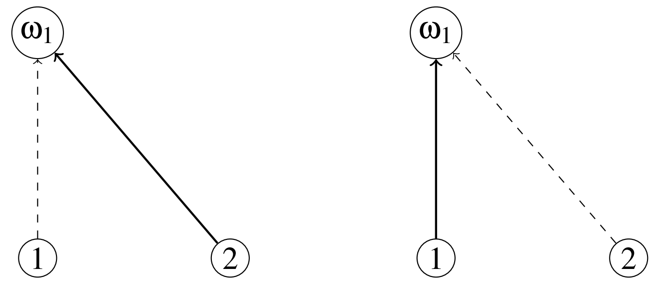
The only change in the algorithm is that $b \in D^1_\omega(p)$, which did not happen under (R'_b, R'_{-b}). Remember that any tie breaking rule selects a maximum ttco at any step of the mechanism. Since ω is the unique most preferred school of b, the edge (b, ω) does not have nodes in common with other edges in $\varsigma(\Upsilon[G^1[R'_b, R'_{-b}]])$. Therefore, (b, ω) must belong to $\varsigma(\Upsilon(G^1[R_b^{\gamma(b)}, R'_{-b}]))$, so $TTCo[R_b^{\gamma(b)}, R'_{-b}](b) = \omega$. $\square$

We extend Lemma 1 of Kojima and Pathak to the family of non-wasteful assignments $TTCo[\varsigma]$, for all tie breaking rule ς. In general, non-wasteful assignments do not satisfy the "Rural Hospital Theorem" (Roth (1984)), which is a central argument in Lemma 1 of [15]). Figure (2.5) shows this fact, the set of single agents is not the same in two different non-wasteful assignments.

2.5.3 Counting Effective schools

Below, we introduce the tools that we use to prove that no student has incentives to deviate from her true ranking in large markets.

Given an assignment Γ, we recall that $\Gamma_\omega[\overline{R}](b) \in O \cup \{\varnothing\}$ is the good assign to b at $\Gamma[\overline{R}]$, for some $\overline{R} \in \mathfrak{R}$. We say that $[\omega]_b = \{\omega' \in O | \omega' \sim_b \omega\}$ is *effective* for the student b if and only if there is a



(a) Non-Wasteful Assignment 1

(b) Non-Wasteful Assignment 2

Figure 2.5: Dotted lines represent two different non-wasteful assignments. These assignments have different sets of unmatched members, so, the “Rural Hospital” Theorem does not hold for our non-wasteful assignment problem.

strategy R'_b such that $\Gamma[R'_b, \tilde{R}_{-b}] \in [\omega]_b$. If there is no preference list $R''_b \in R_b$ such that $\Gamma[R''_b, \tilde{R}_{-b}] \in [\omega]_b$, we say that $[\omega]$ is a *non-effective class*.

Observation 2.5.2. Student b has not incentive to deviate from her true ranking if and only if $TTCo[R'_b, \tilde{R}_{-b}](b) = TTCo[R_b, \tilde{R}_{-b}](b)$, for all dropping strategies

$$R'_b : [\omega^1], [\omega^2], \ldots, [TTCo[R_b, \tilde{R}_{-b}](b)], \ldots [b].$$

The Waiting Algorithm

The importance of Observation 2.5.2 lies in changing our original problem: instead of searching best response correspondences it is sufficient to count how many effective schools each student has. In this section we explain how to count effective schools.

Immorlica and Mahdian (2005) and Kojima and Pathak (2009) also change the original problem into a counting problem. They count stable partners[3] using an algorithm based in *reaction chains*. However, we cannot use reaction chains to count effective schools because our game only considers that one side of the market reports a ranking of the members of the other side of the market.

Example 2.5.1 shows that TTCo is manipulable and Lemma 2.5.1 implies that all profitable manipulations can be gotten through dropping strategies where the corresponding effective school is declared

[3] In papers [11] and [15] the deviations are identified with stable husbands/students, which are equivalent to the effective schools in our model.

as the most preferred school. Based on these observations we construct the *Waiting Algorithm (WA)* to count all the effective schools preferred to $TTCo[R](b)$.

The WA checks if student b can get a school preferred to $TTCo[R](b)$ using dropping strategies while other students are truth-telling. These dropping strategies are truncating $\tilde{R}_b$ in a descending form.

The Waiting Algorithm

Fix a student b, the **Waiting Algorithm** is described below:

Step 0. *Initialization*: Consider $R_b^0 = R_b$ the ranking reported by b, and $X_b^0 = \varnothing$, the set of effective schools of b before starting the assignment procedure. Run the $TTCo$ algorithm with the profile of true preferences $R = (R_b^0, \tilde{R}_{-b})$.

Step t.

1. *Effective schools*: If $TTCo[R_b^{t-1}, \tilde{R}_{-b}](b) \in S$, the WA actualizes the set of effective schools:

$$X_b^t = X_b^{t-1} \cup \{[TTCo[R_b^{t-1}, \tilde{R}_{-b}](b)]\}.$$

2. *Preference actualization*:

Case A. If the dropping strategy R_b^{t-1} is the strategy where the original assignment

$$[TTCo[R_b^0, \tilde{R}_{-b}](b)]$$

is the most preferred school, then the algorithm stops.

Case B. Otherwise, consider the dropping strategy R_b^t as the strategy where the most preferred indifference class in R_b^{t-1} is dropped. Run the $TTCo$ algorithm with the profile of preferences $R^t = (R_b^t, \tilde{R}_{-b})$. Go to $t.1$.

In the step $t.2.B$, we construct the dropping strategy R_b^t by dropping simultaneously all the elements of the most preferred indifference class in R_b^{t-1}. We can construct this dropping strategy deleting school by school from the most preferred class of indifference at R_b^{t-1}. For example, consider

$$R_b : [s, \omega_1, \omega_2], \tilde{s}, TTCo[R_b, \tilde{R}_{-b}](b), b,$$

and suppose that $\tilde{s}$ is an effective school of b, but $[s, \omega_1, \omega_2]$ is not an effective class when other students are truth-telling. The ranking

$$R_b' : [\omega_1, \omega_2], \tilde{s}, TTCo[R_b, \tilde{R}_{-b}](b), b$$

is also a drooping strategy of b. The TTCo algorithm ignores these strategies when the whole class is not effective. In other words, reducing a non-effective class school by school does not affect the final allocation when others tell the truth. This property is justified in the following proposition.

Proposition 2.5.1. *Let $b \in A$ and $\tilde{R} = (\tilde{R}_b, \tilde{R}_{-b})$, a profile of preference lists. Suppose that*

$$\tilde{R}_b : [s^0, \omega^1, \ldots, \omega^k], [TTCo[\tilde{R}](b)], \ldots, [b],$$

for some $k \geq 1$. Consider

$$R'_b : [\omega^1, \omega^2, \ldots, \omega^k]_b, [TTCo[\tilde{R}](b)]_b, \ldots, [b]_b.$$

Then $TTCo[R'_b, \tilde{R}_{-b}](b) = TTCo[\tilde{R}_b, \tilde{R}_{-b}](b)$.

Proof. It is clear that b is removed from the market in the second step of the TTCo algorithm and is assigned to $TTCo[\tilde{R}](b)$.

First, note that each school in $[\omega^0, \omega^1, \ldots, \omega^k]$ is assigned to some student at the end of first step of $TTCo[\tilde{R}]$. To see it, proceed by contradiction. Suppose the existence of ω^t, for some $\tau \in \{1, 2, \ldots, k\}$, such that she is not removed at the end of step 1. We know the following

1. b and ω^t are not removed from the market at the end of step 1, and

2. (ω^t, b) is an edge of G_1.

Then the maximum ttco of G^1 is not a maximum cover, because $\varsigma(\Upsilon[G^1]]) \cup \{(\omega^t, b)\}$ is a larger top trading cover. Consequently $TTCo[\tilde{R}](b)\tilde{R}_b\omega^t = TTCo[\tilde{R}](b)$, which is a contradiction.

Then there exist students b^τ such that $TTCo[\tilde{R}_b, R_{-b}](b^\tau) = \omega^t$, for all $\tau = 0, 1, \ldots, k$. Moreover, the tie breaking rule implies that

$$(b, \omega^t) \prec (\omega^t, b^\tau) \tag{2.2}$$

for all $\tau = 0, 1, \ldots, k$.

Given the profile (R'_b, R_b), the pair (s^0, b) is not an edge of the quasi-bipartite graph $G_1[R'_b, \tilde{R}_b]$, but all pairs (ω^t, b^τ) are edges of $G_1[R'_b, \tilde{R}_b]$ and the equation (2.2) is also true. So, the maximum ttco of $G_1[R'_b, \tilde{R}_b]$ is the same maximum ttco of $G_1[\tilde{R}]$.

Following to the second step, as student b is the unique student who changes her strategy and all schools in $[\omega^1, \omega^2, \ldots, \omega^k]$ are removed at the end of step 1, we have that

$$G_2[\tilde{R}] = G_2[R'_b, \tilde{R}_{-b}].$$

Therefore $TTCo[R'_b, \tilde{R}_{-b}](b) = TTCo[\tilde{R}_b, \tilde{R}_{-b}](b)$. $\square$

Let $x_b = |X_b^T|$ denote the number of effective schools that improve the allocation of b, produced at the end of the WA.

Next example illustrates how the Waiting Algorithm works.

Example 2.5.2. Consider the same market described in Example 2.5.1. We count all the effective schools of student 4 using the WA. We know that $TTCo[R^0](4) = 4$, then $X_b^1 = \varnothing$.
The step 2 of the WA submit $R_4^1 : \omega_2, \omega_3, 4$ to the TTCo algorithm. Then $TTCo[R_4^1, \tilde{R}_{-4}](4) = 4$. WA outputs not effective schools, $X_4^2 = \varnothing$.
Actualizing the ranking, we have $R_4^2 : \omega_3, 4$, consequently $TTCo[R_4^2, \tilde{R}_{-4}] = 3$. WA outputs $X_4^3 = \{\omega_3\}$. In step 4 we have that $R_4^4 : 4 = TTCo[\tilde{R}](4)$, the algorithm stops.
We conclude that ω_3 is the unique effective school of 4. $\square$

We prove formally that the WA outputs all the possible effective class of indifference that represent a profitable deviation.

Proposition 2.5.2. *The Waiting Algorithm outputs, under $\tilde{R}_b$, all the effective classes $[\omega]_b$ for b such that*

$$[\omega]_b P_b TTCo[R_b, \tilde{R}_{-b}](b).$$

Proof. We proceed by contradiction. Suppose there is an effective class $[\omega]_b$ such that the waiting algorithm did not output it and

$$[\omega]_b R_b [TTCo[R_b, \tilde{R}_{-b}](b)]_b.$$

Then $TTCo[R_b^t, \tilde{R}_{-b}](b) \neq [\omega]_b$ for all $t \in \mathbb{N}$.
Since $[\omega]_b$ is an effective class, there is a strategy R'_b such that $TTCo[(R'_b, \tilde{R}_{-b})] = \omega' \in [\omega]_b$. By Lemma 2.5.1, we have that

$$TTCo[R'_b, \tilde{R}_{-b}](b) = TTCo[R_b^{\omega'}, \tilde{R}_{-b}](s),$$

where $R_b^{\omega'}$ is the dropping strategy where ω' is declared the unique most preferred school of b. Now, dropping all the indifference classes preferred to $[\omega]_b$, we get the dropping strategy $R_b^{[\omega]_b}$ such that

$$TTCo[R_b^{\omega'}, \tilde{R}_{-b}](b) \sim_b TTCo[R_b^{[\omega]_b}, \tilde{R}_{-b}](b).$$

By construction of the WA algorithm, in some step $t \in \mathbb{N}$ we have $R_b^t = R_b^{[\omega]_b}$, which is a contradiction. Therefore, the waiting algorithm outputs all affective classes of b. □

Random Markets

To simplify the analysis, we first assume complete information. We have shown that: truth-telling is not a Nash equilibrium of the game described in Subsection 2.2, and students can manipulate the TTCo mechanism. We investigate how likely students manipulate the TTCo algorithm introducing *random markets* as in [11] and [15].

Consider $D = (d_1, d_2, \ldots, d_m)$ a probability distribution over O such that, without loss of generality, $d_j \geq d_{j+1}$ and $d_j > 0$ for all $\omega_j \in S$. We say that school ω_j is **more popular** than school $\omega_{j'}$ if $d_j \geq d_{j'}$. For each student i, a distribution D induces a **random ranking** $\tilde{R}_i$, not necessarily complete, of length $k \in \mathbb{N}$ as follows[4]:

Step 1. Select randomly a school following distribution D. List this school as the most preferred school of i.

Step t. Select randomly a school following distribution D.

t.1 If this school has not been previously drawn in steps 1 through $t-1$, list this school as the t^{th} most preferred school of i, go to $t+1$.

t.2 Otherwise, we select randomly a school following distribution D, go to $t.1$.

The procedure ends at step k.

A **random market** is a tuple $\tilde{\aleph} = (A, O, D, k)$ with an associated profile of random preferences $\tilde{R}$. Given $\varepsilon > 0$, a profile of preferences $\overline{R}^* = (\overline{R}_i^*)_{i \in A}$ is an ε-*Nash equilibrium* if there is no $i \in A$ and $\overline{R}_i'$ such that

$$E[u_i(TTCo[A, O, (\overline{R}_i', \overline{R}_{-i}^*)](i))] > E[u_i(TTCo[A, O, \overline{R}^*](i))] + \varepsilon,$$

[4] This is defined in the same form as in [15].

where the expectation is taken with respect to random preferences.

In random markets, we count all profitable deviations of student b using the WA, assuming that she knows her true ranking R_b and other students report random preferences $\tilde{R}_{-b}$. As a consequence of Proposition 2.5.2, the WA outputs all effective indifference classes $[\omega]_b$ strictly preferred to $TTCo[R_b, \tilde{R}_{-b}](b)$, for all $b \in A$.

Large Markets

A **sequence of random markets** is denoted by $(\tilde{\aleph}^1, \tilde{\aleph}^2, \ldots)$, where each $\tilde{\aleph}^n$ is a random market

$$(A^n, O^n, D^n, k^n)$$

such that $D^n = (d_1^n, d_2^n, \ldots)$, $|A^n| = n$ and $|O^n| = m_n$, for all $n \in \mathbb{N}$.

For each random market $\tilde{\aleph}^n$ in the sequence, we denote the **expected number of students that manipulate** the TTCo mechanism by

$$\begin{aligned} \delta_k(n) \quad = \quad & E[\#\left\{i \in A^n | TTCo\left[R_i'^n, \tilde{R}_{-i}^n\right](i) P_i TTCo\left[R_i^n, \tilde{R}_{-i}^n\right](i)\right\} \\ & \text{for some } R_i^n \text{ in the induced market} | \tilde{\aleph}^n]. \end{aligned}$$

The expectation is taken with respect to the random profile $\tilde{R}^n$ of the random market $\tilde{\aleph}^n$. This represents a difference with the work done by [11] and [15]: we cannot assume that other students are truth-telling because schools do nothing during the TTCo algorithm.

We prove that the expected proportion of students that manipulate the algorithm, $\delta_k(n)/n$, tends to zero as n tends to infinity, i.e., for all $\omega > 0$ there must exists $N \in \mathbb{N}$ such that

$$\left| \frac{\delta_k(n)}{n} - 0 \right| < \omega.$$

for all $n \geq N$. To prove it, we require the following regularity conditions.

Definition 8. A sequence of random markets $(\tilde{\aleph}^1, \tilde{\aleph}^2, \ldots)$ is **regular** if there exists a positive integer k such that

1. $k^n = k$ for all n,

2. $m_n \leq n$ for all n.

Condition (1) assumes that preferences length of each student does not grow as the size of the market increases, i.e., each student has an incomplete preference list because she does not know all the schools in the market; condition (2) requires that the number of schools does not grow much faster than the number of students, the supply is lower than demand.

Regularity conditions are taken from [15], which considers four regularity conditions. The other two conditions are: *the number of college positions is bounded* and *all students are acceptable to any college*, respectively. We do not require a bound of "positions " because we assume that students only can hold one and only one school. On the other hand, they need all students to be acceptable to guarantee the existence of a large number of IR matchings. We do not need it because the existence of a large number of IR assignments is guaranteed by Proposition 2.3.1: there always exists at least one maximum ttco in any quasi-bipartite graph, regardless the tie breaking rule that the assignment procedure uses.

Since we want to know how likely students misreport their true preferences at equilibrium, regularity conditions ensure that markets in the sequence are very similar, they only differ in their associated profile of random preferences. Our first Theorem establishes that $\delta_k(n)$ tends to zero as n tends to infinity. Its proof requires the following lemma, which is the adaptation of Lemma 4.2 in [11] to the characteristics of the TTCo algorithm. With this lemma we can find the bounds to $\delta_k(n)$.

We introduce extra notation to describe the required bounds. Consider a random market $\tilde{\aleph} = (A, O, D)$ and a fixed student $b \in A$, from now on we only write $TTCo(b)$ instead of $TTCo[R_b, \tilde{R}_{-b}](b)$. Let $O_{TTCo}(b) = \{\omega \in O \mid d_\omega \geq d_{TTCo(b)}$ and $TTCo(\omega) = \omega\}$, the set of all schools more popular than $TTCo(b)$ which are not assigned at the end of the algorithm. The number of elements in $O_{TTCo}(b)$ is denoted by $X_{TTCo}(b) = |O_{TTCo}(b)|$. Consider $E_{TTCo}(b) = \{\omega \in O \mid d_\omega > d_{TTCo(b)}$ and $i\tilde{R}_i s$ for all $i \in A\}$, the set of schools more popular than $TTCo(b)$ that do not appear on any ranking $\tilde{R}_{-b}$. The number of elements in $E_{TTCo(b)}$ is denoted by $Y_b = |E_{TTCo}(b)|$.

Lemma 2.5.2. *Consider a regular sequence of random markets* $(\tilde{\aleph}^1, \tilde{\aleph}^2, \ldots)$. *For every* $b > 4k$,

$$E\left[\frac{1}{X_{TTCo}(b)+1}\right] \leq \frac{12e^{8nk/b}}{b}.$$

Moreover, $\frac{\partial g}{\partial b} < 0$ *where* $g(b) \equiv \frac{12e^{8nk/b}}{b}$.

Proof. We divide this proof in the following three affirmations.

Affirmation 2.5.1. For every b, we always have $X_{TTCo}(b) \geq Y_b$.

Proof. Let $\omega \in E_{TTCo}(b)$, by definition, ω does not appear in any ranking and is more popular than $TTCo(b)$. Then, no student does point to s during the $TTCo$ algorithm. Consequently, $TTCo(\omega) = \omega$ for all $\omega \in E_{TTCo}(b)$. Moreover, we have that ω is more popular than $TTCo(b)$ and remains unassigned. So, $\omega \in O_{TTCo}(b)$, which implies that $E_{TTCo} \subseteq O_{TTCo}(b)$. Therefore $X_{TTCo}(b) \geq Y_b$. $\square$

Affirmation 2.5.2. Consider a distribution D. For $b > 4k$, then

$$E[X_{TTCo}(b)] \geq \frac{b}{2} e^{-8nk/g}.$$

Proof. Let $b \in A$ a fixed student, and $Q = \sum_{j=1}^{k} d_j$ the total probability of the first k more popular schools according to D. Suppose that b picks $\omega_1^b, \omega_2^b, \ldots, \omega_{l-1}^b$ as her first $l-1$ preferred schools, with $l \leq k$. Since $d_1 \geq d_2 \geq \ldots \geq d_m$, we have that

$$\begin{aligned}
Q &\geq \sum_{j=1}^{l-1} d_{\omega_j^b} \\
1 - \sum_{j=1}^{l-1} d_{\omega_j^b} &\geq 1 - Q \\
\frac{d_\omega}{1-Q} &\geq \frac{d_\omega}{1 - \sum\limits_{j=1}^{i-1} d_{\omega_j^b}} \\
1 - \frac{d_\omega}{1 - \sum\limits_{j=1}^{i-1} d_{\omega_j^b}} &\geq 1 - \frac{d_\omega}{1-Q}.
\end{aligned}$$

for all schools ω. In words,

$$1 - \frac{d_\omega}{1 - \sum\limits_{j=1}^{l-1} d_{\omega_j^b}}$$

is the probability that student b does not rank school ω as her l'th most preferred school, given that she picks schools ω_j^b. Even more, the probability that a student does not list ω in her ranking is

$$\left(1 - \frac{d_\omega}{(1-Q)}\right).$$

By construction of random preferences, each school is chosen according to D independently of the schools chosen before. Then, the stochastic independence implies that the probability that ω is not listed by any student in her ranking is at least

$$\left(1-\frac{d_\omega}{1-Q}\right)^{nk}. \tag{2.3}$$

Now, consider $\omega=\omega_t$. If $t>k$, there are at least $t-k$ schools who are at least as popular as ω, but not among the k most popular schools[5]. So, d_ω is at most the probability of not being one of the k most popular schools, divided by the total of schools who are at least as popular as her

$$d_\omega \leq \frac{1-Q}{t-k}. \tag{2.4}$$

On the other hand, we know that the limit definition of the exponential function is

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n. \tag{2.5}$$

For $t>2k$, expressions (2.3), (2.4) and (2.5) imply that

$$\begin{aligned} Pr[E_{TTCo}(b)] &\geq \left(1-\frac{d_\omega}{1-Q}\right)^{nk} \\ &\geq \left(1-\frac{1}{t-k}\right)^{nk} \\ &\geq e^{-2nk/(t-k)} \\ &\geq e^{-4nk/t}. \end{aligned}$$

Therefore, decomposing the expectation of Y_b, we obtain

$$\begin{aligned} E[Y_b] &= \sum_{w=1}^{b} Pr[E_{TTCo}(w)] \\ &\geq \sum_{j=2k}^{b} e^{-4nk/j} \\ &\geq \sum_{j=b/2}^{b} e^{-8nk/b} = \frac{b}{2}e^{-8nk/b}, \end{aligned}$$

for every $b>4k$.

[5]Remember that we assume $d_t \geq d_{t+1}$ in distribution D.

By Affirmation 2.5.1, we conclude that

$$E[X_{TTCo}(b)] \geq E[Y_b] \geq \frac{b}{2}e^{-8nk/g}.$$

$\square$

Affirmation 2.5.3. The variance $\sigma^2(Y_b) \leq E[Y_b]$.

Proof. This proof is taken from Immorlica and Mahdian (2005), Lemma 4.4. We show it for pedagogical purposes. We will show that events $E_{TTCo}(i)$ are negatively correlated, proving that

$$Pr[E_{TTCo}(i) \cap E_{TTCo}(j)] \leq Pr[E_{TTCo}(i)]Pr[E_{TTCo}(j)],$$

for all i, $j \in A$.

Let $F_\omega(i)$ be the event that a given school s is not included in the list of preference of i. By stochastic independence, discussed in Affirmation 2.5.2, we know that

$$Pr[E_{TTCo}(i)] \leq (Pr[F_\omega(i)])^n \text{ and}$$
$$Pr[E_{TTCo}(i) \cap E_{TTCo}(j)] \leq (Pr[F_\omega(i)] \cap Pr[F_{\omega'}(j)])^n.$$

On the other hand, the definition of conditional probability is $Pr[X|Y] = \frac{Pr[X \cap Y]}{Pr[Y]}$ then, it is enough to show that for every i and j,

$$Pr[F_\omega(i)|F_{\omega'}(j)] \leq Pr[F_\omega(i)].$$

Let M be an arbitrarily large constant. With the following process, we simulate the selection of one preference list $L = (l_1, l_2, \ldots, l_k)$: Consider Σ the set of $\lfloor p_\omega M \rfloor$, the immediate inferior integer of $p_\omega M$, copies of school ω. Pick a random permutation R of Σ. Let l_i be the i'th distinct name in R. It is clear that, if M tends to ∞, the probability of a given list L in this process converges to its probability under D^k. Therefore, $Pr[F_\omega(i)]$ is equal to the probability that k distinct schools of ω are chosen by i in R when M tends to ∞. Similarly, if Σ' denotes the multiset obtained by removing all copies of school ω' from Σ, then $Pr[F_\omega(i)|F_{\omega'}(j)]$ is equal to the probability that k distinct schools are chosen by i in a random permutation of Σ' as M tends to ∞. However, this is precisely equal to the probability that k distinct names other than ω' occur before ω in a random permutation R of Σ. This implies that k

distinct schools (including ω') occur before ω in R. So, for every R where $F_\omega(i)|F_{\omega'}(j)$ happens, $F_\omega(i)$ also happens. It means that the first event is contained by $F_\omega(i)$, so $Pr[F_\omega(i)|F_{\omega'}(j)] \leq Pr[F_\omega(i)]$. Using the relation between $F_\omega(i)$ and $E_{TTCo}(i)$, we conclude that $E_{TTCo}(i)$ and $E_{TTCo}(j)$ are negative correlated

$$Pr[E_{TTCo}(i) \cap E_{TTCo}(j)] - Pr[E_{TTCo}(i)]Pr[E_{TTCo}(j)] \leq 0.$$

Then

$$\begin{aligned}
\sigma^2(Y_b) &= E[Y_b^2] - E[Y_b]^2 \\
&= \sum_{i=1}^{b} Pr[E_{TTCo}(i)] + 2\sum_{1\leq i<j\leq b} Pr[E_{TTCo}(i)\cap E_{TTCo}(j)] - \sum_{i=1}^{b} Pr[E_i]^2 + \dots \\
\dots &\quad -2\sum_{1\leq i<j\leq b} Pr[E_{TTCo}(i)]Pr[E_{TTCo}(j)] \\
&\leq \sum_{i=1}^{b} Pr[E_{TTCo}(i)] = E[Y_b].
\end{aligned}$$

□

Continuing with the proof of Lemma 2.5.2, this last part is equal to the proof of Lemma 4.1 in [15]. Consider

$$Pr\left[Y_b < \frac{E[Y_b]}{2}\right] = q.$$

Using Chebyshev's inequality, we have that

$$\begin{aligned}
q &\leq Pr\left[|Y_b - E[Y_b]| > \frac{E[Y_b]}{2}\right] \\
&\leq \frac{\sigma^2(Y_b)}{(E[Y_b]/2)^2}.
\end{aligned}$$

Now, applying Affirmation 2.5.3 to the above expression

$$q \leq \tfrac{4}{E[Y_b]}. \tag{2.6}$$

Affirmation 2.5.1 tells us that $Y_b + 1 \leq X_{TTCo}(b) + 1$. Moreover, the expectation operator respect inequalities, then

$$\begin{aligned}
E\left[\frac{1}{X_{TTCo}(b)+1}\right] &\leq E\left[\frac{1}{Y_b+1}\right] \\
&= \sum_{Y_b} \frac{1}{Y_b+1} Pr[Y_b].
\end{aligned}$$

On the other hand, we know that:

$$\frac{1}{Y_b+1} \leq 1 \text{ and } Pr\left[Y_b \leq E\left[\frac{Y_b}{2}\right]\right] = Pr\left[\frac{1}{E[Y_b]/2+1} \leq \frac{1}{Y_b+1}\right].$$

Consequently

$$\begin{aligned}
\sum_{Y_b} \frac{1}{Y_b+1} Pr[Y_b] &\leq \sum_{Y_b} 1 Pr[Y_b] \\
&\leq \sum_{Y_b=0}^{1/(E[Y_b]/2+1)} Pr[Y_b] + \sum_{Y_b=1/(E[Y_b]/2+1)} Pr[Y_b] \\
&\leq Pr\left[\frac{1}{E[Y_b]/2+1} \leq \frac{1}{Y_b+1}\right] \sum_{Y_b=0}^{1/(E[Y_b]/2+1)} 1 + \ldots \\
&\quad + Pr\left[\frac{1}{E[Y_b]/2+1} \geq \frac{1}{Y_b+1}\right] \\
&= (1-q)\frac{1}{E[Y_b]/2+1} + q.
\end{aligned}$$

Using expression (2.6)

$$(1-q)\frac{1}{E[Y_b]/2+1} + q \leq \frac{4}{E[Y_b]} + \frac{2}{E[Y_b]}. \tag{2.7}$$

Finally, expressions (2.6) and (2.7), together with Lemma 2.5.2, imply that

$$\begin{aligned}
E\left[\frac{1}{X_{TTCo}(b)+1}\right] &\leq \frac{6}{E[Y_b]} \\
&\leq \frac{6}{\frac{b}{2}e^{-8nk/b}} = \frac{12e^{8nk/b}}{b}.
\end{aligned}$$

To complete the proof of Lemma 2.5.2, we need to prove that function $g(b) = 12e^{8nk/b}/b$ is decreasing with respect to b:

$$\begin{aligned}
\frac{\partial g(b)}{\partial b} &= \frac{b12e^{8nk/b}(-8nk/b^2) - 12e^{8nk/b}}{b^2} \\
&= -12e^{8nk/b}\left(\frac{b-1}{b^2}\right).
\end{aligned}$$

Since $b > 4k$, we conclude that

$$\frac{\partial g(b)}{\partial b} < 0.$$

□

In the following lemma we prove that the proportion of students with incentives to deviate from their true preference list tends to zero as the size of the market increases. Its proof follows closely the proof of Theorem 4.1 in [11]. The reasoning is the same, however, we avoid the arguments related with reaction chains using the properties of the TTCo algorithm, the Waiting Algorithm and Lemma 2.5.1.

Theorem 2.5.1. *Let* $(\tilde{\aleph}^1, \tilde{\aleph}^2, \ldots)$ *a regular sequence of random markets, and k a fixed positive constant. Suppose that each student has a random ranking chosen according to* D^k. *Then*

$$\lim_{n\to\infty} \frac{\delta_k(n)}{n} = 0.$$

Proof. If b is an student that manipulates the TTCo algorithm, by construction of the WA we have that $x_b > 1$, she has incentives to report a dropping strategy instead of her true ranking, see Lemma 2.5.1 and Definition 7. Consequently, $\delta_k(n)$ is equal to the expected number of students such that $x_b > 1$, then

$$\delta_k(n) = \sum_{b\in B^k} Pr[b \text{ has at least one effective school preferred to } TTCo[R_b, \tilde{R}_{-b}](b)].$$

We search an upper bound for $\delta_k(n)$. First, we bound the probability that b has at least one school preferred to $TTCo[R_b, \tilde{R}_{-b}](b)$. By Proposition 2.5.2, this is the same as bounding the probability that the random variable x_b is more than one, $Pr[x_b > 1]$.

We can divide the WA into two phases: the first phase is from the beginning of the algorithm until it finds the first effective school, and the second phase is from that point until the algorithm terminates. By Proposition 2.3.1, at the end of the first phase there exists an IR assignment, i.e. the assignment $TTCo[R_b, \tilde{R}_{-b}] = TTCo$ always exists. Given the existence of this assignment, we first bound

$$Pr[x_b > 1 | TTCo],$$

the probability that $x_b > 1$ conditioned on the existence of the IR assignment $TTCo$; later, we take the expectation of this bound over $TTCo$ to get $Pr[x_b > 1]$. We know that

$$TTCo(b) \in O \cup \{b\}.$$

Case I. If $TTCo[R_b, \tilde{R}_{-b}](b) = b$, b does not get any school, we have that $Pr[x_b > 1 | TTCo] = 0$. The proof finishes.

Case II. If $TTCo[R_b, \tilde{R}_{-b}] \in S$, remember that

$$O_{TTCo}(b) = \{\omega \in O \mid d_\omega \geq d_{TTCo(b)} \text{ and } TTCo(\omega) = \omega\},$$

the set of schools more popular than $TTCo[R_b, \tilde{R}_{-b}](b)$ that remain unassigned, and $X_{TTCo}(b)$ is the cardinality of this set. Moreover, $x_b > 1$ if and only if for student b there exists a profitable dropping strategy $R_b^{t^*}$, and we know that the WA ends at some t' such that $R_b^{t'}$ is the strategy where $[TTCo[R_b, \tilde{R}_{-b}](b)]$ is declared as the most effective indifference class. So, student b has exhausted all strategies R_b^t strictly preferred to $TTC[R_b, \tilde{R}_{-b}](b)$ in the WA.
Implicitly, we assume that no allocation was made before the WA starts. Since the $TTCo$ always exists at the end of the first phase of the WA, we claim that:

Affirmation 2.5.4. $X_b \cap O_{TTCo}(b) = \varnothing$.

Proof. If there exists $\omega \in X_b \cap O_{TTCo}(b)$ then $TTCo(\omega) = b$ and $TTCo(\omega) = \omega$ which is a contradiction. □

Thus, the WA cannot output an element of $O_{TTCo}(b)$ by Affirmation 2.5.4. Consequently, the probability $Pr[x_b > 1 | TTCo]$ is equal or less than the probability that $TTCo(b)$ appears before any school in $O_{TTCo}(b)$ because dropping strategies in the WA truncate schools in a descending form, so

$$Pr[x_b > 1 | TTCo] \leq Pr[TTCo(b) \text{ appears before any school in } O_{TTCo}(b)]. \tag{2.8}$$

On the other hand, by definition of $O_{TTCo}(b)$, the probability that t is picked is at least as large as the probability that $TTCo(b)$ is picked, for all $t \in O_{TTCo}(b)$. Thus, the probability that $TTCo(b)$ appears before all elements in $O_{TTCo}(b)$, in a sequence of elements picked according to D^k, is at most the probability that $TTCo(b)$ appears first in a random permutation on the elements of $\{TTCo(b)\} \cup O_{TTCo}(b)$ which is $1/(X_{TTCo}(b) + 1)$. So

$$Pr[TTCo(b) \text{ appears before any school in } O_{TTCo}(b)] \leq \frac{1}{X_{TTCo}(b) + 1}. \tag{2.9}$$

Joining expressions (2.8) and (2.9)

$$Pr[x_b > 1 | TTCo] \leq \frac{1}{X_{TTCo}(b) + 1}. \tag{2.10}$$

Applying the expectation operator and the Law of Iterative Expectations

$$\begin{aligned}Pr[x_b > 1] &= E_{TTCo}[Pr[x_b > 1 | TTCo]] \\ &\leq E_{TTCo}\left[\frac{1}{X_{TTCo}(b)+1}\right].\end{aligned} \tag{2.11}$$

Applying Affirmation 2.5.2 for $b \geq \frac{16nk}{\ln(n)}$, equation (2.11) and the fact that $Pr[x_b > 1] \leq 1$, for all $i \in \{1,2,\ldots,\frac{16nk}{\ln(n)}\}$, we get

$$\begin{aligned}\delta_k(n) &= \sum_{i=1}^{\frac{16nk}{\ln(n)}} Pr[x_b > 1] + \sum_{\frac{16nk}{\ln(n)}}^{n} Pr[x_b > 1] \\ &\leq \sum_{i=1}^{\frac{16nk}{\ln(n)}} 1 + \sum_{\frac{16nk}{\ln(n)}}^{n} E_{TTCo}\left[\frac{1}{X_{TTCo}(b)+1}\right] \\ &\leq \frac{16nk}{\ln(n)} + \sum_{b=\frac{16nk}{\ln(n)}}^{n} \frac{12e^{8nk/b}}{b}.\end{aligned}$$

Moreover, by Lemma 2.5.2, we know that $g(b)$ is a decreasing function with respect to b, so

$$\begin{aligned}\frac{16nk}{\ln(n)} + \sum_{b=\frac{16nk}{\ln(n)}}^{n} \frac{12e^{8nk/b}}{b} &\leq \frac{16nk}{\ln(n)} + \sum_{b=\frac{16nk}{\ln(n)}}^{n} \frac{12e^{8nk/\left(\frac{16nk}{\ln(n)}\right)}}{\left(\frac{16nk}{\ln(n)}\right)} \\ &= \frac{16nk}{\ln(n)} + \sum_{b=\frac{16nk}{\ln(n)}}^{n} \frac{3\ln(n)e^{\ln(n)/2}}{4nk} \\ &\leq \frac{16nk}{\ln(n)} + n\left(\frac{3\ln(n)e^{\ln(n)/2}}{4nk}\right) \\ &= \frac{16nk}{\ln(n)} + 3\frac{\sqrt{n}\ln(n)}{4k}.\end{aligned}$$

Dividing between n, we have that

$$\frac{\delta_k(n)}{n} \leq \frac{16k}{\ln(n)} + \frac{3}{4}\frac{\ln(n)}{k\sqrt{n}}.$$

Therefore, the fraction of students with more than one effective school goes to zero as n goes to infinity, for every length k. □

Thickness Condition and Truth-telling at Equilibrium

We have proven that the number of schools with incentives to manipulate tends to zero as the market size increases. A large market is necessary but not sufficient to guarantee that truth-telling is an

$\varepsilon-$Nash equilibrium. The following example shows that.

Example 2.5.3. Consider a market (A,O,D,k) such that $|A| = |O| = n$ and D is a probability distribution over O defined as follows:

$$d_{\omega_1} = d_{\omega_2} = d_{\omega_3} = \frac{1}{4},$$
$$d_{\omega_j} = \frac{1}{4(n-3)} \text{ for all } j > 3.$$

With probability

$$\left[\frac{d_{\omega_1} d_{\omega_2} d_{\omega_3}}{(1-d_{\omega_1})(1-d_{\omega_1}-d_{\omega_2})}\right]^3 = \frac{1}{24^3},$$

preferences of students 1, 2 and 3 are given by:

$$R_1 \ : \ \omega_1, \omega_2, \omega_3, \ldots,$$
$$R_2 \ : \ \omega_1, \omega_2, \omega_3, \ldots,$$
$$R_3 \ : \ \omega_1, \omega_2, \omega_3, \ldots,$$

Under the TTCo algorithm, we have that:

$$TTCo[R] = \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \\ 1 & 2 & 3 \end{pmatrix}.$$

Now, suppose that ω_3 reports

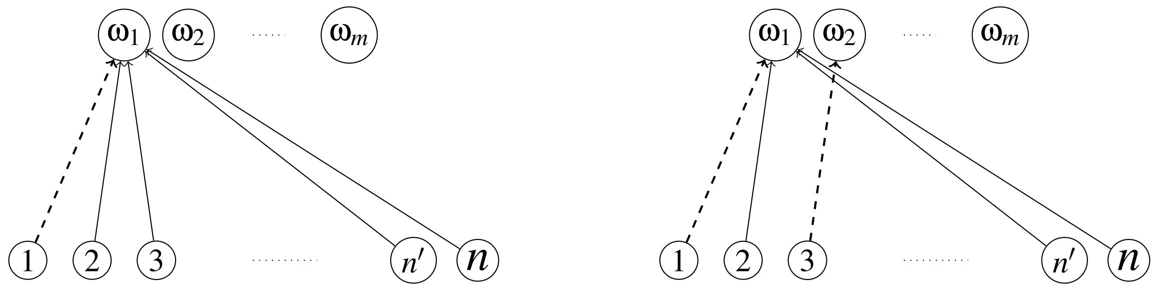
$$R_3' : \omega_2, 3.$$

Figure 2.6(b) illustrates the maximum ttco $\varsigma_L(\Upsilon(G^1[R_3', R_{-3}]))$. Therefore

$$TTCo[R_3', R_{-3}] = \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 & \cdots \\ 1 & 3 & 2 & \cdots \end{pmatrix},$$

where $\omega_2 P_3 \omega_3$. In words, student 3 manipulates the $TTCo$ algorithm with a positive probability, $1/(24^3)$, regardless the size of the market. $\square$

Note that the tie breaking rule ς_L benefits students with the minimum index. Example 2.5.3 shows that ω_1 is *too popular* with respect to other schools. Students with an index far from 1 are aware that it



(a) True preferences

(b) student 3 deviates from her true ranking

Figure 2.6: student 3 has the opportunity to manipulate $TTCo$, in order to get ω_2 regardless the size of the market.

is impossible to get ω_1 and decide to point to some school less popular than ω_1 but in a high position on her preference list. Pointing to an school with a lower popularity implies higher possibilities to be the student with the minimum index. In Example 2.5.3, student 3 is not the student with minimum index between the student that point to school ω_1. However, if she points first to ω_2, she becomes the student with the minimum index of ω_2. Even more, this situation does not change for 3 when the size of the market increases because new elements have indexes far from ω_2, they are greater than 3. Therefore, the deviation R_3' is always profitable to 3 with probability $1/24$. Thus, with positive probability, truth-telling is not the best response of student 3 when others are truth-telling.

Kojima and Pathak (2009) proved that truth-telling is an ε-Nash equilibrium when the sequence of random markets satisfies the *thickness condition*. In Example 2.5.3, the relative probability between ω_1 and ω_n is

$$\frac{d_{\omega_1}}{d_{\omega_n}} = \frac{1/4}{1/4(n-2)} = n-2.$$

So, for each student pointing to ω_n, the number of students that points to ω_1 tends to infinity, i.e. ω_1 is too popular with respect to ω_n, for all $n \geq 4$. If a market satisfies the thickness condition, then no school is too popular in comparison with others, relative probabilities are bounded.

Let $V_T(n) = \left\{ \omega \in O \;\middle|\; \max_{\omega' \in O} \frac{d^n_{\omega'}}{d^n_{\omega}} \leq T \right\}$ the random set that denotes the set of schools sufficiently popular ex-ante. Below, we present the thickness condition of Kojima and Pathak.

Definition 9. A sequence of random markets is **sufficiently thick** if there exists $T \in \mathbb{R}$ such that

$$E[|V_T(n)|] \to \infty,$$

as $n \to \infty$.

Example 2.5.4. A particular random market that satisfies the thickness condition is one where D is a uniform distribution. Consider a sequence of random markets $\tilde{\aleph}^n = (A^n, O^n, D^n)$ such that $|A^n| = n$, $|O^n| = n$ and $d_\omega = 1/n$ for all $\omega \in O$, i.e. $D = (1/n, 1/n, \ldots, 1/n)$. Now, let s, $\omega' \in S$, the maximum of the relative probabilities is

$$\begin{aligned}
\max_{\omega' \in S} \left\{ \frac{d_{\omega'}}{d_\omega} \right\} &= \max_{\omega' \in S} \left\{ \frac{1/n}{1/n} \right\} \\
&= \max_{\omega' \in S} \{1\} \\
&= 1.
\end{aligned}$$

If $T = 2$, then $(\tilde{\aleph}^n)$ is a sequence of random markets sufficiently thick.

In large thick markets, the TTCo algorithm covers all market members at the end of its first step.

Proposition 2.5.3. *Let $(\tilde{\aleph}^1, \tilde{\aleph}^2, \ldots)$ a regular sequence of random markets that satisfy the thickness condition. There exists $N \in \mathbb{N}$ such that the TTCo algorithm removes all schools in $\tilde{\aleph}^n$ at the end of its first step, for all $n \geq N$.*

Proof. Consider $t[TTCo, \tilde{\aleph}^n]$ the step of the TTCo algorithm such that at the end of it there is no remaining schools in the market. We have to prove that

$$\lim_{n \to \infty} t[TTCo, \tilde{\aleph}^n] = 1,$$

in other words, the maximum ttco $\varsigma_L(\Upsilon[G^1[\tilde{\aleph}^n]])$ covers all schools in $\tilde{\aleph}^n$.

We suppose that the sequence of random markets satisfies the thickness condition, then, there exists $T > 0$ such that

$$\max_{\omega' \in O^n} \frac{\{d^n_{\omega'}\}}{d^n_\omega} \leq T \Rightarrow \max_{\omega' \in O^{n'}} \frac{\{d^n_{\omega'}\}}{T} \leq d^n_\omega,$$

and $d^n_\omega > 0$, for all $n \geq N$. In such markets, this implies that ω is with positive probability the most preferred school of at least one student.

Let $d^n_{\omega^M} = \max_{\omega' \in O^n} \{d^n_{\omega'}\}$, then

$$\begin{aligned}
1 &= \frac{d^n_{\omega^M}}{d^n_{\omega^M}} \\
&\leq \frac{d^n_{\omega^M}}{d^n_\omega} \leq T.
\end{aligned}$$

This means that for each student who prefers ω_m to any other school ω there exist between 1 and T students who prefer ω to any other school, ex-ante. Thus, there is a student b_ω such that (b_ω, ω) is an edge of $G_1[\tilde{\aleph}^n]$, for all $\omega \in O^n$ and $n \geq N$.

Moreover, since we assume a regular sequence of random markets, we have that $|O^n| \leq |A^n|$. By the Pigeon-hole principle, there is $N' > N$ such that $b_\omega \neq b_{\omega'}$ for all $\omega \neq \omega'$ in O^n, for all $n \geq N'$.

Let $\theta = \{(b_{\omega_1}, \omega_1), (b_{\omega_2}, \omega_2), \ldots, (b_{\omega_{m_n}}, \omega_{m_n})\}$ and

$$\Theta = \theta \cup \{(b,b) \in A^n \mid b \notin \theta\}.$$

It is clear that Θ is a subset of edges of $G^1(\tilde{\aleph}^n)$, and no edges in Θ have a node in common. Moreover, θ covers all schools in O^n. Therefore, Θ is a maximum ttco of $G^1(\tilde{\aleph}^n)$.

Still $\varsigma_L[\Upsilon(G^1(\tilde{\aleph}^n))]$ is not necessary equal to Θ. However, we have found a maximum top trading cover that covers all students in $\tilde{\aleph}^n$. Consequently, all maximum ttco of $G^1(\tilde{\aleph}^n)$ must cover all students in $\tilde{\aleph}^n$.

Particularly, $\varsigma_L[\Upsilon(G^1(\tilde{\aleph}^n))]$ must cover all students in $\tilde{\aleph}^n$ for all $n \geq N'$. Therefore

$$\lim_{n \to \infty} t[TTCo, \tilde{\aleph}^n] = 1.$$

□

In Example 2.5.3 we show that too popular schools can induce profitable deviations of schools from their true preferences, we call it an *opportunity*[6]. The thickness condition ensures that all schools have a large enough demand, which reduces the possibility to be the the minimum student implying that the opportunity vanishes. Next lemmas explain in more details what are vanishing opportunities.

Consider an student b, and suppose that her true ranking is represented by

$$R_b : [\omega_{b1}]_b, [\omega_{b2}]_b, \ldots, [\omega_{b\eta}]_b, [b]_b.$$

Let $R_b(\omega_{bj})$ be the dropping strategy where $[\omega_{bj}]_b$ is declared as the most preferred class of b and all her strictly preferred classes are declared not IR,

$$R_b(\omega_{bj}) : [\omega_{bj}]_b, [\omega_{b(j+1)}]_b, \ldots, [\omega_{b\eta}]_b, [b]_b.$$

[6]Kojima and Pathak in [15] call it market power of colleges.

Now, let $d(R_b(\omega_{bj}))$ be the probability that the WA outputs a school $TTCo[R_b(\omega_{bj}), \tilde{R}_{-b}](b)$ strictly preferred to $TTCo[R](b)$, conditioned to the matching $TTCo[R]$ and $Y_b(n) > E[Y_b(n)]/2$, i.e. the number of schools more popular than $TTCo[R](b)$ that do not appear in any ranking is bounded inferiorly by $E[Y_b(n)]/2$. So,

$$d(R_b(\omega_{bj})) = Pr\left[TTCo[R_b(\omega_{bj}), \tilde{R}_{-b}](b) P_b TTCo[R](b) \;\middle|\; Y_b(n) > \frac{E[Y_b(n)]}{2}, TTCo\right].$$

If this probability tends to zero, the probability to do a profitable manipulation also tends to zero, then opportunities vanish.

In the next lemma, equivalent to Lemma 9 presented in Appendix B of ([15]), we investigate if the first step of the WA algorithm results in a profitable manipulation, so, we bound the probability that reporting $R_b(\omega_{b2})$ is a profitable manipulation.

Lemma 2.5.3. *Consider $(\tilde{\aleph}^1, \tilde{\aleph}^2, \ldots)$ a regular sequence of sufficiently thick random markets. Let T be such that $E[Y_T(n)] \to \infty$ as $n \to \infty$. Then there is $n \in \mathbb{N}$ large enough such that*

$$d(R_b(\omega_{b2})) \leq \frac{4T\bar{t}}{E[Y_T(n)]},$$

where $\bar{t}$ is the number of schools preferred to $TTCo[R_b, \tilde{R}_{-b}](p)$ under R_b.

Proof. In the first step of the WA, student b reports the dropping strategy $R_b(\omega_{b2})$. In a sequence of thick markets, we know that

$$p^n_{\omega'}(r) \geq \frac{p^n_{\omega_{b2}}(r)}{T},$$

for all $\omega' \in V_T(n)$. The WA algorithm outputs a profitable manipulation at the end of its first step if

$$TTCo[R_b(\omega_{b2})](b) P_b TTCo[R_b](b).$$

Then, ω_{b2} is an school that appears in some ranking (R_i) and is assigned. Then, the probability that $[\omega_{b2}]$ is an effective class of b is at least

$$1 - \frac{1}{Y_T(n)/T + 1},$$

the complement event of some school, more popular than $TTCo[R](b)$, is not picked in any ranking and remains unassigned. Moreover, we are conditioning to $Y_b(n) > E[Y_b(n)]/2$, by definition of

$d(R_b(\omega_{b_j}))$, consequently we have that

$$1-\frac{1}{Y_T(n)/T+1} > 1-\frac{1}{E[Y_T(n)]/2T+1}.$$

Now, if all dropping strategies $R_b(\omega_{bj})$ are profitable, for $j=2,\ldots,r-1<\bar{t}$. Then, there are still at least $Y_T(n)-r+1$ schools more popular than $TTCo[R](b)$ that are still unassigned because at most $r-1$ schools in $V_T(n)$ have been assigned. Therefore, in step r, ω_{br} results to be the less preferred effective school of b with probability of at least

$$1-\frac{1}{E[Y_T(n)]/2T-(r-1)+1}.$$

Since the waiting algorithm has $\bar{t}$ steps, the WA outputs profitable manipulations with probability of at least

$$\begin{aligned}\prod_{r=1}^{\bar{t}}\left(1-\frac{1}{E[Y_T(n)]/2T-(r-1)+1}\right) &\geq \left(1-\frac{1}{E[Y_T(n)]/2T-\bar{t}+2}\right)^{\bar{t}}\\ &\geq \left(1-\frac{1}{E[Y_T(n)]/4T}\right)^{\bar{t}}.\end{aligned}$$

First and second inequalities hold because n is sufficiently large and the sequence of random markets satisfies the thickness condition, also because $r\leq\bar{t}$. Therefore we have that

$$d(R_b(\omega_{b2})) \leq 1-\left(1-\frac{1}{E[Y_T(n)]/4T}\right)^{\bar{t}}.$$

By Bernoulli's inequality we know that $1-yx\leq(1-x)^y$ then $1-(1-x)^y\leq yx$ for any $x\in(0,1)$ and $y\geq 1$. We conclude that

$$d(R_b(\omega_{b2}))\leq\frac{4T\bar{t}}{E[Y_T(n)]}.$$

□

To have incentives to deviate from the true ranking is equivalent to $x_b>1$ at the end of the WA. Consider $\delta_{TTCo[R](b)}(R)=Pr[WA \text{ outputs } x_b>1]$. We bound this probability in the following lemma, which corresponds to Lemma 10 in Appendix B of [15].

Lemma 2.5.4. *Consider* $(\tilde{\aleph}^1,\tilde{\aleph}^2,\ldots)$ *a regular sequence of random markets that satisfies the thickness condition. There exists a large enough* $n\in\mathbb{N}$ *such that*

$$\delta_{TTCo[R](b)}(R)\leq\frac{4[\bar{t}T|[\omega_{b2}]|+1]}{E[Y_T(n)]},$$

for all $b\in A$.

Proof. First, note that

$$\begin{aligned} d(R_b(\omega_{b2})) &\le \sum_{\omega_{bj}\in[\omega_{b2}]} d(R_b(\omega_{bj})) \\ &\le \sum_{\omega_{bj}\in[\omega_{b2}]} \frac{4T\bar{t}}{E[Y_T(n)]} \\ &= \frac{4T\bar{t}}{E[Y_T(n)]} \sum_{\omega_{bj}\in[\omega_{b2}]} 1 \\ &= \frac{4T\bar{t}}{E[Y_T(n)]} |[\omega_{b2}]|, \end{aligned}$$

where the second inequality follows from Lemma 2.5.3.

Previous inequality is conditioned to the $TTCo[R]$ assignment. The same reasoning can be applied to all assignments $TTCo[R_b, \tilde{R}_{-b}]$. Hence

$$\begin{aligned} \Delta(\omega_{bj}) &= Pr\left[x_b > 1 \;\middle|\; Y_b(n) > \frac{E[Y_b(n)]}{2}\right] \\ &\le \frac{4T\bar{t}|[\omega_{b2}]|}{E[Y_T(n)]}. \end{aligned}$$

Remembering what was done in the proof of Affirmation 2.5.3, we know that

$$\begin{aligned} Pr\left[Y_b \le \frac{E[Y_b]}{2}\right] &\le Pr\left[Y_b \le \frac{E[Y_b]}{2}\right] + Pr\left[Y_b \ge \frac{3E[Y_b]}{2}\right] \\ &= Pr\left[|Y_b - E[Y_b]| \ge \frac{E[Y_b]}{2}\right] \\ &\le \frac{Var[Y_b]}{(E[Y_b]/2)^2} \le \frac{4}{E[Y_b]}. \end{aligned}$$

Consequently

$$\begin{aligned} \delta_{TTCo[R](b)}(R) &= \sum_{\omega_{bj}\in X_b} \Delta(\omega_{bj}) Pr\left[Y_T(n) \ge \frac{E[Y_T(n)]}{2}\right] \\ &\le \Delta(\omega_{b2}) Pr\left[Y_T(n) \ge \frac{E[Y_T(n)]}{2}\right] + \sum_{\omega_{bj}\neq\omega_{b2}} \Delta(\omega_{bj}) Pr\left[Y_T(n) \ge \frac{E[Y_T(n)]}{2}\right] \\ &\le Pr\left[Y_T(n) \ge \frac{E[Y_T(n)]}{2}\right] \frac{4T\bar{t}|[\omega_{b2}]|}{E[Y_T(n)]} + Pr\left[Y_T(n) \le \frac{E[Y_T(n)]}{2}\right] \\ &\le \frac{4T\bar{t}|[\omega_{b2}]|}{E[Y_T(n)]} + \frac{4}{E[Y_T(n)]}. \end{aligned}$$

Therefore

$$\delta_{TTCo[R](b)}(R) \le \frac{4(T\bar{t}|[\omega_{b2}]| + 1)}{E[Y_T(n)]}.$$

$\square$

In words, Lemmas 2.5.3 and 2.5.4 say that students opportunities vanish as the market size increases. These bounds imply the $\varepsilon-$Nash equilibrium.

Corollary 2.5.1. *Consider a sequence of random markets* $(\tilde{\aleph}^1, \tilde{\aleph}^2, \ldots)$ *regular and sufficient thick. There exists an* $N \in \mathbb{N}$ *large enough such that truth telling is an* ε*-Nash equilibrium, given the TTCo mechanism.*

Proof. Fixed a student b, an consider $\varepsilon > 0$. As before, the probability that b profitable manipulates the TTCo algorithm is $\delta_{TTCo[R](b)}(R)$. We prove that

$$E\left[u_b\left(TTCo[O^n, A^n, (R'_b, \tilde{R}_{-b})]\right)(b)\right] - E\left[u_b\left(TTCo[A, O, (R_i, \tilde{R}_{-b})]\right)(b)\right] = 0,$$

as n tends to infinity. We know that

$$\begin{aligned}
E\left[u_b\left(TTCo[R'_b, \tilde{R}_{-b}](b)\right)\right] - E\left[u_b\left(TTCo[(R_b, \tilde{R}_{-b})](b)\right)\right] &= \\
E\left[u_b\left(TTCo[R'_b, \tilde{R}_{-b}](b)\right) - u_b\left(TTCo[(R_b, \tilde{R}_{-b})](b)\right)\right] &\leq \\
Pr\left[TTCo[R'_b, \tilde{R}_{-b}](b) P_b TTCo[(R_b, \tilde{R}_{-b})](b)\right] &\leq \delta_{TTCo[R](b)}(R) \times \bar{t} \sup_{n \in \mathbb{N}, \omega \in O^n} u_b(\omega).
\end{aligned}$$

Moreover, the sequence of random markets is regular and satisfies the thickness condition. Then, $E[Y_T(n)] \to \infty$, from Lemma 2.5.2, as n tends to infinity. Hence $1/E[Y_T(n)]$ tends to zero. Moreover, considering

$$\varepsilon' = \frac{\varepsilon}{4(T\bar{t}|[\omega_{b2}]| + 1)} \frac{1}{\sup\limits_{n \in \mathbb{N}, \omega \in O^n} u_b(\omega)},$$

there exists $N \in \mathbb{N}$ such that

$$\frac{1}{E[Y_T(n)]} < \varepsilon', \tag{2.12}$$

for all $n \geq N$. Lemma 2.5.4 and expression (2.12) imply that

$$\begin{aligned}
E\left[u_b\left(TTCo[R'_b, \tilde{R}_{-b}](b)\right)\right] &- E\left[u_b\left(TTCo[R_b, \tilde{R}_{-b}](b)\right)\right] \leq \\
&\leq \delta_{TTCo[R](b)}(R) \sup_{n \in \mathbb{N}, \omega \in O^n} u_b(\omega) \\
&\leq \frac{4(T\bar{t}|[\omega_{b2}]| + 1)}{E[Y_T(n)]} \sup_{n \in \mathbb{N}, \omega \in O^n} u_b(\omega) \\
&\leq 4(T\bar{t}|[\omega_{b2}]| + 1) \sup_{n \in \mathbb{N}, \omega \in O^n} u_b(\omega)\varepsilon' \\
&\leq \varepsilon,
\end{aligned}$$

for all $n \geq N$. We conclude that truth-telling is an ε-Nash equilibrium. $\square$

By Theorem 2.5.1 we know that

$$\delta_k(n) \leq \frac{16nk}{\ln(n)} + \frac{3\sqrt{n}\ln(n)}{4k}.$$

So, the number of students that are not truth-telling at equilibrium tends to zero as the market size increases. Therefore, there exists $n \in \mathbb{N}$ such that truth-telling is the unique $\varepsilon-$Nash equilibrium for all $n \geq N$.

2.5.4 The $\varepsilon-$Bayesian Nash equilibrium

The Bayesian game of stage two is described by (A, O, f, D), where the types are drawn according to f, D is a probability distribution over the set of schools. Since the final allocation is determined by the TTCo algorithm, the pay-off function of each student i is

$$u_i(TTCo(i)) = \begin{cases} v_{\omega i} + \theta((v_{\tau \neq j})) & \text{if } TTCo(i) = \omega \text{ for some } \omega, \\ m_i & \text{otherwise.} \end{cases}$$

The expected utility function of each student i is

$$E[u_i] = \sum_{\omega \in O, \omega R_i i} (m_i - p_\omega + v_{\omega i}) Pr(TTCo_\omega[R_i, \tilde{R}_{-i}](i) = \omega) + m_i Pr_\omega(TTCo[R_i, \tilde{R}_{-i}](i) = i).$$

The ε-Bayesian Nash equilibrium of this stage is a profile of preferences such that a single student cannot improve substantially her final allocation under the $TTCo$ algorithm.

Definition 10. Given $\varepsilon > 0$, a strategy profile $(R_i^*(\hat{v}_i))_{i \in B, \hat{v}_i \in \hat{V}_i}$ is an ε-Bayesian Nash equilibrium if there is no $\overline{R}_i(\hat{v}_i) \in \hat{V}_i$ such that

$$E[u_i(TTCo[R_i^*(\hat{v}_i)](i))] \quad > \quad E[u_i(TTCo[\overline{R}_i(\hat{v}_i)], R^*(\hat{v})_{-i}](i))] + \varepsilon.$$

Students' behaviour at equilibrium is the same as in the complete information case.

Corollary 2.5.2. *Assuming a ranking with length k, the number of students with incentives to deviate from her true ranking tends to zero as the market size increases.*

Proof. This is a direct consequence of Theorem 2.5.1 where we prove this inequality for all possible fixed types. Making the summation over types we get this corollary. □

As before, previous Corollary does not guarantee that truth-telling is an $\varepsilon-$Bayesian Nash equilibrium. However, Corollary 2.5.1 can be generalized to the case of incomplete information for a sequence of regular random markets that satisfies the thickness condition.

Theorem 2.5.2. *Consider a regular sequence of random markets that satisfies the thickness condition. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that truth-telling by every student is an ε-Bayesian Nash equilibrium for any market in the sequence with more than N students.*

Proof. Let $(\tilde{\aleph}^n)_{n \in \mathbb{N}}$ a regular sequence of random markets that satisfy sufficient thickness condition, and $\varepsilon > 0$.
From Corollary 2.5.1 there exists $n \in \mathbb{N}$ such that truth-telling is an ε-Bayesian Nash equilibrium of $\tilde{\aleph}^N$, for all $N \geq n$.
This results holds for any of students type, then we can find N across types realizations. Therefore truth-telling is an ε-Bayesian Nash equilibrium. □

2.6 Concluding Remarks

The Top Trading Cover algorithm induces a family of Pareto efficient mechanisms that recursively respect top preferences and deal with indifferences. Its application is not only restricted to school choice problems. In general, we can use this family of mechanisms whenever we have indivisible objects and agents. For example, we can determine the allocation of apartments and scholarships between households and students.

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2.7 Appendix A1

The Existence of Top Trading Covers

To prove the existence of a maximum Top Trading Cover in each quasi-bipartite graph $G(S', B', \pi; p, v)$, we need more definitions from Graph Theory. Given a top trading cover T of G, a node $r \in S' \cup B'$ is said to be **covered** if there exists $r' \in S' \cup B'$ such that $(r, r') \in T$. A **path** P in G is a succession of nodes

$$\{r_1, t_1, r_2, t_2, \ldots r_n, t_n\}$$

such that $(r_i, t_i) \in E(S' \cup B', \pi; p, v)$ for all $i = 1, 2, \ldots, n$. We say that r_1 and t_n are the **end points** of P.

Definition 11. Let $G = G(S', B', \pi; p, v)$ a quasi-bipartite graph and a top trading cover T of G. A path P is an **augmenting path** for T if:

1. The two end point of P are not covered by T.

2. The edges of P alternate between edges $\in T$ and edges $\notin T$.

Let R a top trading cover of G, and P and augmenting path of R. We define and denote the operation between R and P, as

$$R \oplus P = (A \cup B) - (A \cap B),$$

i.e., we operate ttco's and paths through symmetric difference.

The algorithm to find a maximum top trading cover is described below

Step 1. Consider T a top trading cover of G, could be $T = \varnothing$.

Step 2. While exists an augmenting path P do

2.1 $T = T \oplus P$.

Step 3. While exists $r \in G \setminus T$ such that (r, r) is and edge of G do

3.1 $T = T \cup \{r\}$.

Step 4. Output T.

We have to prove that T is effectively a maximum top trading cover. However, this is clear, because we delete all possible augmenting paths. We conclude this proof applying Berge's Theorem: *"The matching M is maximum if and only if there is no augmenting path"*[7].

[7]For more details consult: Claude Berge (1963). *Topological Spaces.* Oliver and Boyd.

Chapter 3

On uniqueness of equilibrium prices in large assignment games

3.1 Introduction

The diversity of approaches and problems is one distinctive characteristic of the literature on real estate economy. We mention here only a few of them. The fact that properties are heterogeneous has been addressed through hedonic price modelling, both in academic papers (Rosen (1974), Maclennan (1977) (1982), Nelson (1978), ...) and in real life practice, in the U.S., by the Uniform Standards of Professional Appraisal Practice. While the approach is insightful for measuring the contributions of determinants (age, rooms, structure type, neighborhood, ...) in the valuation of a property, it is outperformed by dynamic models to predict other aspects of housing markets like bargaining (Yavaç (1992), Muthoo (1999), Merlo and Ortalo-Magné (2004), Di Pascuale and Wheaton (2012), ...), the role of list prices, their stickiness, and the optimal acceptance and rejection strategy of sellers (Salant (1991), Horowitz (1992), Albrecht et al. (2012), Merlo et al (2013)). These works adopt the tools developed by search theory to model the behavior of one side of the market and do not provide closed-form solutions for prices. Exceptions are Corominas-Bosch (2004) and Polanski (2007)) who elaborate a network framework to study bargaining processes.

Our contribution to this vast literature is to re-examine the heterogeneity feature of real estate markets by modeling them as large auction markets. More specifically, in the Assignment Game introduced in Shapley and Shubik (1971), we study the equilibrium prices when buyers and sellers do not know the valuation of others in large markets. It is intended to capture three features: 1. the stickiness of selling prices: Horowitz (1992) and Merlo et al (2013) provide empirical evidence where prices do not change even after houses have remained unsold for a long period of time or in the presence of different offers; 2. sellers accept to sell to (one of) the first buyers to meet the asking price: Merlo and Ortalo-Magné (2004) present evidence from England where the difference between the list price and the sold price is around 2 per cent, i.e. the bargaining is almost insignificant; and 3. households are not strategic: in an empirical work, Wheaton (1990) shows that buyers search for their favorite house. We model the problem as a three-stage game. First, nature draws the valuation of each agent, namely over the good they own for sellers, or the goods on sale for buyers. At stage two sellers simultaneously set prices, which are observed by buyers before stage three begins, when buyers report their rankings of houses to the Top Trading Cover (TTCo) algorithm, which determines the payoffs. During this assignment procedure, each buyer points to her most favorite house in a descending form and sellers sell their houses to one of the buyers that are willing to pay the list price.

The TTCo algorithm is developed in Chapter 2. The mechanism relies on a graph representation of the market, where a node is associated to all buyers and sellers, and directed edges represent the top choices of buyers. A loop represents a buyer that prefers remaining with her original amount to buy a house. A cover of the graph is selected by a tie breaking rule, transactions between buyers and sellers take place according to the cover and involved agents are withdrawn from the market. The procedure is iterated with the remaining agents, analogically to David Gale's Top Trading Circle; indeed TTCo also encompasses Hierarchical Exchange rules by Papai (2000). Moreover, in Chapter 2 we also prove that at the third stage of the mechanism, when market is sufficiently thick, it is an Epsilon Bayesian Nash equilibrium for buyers to report their true ranking to the TTCo. The thickness condition is reminiscent of the one by Fuhito and Pathak (2009), which specifies that for each seller the number of buyers is balanced, it implies that TTCo ends at the first iteration of the mechanism.

Our main result establishes that the Bayesian equilibrium of the second stage always exists and is unique. The Bayesian equilibrium, as far as we know, is the unique approach interested in establishing the uniqueness of equilibrium prices in the Assignment Game. In contrast, multiplicity of competitive equilibria and core allocations are pervasive in the Assignment Game. Assuming that agents cannot have more than one indivisible good, Quinzii (1984) shows that the core of the economy is non-empty, and not necessarily unique. Also, she analyses the conditions under which the core allocations coincide with competitive equilibrium allocations. Similarly, Demange (1984) proves the existence of at least one competitive equilibrium, which is not always unique, in a model with externalities. Demange and Gale (1985) show that the set of equilibria have a lattice structure in the Assignment game. A generalization of the assignment game is done by Scarf (1990, 1994) through non-convexities. Moreover, Alkan, Demange and Gale (1991) shows that the set of efficient and envy free allocations is non-empty. Even in the presence of multiplicity of fair assignments, they show that it is possible to do some comparative statistics when the amount of money increases. In similar models with multiplicity of fair assignments, Tadenuma and Thomson (1991) study when fair allocations satisfy consistency and Svensson (2009) characterized the set of fair allocation rules that are strategy-proof.

We also provide a closed form solution when valuations are exponentially distributed. The assumption echoes empirical works such as Trippi (1977) who estimates the length of time on market of real estate with data from San Diego, using an exponential distribution. So does I-Chun Tsai (2010) with data from Taiwan and Horowitz (1992) who studies real estate in Baltimore. Unsurprisingly, the relation between the price at equilibrium and its corresponding seller valuation is positive; while the price decreases when the number of sellers and the distribution parameter increases. These results are robust to the cases where agents' preferences are exponentially distributed but not identically. We analyze two cases: different parameter distributions and overlapping valuations.

The paper is organized as follows. The model and the TTCo are presented in Section 2. The analysis of equilibrium incentives of buyers when the market is large under the thickness condition is carried out in Section 3. The uniqueness of equilibrium price vector and its closed-form solution is established in Section 4.

3.2 The Model

We consider a market with indivisible goods, money, and two disjoint sets of agents: a set of sellers, S, and a set of buyers, B. Each member of the market initially has a certain amount of money. Only sellers initially own one and only one indivisible good. A generic amount of money is represented by $\omega \in \mathbb{R}$, and r is used to represent any agent, so $r \in S \cup B$. We use $\varnothing$ whenever a member does not own any indivisible good. Any member owns at most one indivisible good.

Consider $S = \{s_1, s_2, \ldots, s_m\}$, the set of m sellers. To identify a generic seller we use s_j, or s. Each seller initially holds only one indivisible good also identified by s_j; and an amount of money $\overline{\omega}_j$. By simplicity we suppose that $\overline{\omega}_j = 0$. Thus, the initial endowment of seller s_j is the basket $(0, s_j)$. We suppose that each seller s_j has a preference relation represented by the utility function $u_{s_j}(\cdot)$, defined over baskets $(\omega, s) \in \mathbb{R} \times (\{s_j\} \cup \{\varnothing\})$. We assume the following quasi-linear utility function

$$u_{s_j}(\omega, s; v_j) = \begin{cases} \omega + v_j & \text{if } s = s_j, \\ \omega & \text{if } s = \varnothing. \end{cases}$$

The type of s_j is the valuation $v_j \in \mathbb{R}_+$ of her own good. Let $V_j = \mathbb{R}_+$ be the set of all possible types of s_j.

Consider B, the set of n buyers. We identify a generic buyer by i. Each buyer i initially owns only an amount of money $\overline{\omega}_i \geq 0$, and no good. The initial endowment of buyer i is the basket $(\overline{\omega}_i, \varnothing)$. We suppose that each buyer i has a preference relation represented by the utility function $u_i(\cdot)$ over the set of baskets $(\omega, s) \in \mathbb{R} \times (S \cup \{\varnothing\})$. We assume the following quasi-linear utility function

$$u_i(\omega, s; \hat{v}_i) = \begin{cases} \omega + v_{ji} & \text{if } s = s_j, \\ \omega & \text{if } s = \varnothing. \end{cases}$$

Each buyer i has a valuation $v_{ji} \in \mathbb{R}$ of good s_j for all $j \in \{1, 2, \ldots, m\}$. The type of buyer i is the vector $\hat{v}_i = (v_{1i}, \ldots, v_{mi}, \overline{\omega}_i)$. The set of all possible types of buyer i is denoted by $\hat{V}_i \subseteq \mathbb{R}^{m+1}$.

The state of the market is the vector $v = (v_1, \ldots, v_m, \hat{v}_1, \ldots, \hat{v}_n) \in \mathbb{R}^m_+ \times \mathbb{R}^n$. The set of all possible states of the market is the Cartesian product between all sets V_j and the sets $\hat{V}_i$; let $V \equiv \prod_{j=1}^m V_j \times$

$\prod_{i=1}^{n} \hat{V}_i$. We suppose that the state of the market $v \in V$ is drawn according to a probability function f from V to $\mathbb{R}$, of common knowledge.

An *assignment* is a function Γ from $S \cup B$ to $\mathbb{R} \times (S \cup \{\varnothing\})$, i.e Γ assigns to each $r \in S \cup B$ a basket composed of money and a good, or not. The allocation of member r is denoted by $\Gamma(r) = (\Gamma_\omega(r), \Gamma_s(r))$, so $\Gamma_\omega(r) \in \mathbb{R}$ and $\Gamma_s(r) \in S \cup \{\varnothing\}$.

The assignment Γ is *feasible* if it satisfies the following conditions

1. $\sum_{r \in S \cup B} \Gamma_\omega(r) \leq \sum_{i=1}^{n} \overline{\omega}_i$,

2. let r, $r' \in S \cup B$ such that $\Gamma_s(r) = \Gamma_s(r') \in S$ then $r = r'$, and

3. for all $s_j \in S$ there exists some $r \in S \cup B$ such that $\Gamma_s(r) = s_j$.

Conditions 2 and 3 tell us that at Γ, any good in the market is assigned to one and only one agent.

We say that a basket (ω, s) is *individually rational (I.R.)* for agent r if and only if the utility of r by holding (ω, s) is greater or equal than the utility of r by holding her initial endowment. So, Γ is an *individually rational assignment (I.R.)* if each member of the market weakly prefers her allocation under Γ to her initial endowment.

3.2.1 The Game

A Three-step Game

We consider a three-step game. Nature moves first, determining the type of each member of the market according to the probability distribution f. All members of the market observe their type, but do not observe the type of the others. At stage 2, sellers decide simultaneously to set the price of their good. If a seller s_j decides to sell her good, she sets a non-negative price p_j. Otherwise, she sets a price $p_j = +\infty$. Thus, $A_j = \mathbb{R}_+ \cup \{+\infty\}$ is the set of actions of seller s_j. We define a *price vector* $p = (p_1, p_2, \ldots, p_m)$ as a vector in $A_1 \times A_2 \times \cdots A_m$. We denote by $E(p)$ the set of baskets on sale, i.e. $E(p) = \{(p_s, s) \in \mathbb{R}_+ \times S \mid p_s \neq \infty\}$; a basket (p_s, s) in $E(p)$ is denoted by $e_s(p)$.

At stage three buyers report their preferences over baskets. Each buyer observes the price vector p of the second stage and sets her preferences $\overline{\pi}_i(p; \hat{v}_i)$ over the set of baskets in $E(p)$ she can afford and her initial endowment $(m_i, \varnothing)$, we use $E_i(p)$ to represent this set. Naturally, buyers might be indifferent between different baskets. So, for each buyer i, $\overline{\pi}_i(p;\hat{v}_i)$ is a non-strict list of preferences over the set of baskets $E_i(p)$. We write $e_s(p) \sim_i e_{s'}(p)$ when i is indifferent between buying s and s', i.e. $m_i + v_{si} - p_s = m_i + v_{s'i} - p_{s'}$, and $e_s(p)\overline{P}_i e_{s'}(p)$ when i strictly prefers buying s to s', i.e. $m_i + v_{si} - p_s > m_i + v_{s'i} - p_{s'}$. Also, not all baskets in $E_i(p)$ are better than the initial basket $(m_i, \varnothing)$, so, we write $i\overline{\pi}_i e_s(p)$ when i weakly prefers remaining with her initial endowment buying s, i.e. $m_i \geq m_i + v_{si} - p_s$. The ranking of buyer i is represented by

$$\overline{\pi}_i : [e_{s_{i_1}}(p)], [e_{s_{i_2}}(p)], \ldots, [e_{s_{i_k}}(p)], [i], [e_{s_{i_{k+1}}}(p)], [e_{s_{i_{k+2}}}(p)], \ldots, [e_{s_{i_K}}(p)],$$

where $[e_{s_{i_j}}(p)] = \{e_s(p) \in E_i(p) \mid e_s(p) \sim_i e_{s_{i_j}}\}$ and $[i] = \{e_s(p) \in E_i(p) \mid e_s(p) \sim_i (m_i, \varnothing)\}$ are called *classes of indifference*. Each buyer strictly prefers remaining with her initial basket to being assigned to some non I.R. basket. Our analysis only focuses on I.R. baskets.

Let $\Pi_i(p;\hat{v}_i)$ be the set of all possible rankings $\overline{\pi}_i(p;\hat{v}_i)$, thus, $\Pi_i(p;\hat{v}_i)$ is the set of actions of i. The profile of reported rankings is the vector

$$\overline{\pi}(p;v) = (\overline{\pi}_1(p;\hat{v}_1), \overline{\pi}_2(p;\hat{v}_2), \ldots, \overline{\pi}_n(p;\hat{v}_n)).$$

Given the profile $\overline{\pi}(p;v)$ of reported rankings, the payoffs of each member of the market are induced by the Top Trading Cover (TTCo) algorithm, to be introduced in the following section.

The TTCo Mechanism and Payoffs

To describe the TTCo algorithm, we must first record some concepts from Graph Theory (see Appendix A2). In general, the TTCo algorithm removes sequentially maximum top trading covers in every step. Since the maximum ttco is not always unique, we use tie breaking rules to choose one of them. Formally, let G be a quasi-bipartite graph and $\Upsilon[G] = \{T \subseteq E[G] \mid T \text{ is a ttco of G}\}$ the set of all top trading covers of G. Note that $\varnothing \in \Upsilon[G]$, i.e., the picking rule can decide not to pick any cover in some iterations of the ttco algorithm. The set of elements which are subsets of $\Upsilon[G]$ is denoted by $2^{\Upsilon[G]}$. A *tie breaking rule* is a function $\varsigma : \Upsilon[G] \to \Upsilon[G]$ such that $\varsigma(T)$ is a maximum ttco of G, for all $T \in 2^{\Upsilon[G]} \backslash \{\varnothing\}$.

The following questions immediately arise: how do tie breaking rules affect the final assignment? Do tie breaking rules modify agent's behaviour? In Chapter 1, we prove that the assignments produced by tie-breaking rules are characterized by Pareto efficiency and that recursively respects top Rankings. Also, we prove that seller's behaviour at equilibrium is not affected by tie-breaking rules. In other words, equilibrium prices are invariant with respect to tie-breaking rules.

The above results allow us to fix a tie-breaking rule without loss of generality. The tie-breaking rule that we use in the assignment procedure is based on the lexicographic order defined over the set of edges of a quasi-bipartite graph G.

Definition 12. Let $G = (S' \cup B', E(S' \cup B', p, \overline{\pi}(p;v)))$ be a quasi-bipartite graph, and $(s_j, i), (s_{j'}, i') \in E(S' \cup B', p, \overline{\pi}(p;v)) \cap (S \times B)$. The **lexicographic order** $\preccurlyeq_L$ over $E(S' \cup B', p, \overline{\pi}(p;v)) \cap (S \times B)$ is defined as follows

$$(s_j, i) \preccurlyeq_L (s_{j'}, i') \text{ if and only } \begin{cases} j < j', \text{ or} \\ j = j' \text{ and } i \leq i'. \end{cases}$$

We recall that buyers and sellers are indexed by the set of natural numbers, a well-ordered set, which implies the veracity of the next observation.

Observation 3.2.1. Each subset $\overline{E}$ of $E(S' \cup B', p, \overline{\pi}(p;v)) \cap (S \times B)$ has a minimum element, i.e. there exists (s, i) such that $(s, i) \preccurlyeq_L (s', i')$ for all $(s', i') \in \overline{E}$. We write $(s, i) = \min \overline{E}$.

By Proposition 2.3.1 we can ensure the existence of at least one non-empty maximum top trading cover regardless the quasi-bipartite graph. Thus, we can proceed to define the lexicographic tie-breaking rule over $\Upsilon[G]$ using Definition 12 and Observation 3.2.1.

Definition 13. Let G be a quasi-bipartite graph, and $T = \{T_\eta \mid T_\eta \in \Upsilon[G] \text{ and } T_\eta \text{ is maximum for all } 1 \leq \eta \leq K\}$ a finite subset of maximum top trading covers of G. Consider $T'_\eta = T_\eta \cap (S \times B)$ for all $\eta \in \{1, 2, \ldots, K\}$. The **lexicographic tie breaking rule** $\varsigma_{\mathbf{L}}$ is the function $\varsigma_L : 2^{\Upsilon[G]} \to \Upsilon[G]$ such that $\varsigma_L(T) = T_\kappa$ if and only if

$$\min \left[T'_\kappa \setminus \bigcap_{\eta=1}^{K} T'_\eta \right] \preccurlyeq_L \min \left[T'_t \setminus \bigcap_{\eta=1}^{K} T'_\eta \right],$$

for all $t \neq \kappa$.

The Top Trading Cover Algorithm

Consider a market $(S, B, \overline{\pi}(p; v))$ and a tie breaking rule ς. The assignment proceed as follows:

Step 0: Let $S^0 = \{s_j \in S | p_j \neq \infty\}$ and $B^0 = B$. If $S^0 = \varnothing$, the algorithm finishes and all agents in the market receive their original basket. Otherwise, $S^0 \neq \varnothing$, the procedure continues to the iterative step t.

Step t: Let $G^t = G^t(S^{t-1}, B^{t-1}, p, \overline{\pi}(p, v))$ be the $t-$th quasi-bipartite graph. We choose a maximum TTCo of G^t according to ς. For all pairs $(s, i) \in \varsigma(\Upsilon[G^t])$, seller s sells her good to buyer i. Then, baskets $(p_s, \varnothing)$ and $(m_i - p_s, s)$ are assigned to s and i, respectively. The maximum ttco $\varsigma(\Upsilon[G^t])$ is removed from the market. Let S^t and B^t be the sets of buyers and sellers remaining in the market after removing this maximum ttco. If both are non-empty, we iterate the procedure. Otherwise, the algorithm stops.

Agents that sell/buy a good receive a payoff of $p_s/m_b - p_s + v_{sb}$ respectively. Agents that do not sell, or buy, any good during the assignment procedure are assigned to their initial basket. The payoffs of these sellers and buyers are v_j and m_i, respectively.

The final allocation produced by the algorithm described above depends on the price vector p, the profile of reported rankings $\overline{\pi}$, the state of nature v and the tie breaking rule ς that we use. We sometimes write the final allocation as $TTCo[S, B, \overline{\pi}; p, v, \varsigma]$. We use $TTCo[S, B, \overline{\pi}; p, v, \varsigma](r)$ to represent the assignment given to $r \in S \cup B$ under the assignment procedure. If there is no confusion we refer to the assignment only by $TTCo$.

From now on, we fix the lexicographic tie breaking rule ς_L. Below, we provide an example to show how the $TTCo[\varsigma_L]$ algorithm works.

Example 3.2.1. Consider the set of seller $S = \{s_1, s_2, s_3\}$, and the set of buyers $B = \{1, 2, 3, 4\}$. We assume that sellers have the same valuation $v_j = 0$ for all $j \in \{1, 2, 3\}$, and buyers also have the same valuation vector:

$$\hat{v}_1 = \hat{v}_2 = \hat{v}_3 = \hat{v}_4 = (5, 5, 5).$$

In stage 2, all sellers set the same price, $p_1 = p_2 = p_3 = 1$; and the amount of money given to each buyer is $m_1 = m_2 = 2.5$ and $m_3 = m_4 = 3.5$.

Suppose buyers report the following rankings at the end of the stage 3 (which are not necessarily equilibrium strategies):

$$
\begin{aligned}
\overline{\pi}_1(p,\hat{v}_1) &: \ e_{s_1}(p), e_{s_2}(p),\\
\overline{\pi}_2(p,\hat{v}_2) &: \ e_{s_1}(p), e_{s_2}(p),\\
\overline{\pi}_3(p,\hat{v}_3) &: \ e_{s_1}(p), e_{s_2}(p), e_{s_3}(p),\\
\overline{\pi}_4(p,\hat{v}_4) &: \ e_{s_1}(p), e_{s_2}(p), e_{s_3}(p).
\end{aligned}
$$

Figure 3.1 illustrates the step 1 of the TTCo algorithm. The dotted line is the maximun ttco chosen by ς_L, so the pair $(s_1, 1)$ is removed from the market.

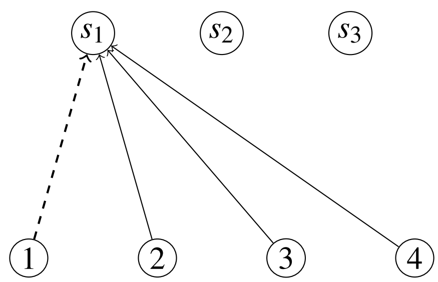


Figure 3.1: First step of Top Trading Cover algorithm

In the step 2 (Figure 3.2) the set $\{(2, s_2)\}$ is the maximum ttco removed from the market.

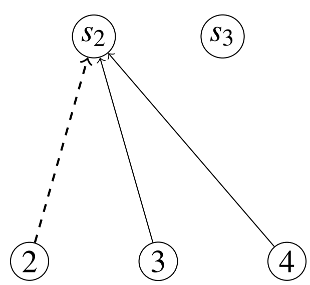


Figure 3.2: Second step of Top Trading Cover algorithm

Now, the ttco removed from the market in the Step 3 is $\{(3, s_3)\}$ (see Figure 3.3). Consequently, the TTCo algorithm stops because there are no sellers remaining in the market after removing s_3. So, buyer 4 is assigned to her original basket.

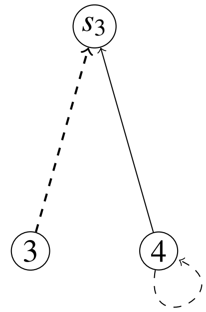


Figure 3.3: Third and last step of Top Trading Cover algorithm

Therefore, the final allocation is

$$TTCo = \begin{pmatrix} s_1 & s_2 & s_3 & 1 & 2 & 3 & 4 \\ (p_1,\varnothing) & (p_2,\varnothing) & (p_3,\varnothing) & (1.5,s_1) & (1.5,s_2) & (2.5,s_3) & (3.5,\varnothing) \end{pmatrix}.$$

Properties of the TTCo algorithm

Consider $\overline{\pi}(p;v)$ a profile of rankings, an assignment Γ and a pair $(i,s_j) \in B \times S$. We say that Γ is *blocked by* $(\mathbf{i},\mathbf{s_j})$ *with respect to* $\overline{\pi}(\mathbf{p};\mathbf{v})$, if and only if

- $\Gamma(s_j) = (0,s_j)$ and $p_j \neq \infty$, and
- $e_{s_j}(p)\overline{P}_i(p;\hat{v}_i)e_{\Gamma(i)}(p)$.

If Γ is blocked by the pair (i,s_j), we say that it is a *blocking pair* of Γ. An assignment Γ is *non-wasteful with respect to* $\overline{\pi}(\mathbf{p};\mathbf{v})$ if and only if it is individually rational and it is not blocked by any pair. In words, an assignment is non-wasteful if there is no buyer such that she strictly prefers an unassigned basket to her allocation.

The non-wastefulness property will be useful in the section of the Large Markets to prove that all assignments can be generated by the *TTCo* algorithm through *dropping strategies*. This in turn implies that any deviation is identified through *effective sellers*. Buyers with at most one effective seller do not have incentives to deviate from her true ranking. Given a profile of reported rankings $\overline{\pi}$, and a non-wasteful assignment Γ with respect to $\overline{\pi}$, $s' \in S$ is an *effective seller* for buyer $b \in B$ if and only if $\Gamma_s[\overline{\pi}](b) = s'$. Analogously, we define an *effective buyer* for any seller s.

We say that an assignment satisfies Pareto efficiency if there is no other assignment that assigns each agent in the market a weakly preferred basket and at least one agent a strictly preferred basket. So, it is impossible to make an agent better off without making another agent worse off. An assignment Γ is *Pareto efficient* at rankings profile $\overline{\pi}$ if there is no other feasible assignment Γ' such that $u_r(\Gamma'[\overline{\pi}](r)) \geq u_r(\Gamma[\overline{\pi}](r))$ for all $r \in S \cup B$, with strict preference for some r'. The $TTCo[\varsigma]$ assignment is Pareto efficient for all tie-breaking rules ς.

Proposition 3.2.1. *The assignment* $TTCo[A,B,\overline{\pi};p,v,\varsigma]$ *is individually rational and non-wasteful.*

Proof. See Chapter 1. □

3.2.2 Solution Concept

We need some extra concepts and notations. A *decision rule* of seller s_j is a function $\alpha_j : V_j \to A_j$ mapping a type into an action. A *pure strategy* for seller s_j is an element $\sigma_j \in S_j = \{\alpha : \alpha : V_j \to A_j\}$. Given a price vector p and the realization of her type $\hat{v}_i$, an action of buyer i is a ranking $\overline{\pi}_i(p,\hat{v}_i)$ over the set of baskets $\{(p_s,s)|p_s \neq \infty\} \cup \{(m_i,\varnothing)\}$. A *decision rule* $\beta_i(p,\hat{v}_i)$ for buyer i is a function that maps a price vector and types into rankings.

Definition 14. Let $\sigma^* = (\sigma_1^*,\ldots,\sigma_m^*)$ and $\beta^* = (\beta_1^*,\ldots,\beta_n^*)$ be profiles of strategies for sellers and buyers respectively, and $\varepsilon > 0$. A $\varepsilon-$**sub-game perfect Bayesian Nash equilibrium** is a profile of pure strategies

$$(\sigma_1^*,\ldots,\sigma_m^*),\beta_1^*(\sigma^*),\ldots,\beta_n^*(\sigma^*)) = (\sigma_1^*,\ldots,\sigma_m^*,\beta_1^*,\ldots,\beta_n^*)$$

such that

1. For all $s_j \in S$,

$$E[u_{s_j}(TTCo(\sigma_j^*,\sigma_{-j}^*,\beta^*(\sigma_j^*,\sigma_{-j}^*))(s_j)] + \varepsilon \geq E[u_{s_j}(TTCo(\sigma_j,\sigma_{-j}^*,\beta(\sigma_j,\sigma_{-j}^*))(s_j)].$$

2. For all $i \in B$,

$$E[u_i(TTCo(\sigma^*,\beta_i^*(\sigma^*),\beta_{-i}^*(\sigma^*))(i)] + \varepsilon \geq E[u_i(TTCo(\sigma^*,\beta_i'(\sigma^*),\beta_{-i}^*(\sigma^*))(i)].$$

3.3 Equilibrium Analysis of Third Stage

To analyse the set of $\varepsilon-$sub-game perfect Bayesian Nash equilibria, we proceed by backward induction. We suppose that buyers observe the price vector p set at the end of stage two. Also, we consider that $f = f_S f_B$, where f_S is the type distribution of sellers and f_B is the type distribution of buyers. In other words, we assume that buyers behaviour is independent from sellers behaviour.

The third stage is a simultaneous game where each buyers only knows her own preference relation and reports a ranking of sellers to the TTCo algorithm. Moreover, sellers wait until someone is interested in buying their good. So, an $\varepsilon-$Bayesian Nash equilibrium of the third stage is a profile of decision rules such that every buyer does not have incentives to deviate from her equilibrium strategy. This game coincides with the game described in Chapter 1. In this paper we proved the existence of a unique $\varepsilon-$Bayesian Nash equilibrium where all agents report their true ranking in large markets that satisfies the thickness condition. Moreover, the TTCo algorithm ends at step 1 in these kind of markets, i.e. all sellers sell their good. Before to present previous result, we remember the thickness condition.

Assume that all sellers want to sell her good, i.e. $p_s \neq \infty$ for all $s \in S$. Consider $D = (d_1, d_2, \ldots, d_m)$ a probability distribution over S such that, without loss of generality, $d_j \geq d_{j+1}$ and $d_j > 0$ for all $s_j \in S$. We say that seller s_j is **more popular** than seller $s_{j'}$ if $d_j \geq d_{j'}$, that is to say, s_j is top ranked in more preference list that $s_{j'}$.

A **random market** is a tuple $\tilde{\aleph} = (B, S, D, k)$ with an associated profile of random rankings $\tilde{\pi}$[1]. Given $\varepsilon > 0$, a profile of rankings $\overline{\pi}^* = (\overline{\pi}_i^*)_{i \in B}$ is an ε-*Nash equilibrium* if there is no $i \in B$ and $\overline{\pi}_i'$ such that

$$E[u_i(TTCo[S,B,(\overline{\pi}_i^*, \tilde{\pi}_{-i}^*)](i))] + \varepsilon > E[u_i(TTCo[S,B,(\overline{\pi}_i', \tilde{\pi}_{-i}^*)](i))],$$

[1]**Step 1.** Select randomly a seller following distribution D. List this seller as the most preferred seller of i.

Step t. Select randomly a seller following distribution D.

t.1 If this seller has not been previously drawn in steps 1 through $t-1$, list this seller as the t^{th} most preferred seller of i, go to $t+1$.

t.2 Otherwise, we select randomly a seller following distribution D, go to $t.1$.

The procedure ends at step k.

where the expectation is taken with respect to random rankings.

A **sequence of random markets** is denoted by $(\tilde{\aleph}^1, \tilde{\aleph}^2, \ldots)$, where each $\tilde{\aleph}^n$ is a random market

$$(S^n, B^n, D^n, k^n; p^n)$$

such that $D^n = (d_1^n, d_2^n, \ldots)$, $|B^n| = n$ and $|S^n| = m_n$, for all $n \in \mathbb{N}$. Also, some regularity conditions over sequences of random markets are necessary to define thickness condition.

Definition 15. A sequence of random markets $(\tilde{\aleph}^1, \tilde{\aleph}^2, \ldots)$ is **regular** if there exists a positive integer k such that

1. $k^n = k$ for all n,

2. $m_n \leq n$ for all n.

Let $V_T(n) = \left\{ s \in S \;\middle|\; \max_{s' \in S} \frac{d_{s'}^n}{d_s^n} \leq T \right\}$ the random set that denotes the set of sellers sufficiently popular ex-ante. Below, we present the thickness condition of Kojima and Pathak.

Definition 16. A sequence of random markets is **sufficiently thick** if there exists $T \in \mathbb{R}$ such that

$$E[|V_T(n)|] \to \infty,$$

as $n \to \infty$.

Now, we present the results from Chapter 2 that we use to compute the set of $\varepsilon-$Bayesian Nash equilibria. The first result is a property of the TTCo algorithm. The following proposition establishes that all sellers sell their good in the first step of the TTCo algorithm, regardless the tie breaking rule, when the market satisfies the thickness condition.

Proposition 3.3.1. *Let $(\tilde{\aleph}^1, \tilde{\aleph}^2, \ldots)$ a regular sequence of random markets that satisfy the thickness condition. There exists $N \in \mathbb{N}$ such that the TTCo algorithm removes all sellers in $\tilde{\aleph}^n$ at the end of its first step, for all $n \geq N$.*

Finally, with the thickness condition we can ensure the existence of a unique $\varepsilon-$Bayesian Nash equilibrium where all buyers are truth-telling

Theorem 3.3.1. *Consider a regular sequence of random markets that satisfies the thickness condition. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that truth-telling by every buyer is an ε-Bayesian Nash equilibrium for any market in the sequence with more than N buyers.*

3.4 Equilibrium Price Vector of Second Stage

To compute the equilibrium of the second stage we assume a large thick market $\tilde{\aleph}$ and we suppose that agents' types are drawn according to a probability function f of common knowledge. The Bayesian Nash equilibrium of this stage is a profile of decision rules $(p_s(v_s))_{s\in S}$ such that each seller s maximizes her expected utility function. Remember, a decision rule maps sellers' types into prices.

3.4.1 Invariance with respect to tie breaking rules

We prove that the equilibrium price is independent of the tie breaking rule used in the TTCo algorithm. For a market $(S,B;v)$, the payoff function of each seller s is the following:

$$u_s = \begin{cases} p_s & \text{if } s \text{ sells her good,} \\ v_s & \text{otherwise.} \end{cases}$$

Moreover, the expected utility function of s is:

$$E[u_s](p_s,p_{-s};\varsigma) = p_s Pr[\text{Selling}] + v_s Pr[\text{not selling}] = (p_s - v_s)Pr[\text{Selling}] + v_s, \tag{3.1}$$

for all type $v_s \in V_s$. At equilibrium, each seller s maximizes $E[u_s]$.

Now, we know that seller s sells her good if and only if there exists a buyer i such that $TTCo_s(i) = s$, then

$$Pr[\text{Selling}] = Pr[TTCo[\overline{\pi};(p_s(v_s),p_{-s}(v_{-s})),\varsigma](s) = i, \text{ for some } i \in B].$$

The decision rule at equilibrium can be written as

$$p_s^* = p_s^*(TTCo[S,B,\overline{\pi};p_i^*,p_{-i}^*,v,\varsigma]).$$

Proposition 3.4.1. ***Invariance with respect to the tie breaking rule.*** *Let $s \in S$, then*

$$p_s^* = p_s^*(TTCo[S,B,\overline{\pi};(p_s^*,p_{-s}^*),v,\varsigma_L]) = p_s^{**}(TTCo[S,B,\overline{\pi};(p_s^{**},p_{-s}^*),v,\varsigma]) = p_s^{**},$$

for all tie breaking rules ς.

Proof. By definition, p_s^* is the decision rule that maximizes $E[u_s]$ when the algorithm uses the tie breaking rule ς_L, then

$$E[u_s](p_s^*, p_{-s}^*; \varsigma_L) \geq E[u_s](p_s, p_{-s}^*; \varsigma_L),$$

for all $p_s \in S_s$, and p_{-s}^* denotes the decision rules at equilibrium of sellers different from s. Particularly

$$E[u_s](p_s^*, p_{-s}^*; \varsigma_L) \geq E[u_s](p_s^{**}, p_{-s}^*; \varsigma_L),$$

where p_s^{**} is the decision rule at equilibrium when the TTCo algorithm uses ς. Substituting equation (3.1) in the above inequality, we have that

$$(p_s^* - v_s)Pr[TTCo[\overline{\pi}; (p_s^*, p_{-s}), v, \varsigma_L](s) = i] + v_s \geq$$
$$(p_s^{**} - v_s)Pr[TTCo[\overline{\pi}; (p_s^{**}, p_{-s}), v, \varsigma_L](s) = i] + v_s.$$

So

$$\begin{aligned} &(p_s^* - v_s)Pr[TTCo[\overline{\pi}; (p_s^*, p_{-s}^*), v, \varsigma_L](s) = i] \geq \\ &(p_s^{**} - v_s)Pr[TTCo[\overline{\pi}; (p_s^{**}, p_{-s}^*), v, \varsigma_L](s) = i]. \end{aligned} \tag{3.2}$$

Analogously, for any tie breaking rule ς we get that

$$\begin{aligned} &(p_s^{**} - v_s)Pr[TTCo[\overline{\pi}; (p_s^{**}, p_{-s}^*), v, \varsigma](s) = i] \geq \\ &\quad (p_s^* - v_s)Pr[TTCo[\overline{\pi}; (p_s^*, p_{-s}^*), v, \varsigma](s) = i]. \end{aligned} \tag{3.3}$$

We claim that

Affirmation 3.4.1. For any price vector p, we have that

$$Pr[TTCo[\overline{\pi}, p; v, \varsigma_L](s) = i] = Pr[TTCo[\overline{\pi}, p; v, \varsigma](s) = i],$$

for any tie breaking rule ς.

Proof. In Chapter 1 we prove that the assignment $TTCo[\varsigma]$ partitions the quasi-bipartite graph $G[S, B, \overline{\pi}; p, v]$ in disjoint subsets $TTCo[\varsigma]^k$ such that

$$(s,b) \in TTCo[\varsigma]^k \text{ if and only if } s \text{ is the most preferred seller of } b \text{ in } G^k,$$

i.e. seller s sells her good if and only she belongs to some $TTCo[\varsigma]^k$ for $k \in \{1,2,\ldots,K^\varsigma\}$. Consequently

$$\begin{aligned}
Pr[\text{selling}] &= \sum_{k=1}^{K^\varsigma} Pr[\text{ selling} \mid s \in TTCo^k] \cdot Pr[s \in TTCo^k] \\
&= \sum_{k=1}^{K^\varsigma} 1 \cdot Pr[s \in TTCo^k] \\
&= \sum_{k=1}^{K^\varsigma} Pr[(s,b) \in TTCo^k] \\
&= \sum_{k=1}^{K^\varsigma} Pr[v_{si} - p_s = \{v_{s'b} - p_{s'} \mid s' \in S\}_{(k)} \geq 0] \\
&= Pr[TTCo_s[\overline{\pi} : (p_s^*, p_{-s}), v, \varsigma](b) = s \text{ for some } b],
\end{aligned}$$

where $\{v_{s'b} - p_{s'} — s' \in S\}_{(k)}$ is the statistic of order k and K^ς is the number of elements in the partition induced by $TTCo[\varsigma]$. However, seller s does not know buyers rankings in stage two, whereby she does not know K_ς, then we must consider $E_s[K_\varsigma]$, the expectation of K_ς, to compute the expected utility $E[u_s]$.

Let $l_{\max}$ the maximum length of buyers rankings (if $S = m$, then $l_{\max} \leq m$). It is clear that the number of elements in the partition is equivalent to the number of steps of the TTCo algorithm, consequently

$$1 \leq K_s^\varsigma \leq l_{\max}, \tag{3.4}$$

the TTCo algorithm can terminate in the first step, the maximum ttco of G^1 covers all agents in the market, or the algorithm can finish in $l_{\max}$ iterations. Then

$$E_s[K^\varsigma] = \sum_{k=1}^{l_{\max}} kPr[\text{TTCo terminates in } K^\varsigma \text{ steps.}]. \tag{3.5}$$

It is important to note that expression (3.4) holds for any tie breaking rule ς, that is to say

$$E_s[K^\varsigma] = E_s[K^{\varsigma_L}].$$

Moreover

$$\begin{aligned}
Pr[TTCo[\overline{\pi}, p; v, \varsigma_L](b) = s] &= Pr[(s,b) \in TTCo^k[\varsigma_L]] \\
&= \frac{|\{T \in \Upsilon[G^k] \mid (s,b) \in T \text{ and } T \text{ is maximum ttco}\}|}{|\Upsilon[G^k]|} \\
&= Pr[(s,b) \in TTCo^k[\varsigma] \\
&= Pr[TTCo[\overline{\pi}, p; v, \varsigma](b) = s]. \qquad (3.6)
\end{aligned}$$

$\square$

Therefore, expressions (3.5) and (3.6) imply that

$$Pr[TTCo[\overline{\pi};(p_s^*,p_{-s}^*),v,\varsigma_L](s)=i] \quad = \quad Pr[TTCo[\overline{\pi};(p_s^*,p_{-s}^*),v,\varsigma](s)=i]=X \tag{3.7}$$

and

$$Pr[TTCo[\overline{\pi};(p_s^{**},p_{-s}^*),v,\varsigma_L](s)=i] \quad = \quad Pr[TTCo[\overline{\pi};(p_s^{**},p_{-s}^*),v,\varsigma](s)=i]=Y, \tag{3.8}$$

for all tie breaking rules ς. Using expressions (3.2), (3.3), (3.7) and (3.8) we have that

$$\begin{aligned}(p_s^*-v_s)X &\geq (p_s^{**}-v_s)Y \text{ and}\\ (p_s^*-v_s)X &\leq (p_s^{**}-v_s)Y.\end{aligned}$$

Above expression implies

$$(p_s^*-v_s)X=(p_s^{**}-v_s)Y,$$

for all $v_s \in V_s$. By the Bapat-Beg Theorem (see Appendix C2), the probabilities X and Y are polynomials from the variable v_s because we use the permanent to compute it. Thus, the equality between polynomials implies that $X=Y$. Therefore, $p_s^*=p_s^{**}$ for all tie breaking rule ς. $\square$

Summarizing, we use the TTCo assignment procedure because the assignment $TTCo[\varsigma]$ is non-wasteful and Pareto efficient for any tie breaking rule ς. Even more, Proposition 3.4.1 allows us to fix the tie breaking rule ς_L without loss of generality. In Chapter 1 we characterize the TTCo mechanism and study other properties that satisfies it.

3.4.2 Sellers Behaviour at equilibrium

Indifferences are pervasive in the market. Moreover, in large markets, effective buyers are assigned their favourite good (Proposition 3.3.1). We show that under the thickness condition, the expected utility of sellers is independent on the pattern of indifferences.

We assume a quasi-linear utility function for each seller s and we consider that buyers report non-strict rankings over baskets $e_s(p)$, at the end of stage three. So, a buyer i can be indifferent between seller s and at most $m-1$ other sellers. This motivates the introduction of tie breaking rules during the TTCo assignment procedure to choose only one maximum top trading cover. To compute the expected utility $E[u_{s_j}]$ we need to describe the situation where the buyer that is assigned s_j is indifferent between s_j and other sellers. Suppose that i buys s_j, we define the following events to consider the case of indifferences.

- $A^i_{j0} = \{(v,p) \mid v_{ji} - p_j \geq \max\{v_{ti} - p_t\} > 0\}$.
- Let there be $s_{j_1} \in S \setminus \{s_j\}$, we define the set $A^i_{j1} = \{(v,p) \mid v_{ji} - p_j = v_{j_1 i} - p_{j_1} \geq \max\{v_{ti} - p_t\} > 0 \text{ and } v_j > v_{j_1}\}$.
- Let $\kappa \in \mathbb{N}$ such that $\kappa \leq m-1$ and $s_{j_1}, s_{j_2}, \ldots, s_{j_\kappa} \in S$. Also, suppose that $s_{j_\tau} \neq s_j$ for each $\tau = 1, \ldots, \kappa$. We define the set $A^i_{j\kappa} = \{(v,p) \mid v_{ji} - p_j = v_{j_1 i} - p_{j_1} = \ldots = v_{j_\kappa i} - p_{j_\kappa} \geq \max\{v_{ti} - p_t\} > 0 \text{ and } v_j > v_{j\tau} \text{ for all } \tau\}$.

So, the event $A^i_{j\kappa}$ is the set of all pairs (v,p), a state of the market v and a price vector p, that satisfy:

1. $e_{s_j}(p)$ is the most preferred basket of buyer i, and
2. there exists κ sellers in S such that i is indifferent between s_j and s_{j_t}, for all $t = 1, \ldots, = \kappa$.

The payoff function of s_j is

$$u_{s_j}(\omega, s; v_j) = \begin{cases} p_j & \text{if } TTCo(i) = s_j \text{ and } (p,v) \in \bigcup_{\kappa=0}^{m-1} A^i_{j\kappa} \text{ for some } i, \\ v_j & \text{otherwise.} \end{cases}$$

Seller s_j earns p_j if the basket $e_{s_j}(p)$ is assigned to some buyer i in the first step of the TTCo algorithm.

A pair (v,p) belongs to $A^i_{j\kappa}$ if and only if $e_{s_j}(p)$ is the most preferred basket of i and she is indifferent between s_j and κ different sellers. Then, the first condition ensures that $(v,p) \in A^i_{j0}$. Therefore, $A^i_{j\kappa} \subseteq A^i_{j0}$ for all $\kappa \in \{1, 2, \ldots, m-1\}$. This implies that

$$\bigcup_{\kappa=0}^{m-1} A^i_{j\kappa} = A^i_{j0}.$$

So, the payoff function is

$$u_{s_j}(\omega, s; v_j) = \begin{cases} p_j & \text{if } TTCo(i) = s_j \text{ and } (p, v) \in A^i_{j0}, \text{ for some } i, \\ v_j & \text{otherwise.} \end{cases}$$

In order to simplify the algebra, we consider the following monotonic transformation of the payoff function u_{s_j}

$$\overline{u}_{s_j}(\omega, s; v_j) = \begin{cases} p_j - v_j & \text{if } TTCo(i) = s_j \text{ and } (p, v) \in A^i_{j0}, \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\overline{u}_{s_j} = u_{s_j} - v_j$. Then, to maximize the expected utility of seller s_j, we simply maximize the expected utility $E[\overline{u}_{s_j}]$, which is

$$E[\overline{u}_{s_j}] \;=\; \left(p_j - v_j\right) Pr\left[v_{ji} - p_j \geq \max_{s_\tau \in S}\{v_{\tau i} - p_\tau\} \geq 0\right]. \tag{3.9}$$

To find the decision rule at equilibrium, we proceed computing the best response correspondence of each seller. To do that, we first compute the probability that $e_{s_j}(p)$ is the I.R. most preferred basket of i, the probability indicated in expression (3.9). Remember that, by Proposition 3.3.1, buyers get their most preferred good in large markets that satisfy the thickness condition. Since sellers set simultaneously the price of their good and only know their own type, we compute this probability following an Auction Theory technique. We investigate if there is a symmetric Bayesian Nash equilibrium in which all sellers set a linear price

$$p_\tau = \alpha v_\tau,$$

where α is a non-negative constant, for all $\tau \neq j$. Consequently

$$\begin{aligned} Pr[A^i_{j0}] &= Pr\left[v_{ji} - p_j \geq \max_{s_\tau \in S}\{v_{\tau i} - p_\tau\} \geq 0\right] \\ &= Pr\left[v_{ji} - p_j \geq \max_{s_\tau \in S}\{v_{\tau i} - p_\tau\}, v_{ji} - p_j \geq 0\right] \\ &= Pr\left[v_{ji} - p_j \geq \max_{s_\tau \in S}\{v_{\tau i} - p_\tau\}\right] Pr\left[v_{ji} - p_j \geq 0\right]. \end{aligned}$$

On the other hand, we know that the probability of the largest order statistic is defined as

$$Pr[\max x \leq x_0] = Pr[x_1 \leq x_0, x_2 \leq x_0, \ldots, x_n \leq x_0].$$

To simplify it, we assume that random variables V_τ and $V_{\tau i}$ are statically independent and identically distributed for all $s_\tau \in S$ and $i \in B$. Then

$$\begin{aligned} Pr[A^i_{j0}] &= \prod_{\tau \neq j} Pr\left[v_{\tau i} - \alpha v_\tau \leq v_{ji} - p_j\right]\left(1 - Pr\left[v_{ji} \leq p_j\right]\right) \\ &= Pr\left[v_{\tau i} - \alpha v_\tau \leq v_{ji} - p_j\right]^{m-1}\left(1 - Pr\left[v_{ji} \leq p_j\right]\right). \end{aligned}$$

So, the expected utility function can be written as follows

$$E[\bar{u}_{s_j}] = (p_j - v_j) Pr\left[v_{\tau i} - \alpha v_\tau \leq v_{ji} - p_j\right]^{m-1}\left(1 - Pr\left[v_{ji} \leq p_j\right]\right). \tag{3.10}$$

Moreover, we assume that agents types are drawn according to the distribution

$$f(v) = f(v_1, \ldots, v_m, \hat{v}_1, \ldots, \hat{v}_n),$$

thus, we denote by f_{v_τ} and $f_{v_{\tau i}}$ the marginal distributions of variables V_τ and $V_{\tau i}$ for all $\tau \in \{1, 2, \ldots, m\}$ and $i \in \{1, 2, \ldots, n\}$, respectively.

Now, consider the random vector

$$g_{j,\tau,i} = \begin{pmatrix} V_{\tau i} - \alpha V_\tau - V_{ji} \\ V_{\tau i} \\ V_\tau \end{pmatrix} = \begin{pmatrix} Z \\ X \\ Y \end{pmatrix}.$$

It is clear that $g_{j,\tau,i}$ is a linear transformation of the random vector $(V_{\tau i}, V_\tau, V_{ji})$. The inverse transformation $g^{-1}_{j,\tau,i}$ is given by

$$\begin{pmatrix} V_{\tau i} \\ V_\tau \\ V_{ji} \end{pmatrix} = \begin{pmatrix} X \\ Y \\ X - \alpha Y - Z \end{pmatrix} = g^{-1}_{j,\tau,i}.$$

Since the inverse transformation exists, the Jacobian of the transformation $g_{j,\tau,i}$ is not zero.

Observation 3.4.1. Let $X = (X_1, X_2, \ldots, X_p)$ be a continuous random vector with joint distribution $f_X(x_1, \ldots, x_p)$. Let $h = (h_1(x_1, \ldots, x_p), \ldots, h_p(x_1, \ldots, x_p))$ be a one-to-one transformation of X. Consider $A = \{(x_1, \ldots, x_p) \in \mathbb{R}^p \mid f_X(x_1, \ldots, x_p) > 0\}$, the domain of distribution f, and $B = \{(u_1, \ldots, u_p) \in \mathbb{R}^p \mid u_l = h_l(x_1, \ldots, x_p)\ \forall l = 1, 2, \ldots, p$, and $(x_1, \ldots, x_p) \in A\} = h(A)$, the image of

A under transformation h. Since we assume that h is one-to-one, its inverse transformation exists and is denoted by $x_l = H_l(u_1, \dots, u_p)$, for all $l \in \{1,2,\dots,p\}$. The *Jacobian of the transformation* $H = (H_1, \dots, H_p)$ is defined as

$$J(H) = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_p} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_p} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_p}{\partial u_1} & \frac{\partial x_p}{\partial u_2} & \cdots & \frac{\partial x_p}{\partial u_p} \end{vmatrix},$$

the determinant of a matrix of partial derivatives. The Jacobian of H is not zero because h is a one-to-one transformation. Therefore, the joint probability distribution of $U = (U_1, U_2, \dots, U_p)$ on the set B is given by

$$f_U(u_1, \dots, u_p) = f_X(H_1(U), H_2(U), \dots, H_p(U))|J(H)|.$$

Observation 3.4.1 and the independence assumption imply that the probability distribution of $g_{j,\tau,i}(V)$ is

$$\begin{aligned} f_{g_{j,\tau,i}}(x,y,z) &= f_{v_{ji}}(x - \alpha y - z) f_{v_{\tau i}}(y) f_{v_\tau}(z) |J(g_{j,\tau,i})| \\ &= f_{v_{ji}}(x - \alpha y - z) f_{v_{\tau i}}(y) f_{v_\tau}(z). \end{aligned}$$

Let $X_{j,\tau,i} \equiv X_{\tau i} - \alpha X_\tau - V_{ji}$, the probability distribution of $X_{j,\tau,i}$ is given by

$$f_{v_{\tau i} - v_\tau - v_{ji}}(x) = \int \int_{y,z} f_{v_{ji}}(x - \alpha y - z) f_{v_{\tau i}}(y) f_{v_\tau}(z) dy dz, \tag{3.11}$$

where $F_{X_{j,\tau,i}}$ and $F_{V_{ji}}$ denote the cumulative densities functions of variables $X_{j,\tau,i}$ and V_{ji}, respectively. Rewriting the expected utility in (3.10), we have that

$$E[\overline{u}_{s_j}] = (p_j - v_j) F_{X_{j,\tau,i}}(-p_j)^{m-1} \left(1 - F_{V_{ji}}(p_j)\right). \tag{3.12}$$

Sellers problem is to compute the price of her good that maximises her expected utility, function (3.12). Assuming that $F_{X_{j,\tau,i}}$ and $F_{V_{ji}}$ are differentiable on an open interval, the first order condition is

$$\begin{aligned} \frac{\partial E[\overline{u}_{s_j}]}{\partial p_j} = F_{V_{ji}}(p_j) F_{X_{j,\tau,i}}(-p_j)^{m-1} + (p_j - v_j) F_{X_{j,\tau,i}}(-p_j)^{m-1} F'_{V_{ji}}(p_j) &\quad + \\ -(m-1)(p_j - v_j) F_{V_{ji}}(p_j) F_{X_{j,\tau i}}(-p_j)^{m-2} F'_{X_{j,\tau,i}}(-p_j) &= 0. \end{aligned} \tag{3.13}$$

Consider $\gamma(p_j) = \partial E[\overline{u}_{s_j}]/\partial p_j$, which is not necessarily a linear function from $\mathbb{R}$ to $\mathbb{R}$.

The Theory of Non-Linear Equations define a **root** of γ as a point $x^* \in \mathbb{R}$ such that $\gamma(x^*) = 0$. Newton's method, that we describe below, is used to find these roots.

Newton's Method.

Assume that $\gamma(x)$ has at least one real root.

Step 0. Start with an initial guess $x_0 \in \mathbb{R}$, for the location of the root.

Step t. Finding a root is given by iterating repeatedly next expression

$$x_{t+1} = x_t - \frac{\gamma(x_t)}{\gamma'(x_t)}.$$

A root of the equation $\gamma(x) = 0$ is $x^* = \lim_{t\to\infty} x_t$.

If the Newton's succession $\{x_t\}_{t\in\mathbb{N}}$ converges, the Newton's method implies that

$$x^* = x^* - \frac{f(x^*)}{f'(x^*)}.$$

In other words, the Newton's method computes a *fixed point* of function

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

However, the convergence of this succession is not generally guaranteed and there is not a unique criteria to choose the starting point.

There exist functions that ensure the convergence of the Newton's succession regardless the starting point. Even more, extra conditions guarantee the uniqueness of the root. One of these is the *Lipschitz condition*, defined below.

Definition 17. Assume that $g(x)$ is a continuous function in $[a,b]$. Then $g(x)$ is a *contraction* if there exists a constant L such that $0 < L < 1$ for which any $x, y \in [a,b]$:

$$|g(x) - g(y)| \leq L|x - y|.$$

The constant L is the *Lipschitz constant*.

Above condition ensures the existence of a unique zero of the equation $g(x) = x$ and the converges to the Newton's succession to it, regardless the starting point.

Theorem 3.4.1. ***(Contracting Map)*** *Let $g : [a,b] \to \mathbb{R}$ a continuous function. Assume that $g(x)$ satisfies the Lipschitz condition (17), and that $g([a,b]) \subset [a,b]$. Then $g(x)$ has a unique fixed point $x^* \in [a,b]$. Also, the Newton's succession converges to x^* as $n \to \infty$ for any $x_0 \in [a,b]$.*

We impose some conditions to the distribution f to guarantee the existence of a unique price vector. To maximize the expected utility of seller s_j, we have to compute her best response through the first order condition

$$\frac{\partial E[\overline{u}_{s_j}]}{\partial p_j} = 0.$$

To solve this problem is the same as finding the fixed point of

$$\frac{\partial E[\overline{u}_{s_j}]}{\partial p_j} + p_j = p_j.$$

Let

$$\gamma(p_j) = \frac{\partial E[\overline{u}_{s_j}]}{\partial p_j} + p_j.$$

To get a unique fixed point, γ must satisfy the Lipschitz condition in (17). Consequently, the first derivative of $E[\overline{u}_{s_j}]$ must satisfy

$$\frac{\partial E[\overline{u}_{s_j}]}{\partial p_j}(x) - \frac{\partial E[\overline{u}_{s_j}]}{\partial p_j}(y) \quad \leq \quad L|x-y|, \tag{3.14}$$

for some $0 < L < 1$ and any $x, y \in [a,b]$, where $[a,b]$ is an interval where $\gamma(x)$ is continuous. The following theorem summarizes previous discussion.

Theorem 3.4.2. *Let $f \in C[a,b]$. If the first derivative of $E[\overline{u}_{s_j}]$ satisfies condition (3.14), then the Newton's succession*

$$x_{n+1} = x_n - \frac{\gamma(x_n)}{\gamma(x_{n+1})}$$

converges to a unique root from every starting point.

Proof. This is an immediate consequence of Theorem 3.4.1 and the discussion made in above paragraphs. □

Theorem 3.4.2 implies the existence of a unique best response for every seller. In next subsection we show that the exponential distribution satisfies the Lipschitz condition established in (3.14).

3.4.3 Equilibrium Characterization for the Exponential Case

The price p_j is implicitly defined in the non-linear equation (3.13). Theorem 3.4.2 establishes the conditions over the probability function f to compute a unique price vector, even more, we have a method to compute it (Newton's Method).

Assume that all random variables V_j, V_{ji} are independent and exponentially distributed with parameter $\lambda > 0$. So, their probability distributions are

$$f_{V_{ji}}(x) = f_{V_j}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

For all $j \in \{1,2,\ldots,m\}$ and $i \in \{1,2,\ldots,n\}$.

We compute the probability distribution of the random variable $X_{j,\tau,i}$ assuming that $\alpha > 1$, i.e. sellers do not set a price lower than its valuation. To do that, we follow the methodology described in Observation 3.4.1. We know that $g_{j,\tau,i}(V_{\tau i}, V_\tau, V_{ji})$ is a linear transformation of the vector

$$\begin{pmatrix} V_{\tau i} \\ V_\tau \\ V_{ji} \end{pmatrix} = g^{-1}(X_{j,\tau,i}, Y, Z),$$

where each probability is exponentially distributed with parameter $\lambda > 0$. Then, the joint distribution of $g_{j,\tau,i}^{-1}$ is

$$f_{V_{\tau i}, V_\tau, X_{j,\tau,i}}(v_{\tau i}, v_\tau, x) = \lambda^3 e^{-\lambda(x - 2v_{\tau i} + (\alpha-1)v_\tau)}.$$

By Observation 3.4.1 and expression (3.11), the probability distribution of $X_{j,\tau,i}$ is

$$f_{X_{j,\tau,i}}(x) = \int_0^\infty \int_{\alpha v_\tau + x}^\infty \lambda^3 e^{-\lambda(2v_{\tau i} - x + (1-\alpha)v_\tau)} dv_{\tau i} dv_\tau.$$

If $x \leq 0$, we have that

$$f_{X_{j,\tau,i}}(x) = \frac{\lambda(1+\alpha)e^{-\lambda x}}{2(1+\alpha)}.$$

Otherwise,

$$\begin{aligned} f_{X_{j,\tau,i}}(x) &= \int_0^\infty \int_{\alpha v_\tau + x}^\infty \lambda^3 e^{\lambda(2v_{\tau i} - x + (1-\alpha)v_\tau)} dv_{\tau i} dv_\tau \\ &= \frac{\lambda(1+\alpha)e^{-\lambda x}}{2(1+\alpha)}. \end{aligned}$$

Therefore,

$$f_{X_{j,\tau,i}}(x) = \frac{\lambda e^{-\lambda|x|}}{2}$$

where $X_{j,\tau,i} \in (-\infty, \infty)$. Then

$$\begin{aligned} Prob[X_{j,\tau,i} \leq -p_j] &= Prob[-X_{j,\tau,i} \geq p_j] \\ &= \int_{p_j}^{\infty} \frac{\lambda e^{-\lambda x}}{2} dx \\ &= \frac{e^{-\lambda p_j}}{2}. \end{aligned}$$

Consequently, the expected utility is given by

$$E[\overline{u}_{s_j}] = (p_j - v_j)e^{-\lambda p_j} \left(\frac{e^{-\lambda p_j}}{2} \right)^{m-1}. \tag{3.15}$$

Before to solve the first order condition, we verify that $E[\overline{u}_{s_j}]$ satisfies the uniqueness condition (3.14) of Theorem 3.4.2, and the thickness condition.

Observation 3.4.2. (**Leibnitz Condition**.) First, since each random variable is exponentially distributed, it is clear that $\partial E[\overline{u}_{s_j}]/\partial p_j$ is bounded and differentiable on an interval $[a, b]$, because

$$0 \leq e^{-\lambda p_j}, \left(\frac{e^{-\lambda(p_j + r)}}{2} \right)^{m-1} \leq 1,$$

then

$$\frac{\partial E[\overline{u}_{s_j}]}{\partial p_j}([0, \beta]) \subset [0, \beta],$$

for β large enough, where

$$\frac{\partial E[\overline{u}_{s_j}]}{\partial p_j} = -e^{\lambda r}(-1 + \lambda m(p_j - v_j)) \left(\frac{e^{-\lambda p_j}}{2} \right).$$

Since we are searching the best response of s_j we can assume that $a \geq 0$. Then, for any $x, y \in [a, b]$ we have that

$$\begin{aligned} \frac{\left| \frac{E[\overline{u}_{s_j}]}{\partial p_j}(x) - \frac{E[\overline{u}_{s_j}]}{\partial p_j}(y) \right|}{|x - y|} &= \frac{\frac{E[\overline{u}_{s_j}]}{\partial p_j}(x) - \frac{E[\overline{u}_{s_j}]}{\partial p_j}(y)}{x - y} \\ &= 2 \frac{(1 - x\lambda m)\left(\frac{e^{-\lambda x}}{2}\right)^m - (1 - y\lambda m)\left(\frac{e^{-\lambda y}}{2}\right)^m}{x - y}. \end{aligned}$$

Moreover, we know that

$$e^{-\lambda x}, e^{-\lambda y} < e^{-\lambda 0} = 1, \text{ and } x > y,$$

then

$$\begin{aligned} 2\frac{(1-x\lambda m)\left(\frac{e^{-\lambda x}}{2}\right)^m - (1-y\lambda m)\left(\frac{e^{-\lambda y}}{2}\right)^m}{x-y} &< \\ 2\frac{(1+x\lambda m)\left(\frac{1}{2}\right)^m - (1+y\lambda m)\left(\frac{1}{2}\right)^m}{x-y} &= \\ \frac{2\lambda m}{2^{m-1}}. \end{aligned}$$

Let

$$L \equiv \frac{2\lambda m}{(2+2\alpha)^{m-1}},$$

for $\alpha \geq \lambda m$ and $m > 1$, we have that $0 < L < 1$. So,

$$\left| \frac{E[\bar{u}_{s_j}]}{\partial p_j}(x) - \frac{E[\bar{u}_{s_j}]}{\partial p_j}(y) \right| < L|x-y|,$$

i.e., the expected utility satisfies the condition (3.14).

Also, it is important to note that the exponential function satisfies the thickness condition.

Observation 3.4.3. (**Thickness Condition**.) Since valuations are independent and identically distributed, we have that $d_j = e^{-n\lambda p_j}$, then

$$\begin{aligned} \frac{d_j}{d_k} &= \frac{e^{-n\lambda p_j}}{e^{-n\lambda p_k}} \\ &= e^{-n\lambda(p_j - p_k)}. \end{aligned}$$

If s_j is the most popular seller, then $d_j \geq d_k$ implies that $p_j \leq p_k$, i.e $p_j - p_k \leq 0$. Moreover, we have a finite set of sellers that set prices in the second stage, then $\frac{d_j}{d_k}$ is bounded by $e^{-n\lambda(p_j - p_{\max})}$, where p_{max} is the highest price set at the second stage. Therefore, the exponential function satisfies the thickness condition.

Therefore, sellers problem has a unique solution given by the first order condition

$$\begin{aligned} 0 &= \frac{\partial E[\bar{u}_{s_j}]}{\partial p_j} \\ &= -e^{\lambda r}(-1 + \lambda m(p_j - v_j))e^{-\lambda p_j}. \end{aligned}$$

The best response of s_j when other sellers set a linear price is

$$\begin{aligned} p_j^* &= \frac{1+\lambda m v_j}{\lambda m} \\ &= v_j + \frac{1}{\lambda m}. \end{aligned}$$

Verifying the second order condition, we have that

$$\begin{aligned} \left.\frac{\partial^2 E[\overline{u}_{s_j}]}{\partial p_j^2}\right|_{p_j=p_j^*} &= \left. 2\lambda m e^{\lambda r}(-2+\lambda m(p_j - v_j))\left(\frac{e^{-\lambda(p_j+r)}}{2}\right)^m \right|_{p_j=p_j^*} \\ &= -2\lambda m e^{\lambda r}\left(\frac{e^{-\lambda\left(r+\frac{1+\lambda m v_j}{m}\right)}}{2}\right)^m < 0. \end{aligned}$$

Therefore p_j^* is the price of s_j at equilibrium. Previous discussion is summarized in the following theorem.

Theorem 3.4.3. *Suppose that V_j and V_{ji} are independent and exponentially distributed with parameter $\lambda > 0$. For the second stage, the price vector at symmetric equilibrium exists and is unique. Even more, the price that each seller s_j sets at equilibrium is*

$$p_j^* = v_j + \frac{1}{m\lambda},$$

for all $s_j \in S$.

By the above theorem, we have shown that the price vector at equilibrium is unique. Consequently, we can do some comparative statics.

Corollary 3.4.1. *Let p_j^* be the unique price at the symmetric equilibrium found. Then*

- *The relation between p_j^* and v_j is positive, and*
- *The relation between p_j^* and m is negative.*

Proof. Note that

$$p_j^* = \frac{1}{\lambda m} + v_j.$$

Its derivative with respect to v_j is

$$\frac{\partial p_j^*}{\partial v_j} = 1 > 0.$$

The derivative of p_{j^*} with respect to m is

$$\begin{aligned}\frac{\partial p_j^*}{\partial m} &= -\frac{1}{\lambda m^2} + \frac{v_j}{m} - \frac{v_j}{m} \\ &= -\frac{1}{\lambda m^2} < 0.\end{aligned}$$

□

We compute in Appendix D2 two extensions that shows the robustness of previous results: the case where parameters of the exponential distributions vary, then we assume that valuations overlap. Finally, in the Appendix E2 we discuss the uniform distribution case and computational difficulties that it entails.

Concluding Remarks

3.4.4 Small Markets

When markets are small, uniqueness of equilibrium price vector is not guaranteed any more, as shows the following example.

Example 3.4.1. Consider a market such that $S = \{s_1, s_2, s_3, s_4\}$ and $B = \{1, 2, \ldots, n\}$, with $n < \infty$; and the random variables $V_{\tau i}$, V_τ are independent and exponentially distributed with parameter $\lambda > 0$. Note that Proposition 3.3.1 is not necessarily true because we do not have a large thick market. So, the top trading cover algorithm can stop in four steps. Denote by $X^i_{(k)}$ the statistic of order k. So, if $v_{\tau i} - p_\tau = X^i_{(k)}$, this means that s_τ is the $k-$most preferred good given de price vector p, where $k \in \{1, 2, 3, 4\}$. Note that $X_{(1)} = \max\{v_{\tau i} - p_\tau\}$ and $X_{(4)} = \min\{v_{\tau i} - p_\tau\}$. Consequently, the payoff function of each seller is the following.

$$u_{s_j} = \begin{cases} p_j & \text{if } v_{ji} - p_j = X^i_{(k)} \text{ and } X^i_{(1)} \geq 0, \forall k \in \{1,2,3,4\} \text{ and for some } i \in B, \\ v_j & \text{otherwise.} \end{cases}$$

As before, in order to simplify the algebra, we consider the linear transformation $\overline{u}_{s_j} = u_{s_j} - v_j$. Hence, the expected utility function of s_j is

$$E[u_{s_j}] \;=\; (p_j - v_j)Pr[v_{ji} - p_j \geq 0]\sum_{k=1}^{4} Pr[X_{(k)} \leq v_{ji} - p_j].$$

Using the random variable $X_{j,\tau,i}$, we know that the cumulative density function of the statistic of order k is given by

$$F_{(k)}(x) \;=\; \sum_{r=k}^{3} \begin{pmatrix} 3 \\ r \end{pmatrix} F_{X_{j,\tau,i}}(x)^r (1 - F_{X_{j,\tau,i}}(x))^{3-r}.$$

On the other hand, a well-known result is that

$$\begin{aligned}\sum_{\tau=t}^{m} \begin{pmatrix} m \\ \tau \end{pmatrix} p^{\tau}(1-p)^{m-\tau} &= I_p(t, n-t+1) \\ &= \frac{\int_0^p r^{k-1}(1-r)^{m-k} dr}{\int_0^1 r^{k-1}(1-r)^{m-k} dr},\end{aligned}$$

where $I_p(t, m-t+1)$ is the *incomplete beta function* defined as follows.

By previous discussion, the expected utility $E[\overline{u}_{s_j}]$ is given by

$$\begin{aligned}E[\overline{u}_{s_j}] &= (p_j - v_j)Pr[v_{ji} \geq p_j]\sum_{k=1}^{4} I_{F_{X_{j,l,i}}(-p_j)}(k, n-k+1) \\ &= (p_j - v_j)e^{-\lambda p_j}\left(\frac{3}{2}e^{-\lambda p_j} + \frac{3}{8}e^{-2\lambda p_j} + \frac{15}{8}e^{-3\lambda p_j} - \frac{47}{64}e^{-4\lambda p_j}\right).\end{aligned}$$

The first order condition is the following

$$\begin{aligned}&\frac{e^{-5\lambda p_j}}{64}\Big((24 - 72\lambda(p_j - v_j))e^{2\lambda p_j} \\ &- 96(-1 + 2\lambda(p_j - v_j)) - 120(-1 + 4\lambda(p_j - v_j))e^{\lambda p_j} \\ &+ 47(-1 + 5\lambda(p_j - v_j))\Big) = 0.\end{aligned}$$

Note that the expression

$$\begin{aligned}24 - 72\lambda(p_j - v_j))e^{2\lambda p_j} &- 96(-1 + 2\lambda(p_j - v_j))e^{3\lambda p_j} \\ &- 120(-1 + 4\lambda(p_j - v_j))e^{\lambda p_j} 47(-1 + 5\lambda(p_j - v_j))\end{aligned}$$

can be seen as a polynomial of degree 3 with respect to $z = e^{\lambda p_j}$. This implies that it is not bounded, consequently, it does not satisfy the Lipschitz condition. Therefore, we cannot guarantee that the Newtons succession converges to a unique price p_j.

3.4.5 Public Policies

Mexican government has included strengthening the housing sector as a public policy priority since 2000. Public housing policy is outlined in National Housing Programs (Programas Nacionales de Vivienda, PNV), which aim to help those most in need by providing access to mortgages and loans, and by encouraging the construction of affordable housing. To achieve this goal, Mexican government provides credits to both buyers and sellers through different institutions. For example, home-builders can request support from the Federal Mortgage Society (Sociedad Hipotecaria Federal, SHF), encourages construction through direct loans and guarantee of bank loans; and the National Institute of Worker Housing (Instituto Nacional de la Vivienda para los Trabajadores, INFONAVIT) provides loans to workers to build, purchase or repair a house. Moreover, there exists institutions like the National Fund for Popular Housing (Fideicomiso Fondo Nacional de Habitaciones Populares, FONHAPO) subsidizes the purchase or the construction of low income housing.

Public intervention affects both valuations and prices. It is not clear, especially in stressed market, which side of the market benefits more from public spending. Establishing a closed-form solution is first step to empirically answer the question.

3.5 Appendix A2

Graph Theory Concepts

Consider a set of sellers, $S' \subseteq S$; a set of buyers, $B' \subseteq B$; and a profile of rankings at p, $\overline{\pi}(p;v) = (\overline{\pi}_{i_1}(p;v),\ldots,\overline{\pi}_{i_k}(p;v))$, where $i_j \in B'$ for all $j = 1,\ldots,k$. We define the **bipartite directed graph** $\overline{G}(S',B',p,\overline{\pi}(p;v))$ as a pair $(S' \cup B', \overline{E}(S' \cup B', p, \overline{\pi}(p;v)))$, where $S' \cup B'$ is the set of nodes; and $E(S' \cup B', p, \overline{\pi}(p;v))$ is the set of all directed edges $(s,i) \in S' \times B'$, such that $(s,i) \in E$ if and only if buyer i prefers buying the basket $e_s(p)$ to any other basket (p_s,s) in $E_i(p)$. The reported rankings are non-strict; the most preferred basket of a buyer i is not necessarily unique. Thus, there are more than one edge from i to the set of sellers, which represents indifference between baskets.

We represent the case where a buyer i prefers her initial endowment to any basket $e_s(p)$ by a **loop** (i,i), which is an edge from B' to B'. Loops, however, are not admitted in bipartite graphs, that is why the TTCo makes use of quasi-bipartite graphs. We define a **quasi-bipartite directed graph** $G(S',B',p,\pi(p;v))$ as a pair $(S' \cup B', E(S' \cup B', p, \overline{\pi}(p;v)))$, where $S' \cup B'$ is the set of nodes; and $E(S' \cup B', p, \overline{\pi}(p;v)))$ is the set of all directed edges $(s,i) \in S' \times B'$ and loops $(j,j) \in B' \times B'$, such that:

- buyer i prefers the basket $e_s(p)$ to any other basket, and
- buyer j prefers her initial endowment $(m_i, \varnothing)$ to any basket $e_s(p)$.

An arbitrary element of $E(S' \cup B', p, \overline{\pi}(p;v)))$ is denoted by $\vec{a}$.

The quasi-bipartite graph $G(S',B',p,\overline{\pi}(p;v))$, or only G whenever there is no confusion, represents the situation when each buyer i in B' points to the owner of her favorite basket. The following example shows the construction of a quasi-bipartite graph.

Example 3.5.1. Consider $S = \{s_1,s_2,s_3\}$, $B = \{1,2,3\}$. For some $p \in \prod_{j=1}^{3} A_j$ and $v \in V$, suppose that the rankings reported by $1,2$ and 3 are the following:

$$\begin{aligned}\overline{\pi}_1(p;v) &: [e_{s_1}, e_{s_3}], e_{s_2}, 1,\\ \overline{\pi}_2(p;v) &: [e_{s_2}, 2],\\ \overline{\pi}_3(p;v) &: e_{s_3}, 3.\end{aligned}$$

Figure 3.4 shows the quasi-bipartite graph $G(B,S,\overline{\pi};v)$.

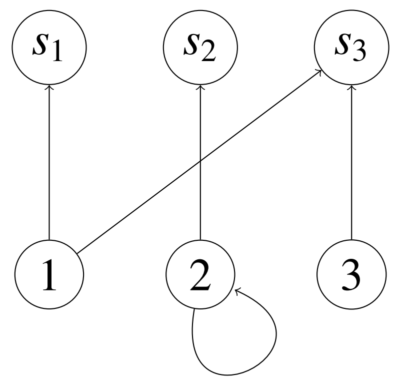


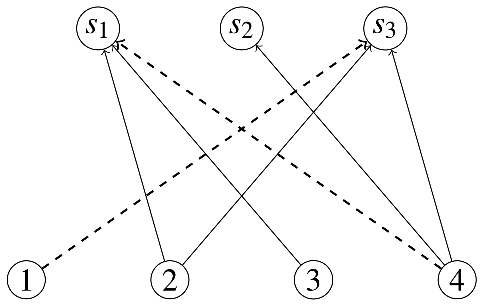
Figure 3.4: A Quasi-bipartite Graph

Given $G(S',B',p,\overline{\pi}(p;v))$, a **Top Trading Cover (ttco)** is a subset

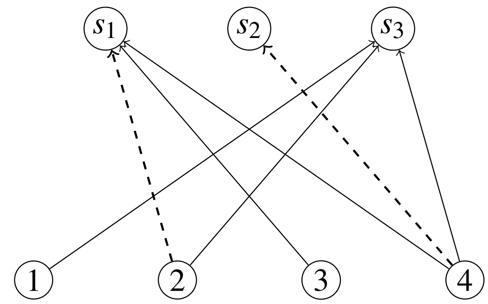
$$T = \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k\}$$

of $E[G]$, such that no two edges in T have a common node. In particular, the empty set is top trading cover.

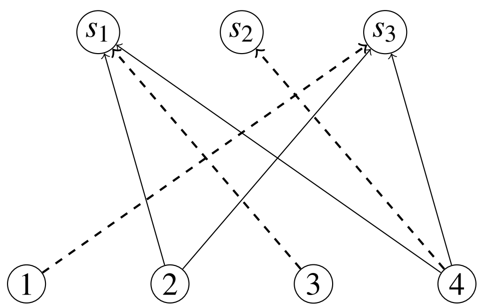
A ttco T is **maximal1** if it is no longer a ttco when an edge, not in T, is added to T. In other words, a maximal ttco is not a proper subset of any other ttco of the quasi-bipartite graph G. A **maximum top trading cover** is a ttco that covers the largest possible number of nodes. It is clear that all maximum top trading covers are also maximal; however, a maximal ttco is not always maximum. Figure 3.5 shows four different top trading covers in dotted lines for the same quasi-bipartite graph. The ttco in 3.5(a) is maximal, because if an extra edge is added to it, the resulting subset of edges is not a ttco; 3.5(a) is not a maximum ttco because there exists 3.5(c), which covers the largest possible number of nodes, 6, a number larger than the number of nodes cover by 3.5(a), 4. On the other hand, in 3.5(b) we can add the edge $(1, s_3)$ and get a larger top trading cover, so 3.5(b) is non-maximum and non-maximal.



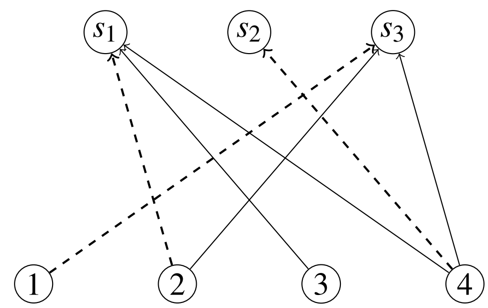
(a) Maximal TTCo



(b) Non-Maximum and Non-Maximal TTCo



(c) Maximum TTCo 1



(d) Maximum TTCo 2

Figure 3.5: Top Trading Covers in dotted lines for the same Quasi-Bipartite Graph

We can ensure the existence of at least one maximum ttco for any quasi-bipartite graph.

Proposition 3.5.1. *Let $G(B', S', \overline{\pi}; p, v)$ a quasi-bipartite graph, a maximum $TTCo$ always exists.*

Proof. See Appendix B2. □

Note that the maximum top trading cover is not unique. Figure 3.5(d) shows a ttco that covers the same number of nodes than the ttco in 3.5(c), but does not have the same edges. That is, in 3.5(d) we have a maximum ttco different from the maximum ttco in 3.5(c).

3.6 Appendix B2

The Existence of Top Trading Covers

To prove the existence of a maximum Top Trading Cover in each quasi-bipartite graph $G(S', B', \pi; p, v)$, we need more definitions from Graph Theory. Given a top trading cover T of G, a node $r \in S' \cup B'$ is said to be **covered** if there exists $r' \in S' \cup B'$ such that $(r, r') \in T$. A **path** P in G is a succession of nodes

$$\{r_1, t_1, r_2, t_2, \ldots r_n, t_n\}$$

such that $(r_i, t_i) \in E(S' \cup B', \pi; p, v)$ for all $i = 1, 2, \ldots, n$. We say that r_1 and t_n are the **end points** of P.

Definition 18. Let $G = G(S', B', \pi; p, v)$ a quasi-bipartite graph and a top trading cover T of G. A path P is an **augmenting path** for T if:

1. The two end point of P are not covered by T.

2. The edges of P alternate between edges $\in T$ and edges $\notin T$.

Let R a top trading cover of G, and P and augmenting path of R. We define and denote the operation between R and P, as

$$R \oplus P = (A \cup B) - (A \cap B),$$

i.e., we operate ttco's and paths through symmetric difference.

The algorithm to find a maximum top trading cover is described below

Step 1. Consider T a top trading cover of G, could be $T = \varnothing$.

Step 2. While exists an augmenting path P do

2.1 $T = T \oplus P$

Step 3. While exists $r \in G \setminus T$ such that (r, r) is and edge of G do

3.1 $T = T \cup \{r\}$

Step 4. Output T.

We have to prove that T is effectively a maximum top trading cover. However, this is clear, because we delete all possible augmenting paths. We conclude this prove applying Berge's Theorem: *"The matching M is maximum if and only if there is no augmenting path"*[2].

3.7 Appendix C2

Bapat-Beg Theorem

The **permanent** of a matrix of 2×2 is denoted and defined by

$$Perm \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad + bc.$$

Let $A = (a_{ij})$ a matrix of $n \times n$, then

$$Perm(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

Theorem 3.7.1. ***Bapat-Beg Theorem*** *$X_1, X_2, \ldots, X_n$ be independent random variables with distributions functions $F_1, \ldots, F_n$ respectively. Then the distribution function of the statistic of order r, $1 \leq r \leq n$, is given by*

$$P[Y_r \leq y] = \sum_{i=r}^{n} \frac{1}{i!(n-i)!} Per \begin{bmatrix} F_1(y) & 1-F_1(y) \\ \vdots & \vdots \\ \underbrace{F_n(y)} & \underbrace{1-F_n(y)} \\ i\ times & n-i\ times \end{bmatrix}, -\infty < y < \infty. \tag{3.16}$$

3.8 Appendix D2

A Unique Fixed Point

We explain some concepts from the theory of non-linear equations what we use to get the uniqueness of the price vector. In the main text we use the one dimensional Newton method given by the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ x_0 \in [a,b]\ n \geq 0 \tag{3.17}$$

[2]For more details consult: Claude Berge (1963). *Topological Spaces.* Oliver and Boyd.

assuming $f' \neq 0$ on $[a,b]$.

First, we must guarantee the existence of at least one root. This is possible by Browser's Theorem:

Theorem 3.8.1. ***(Brouwer's Fixed Point Theorem)*** *Assume that $g(x)$ is continuous on the closed interval $[a,b]$. Assume that the interval $[a,b]$ is mapped to itself by $g(x)$, i.e., for any $x \in [a,b]$, $g(x) \in [a,b]$. Then there exists a point $c \in [a,b]$ such that $g(c) = c$. The point c is a fixed point of $g(x)$.*

Definition 19. Assume that $g(x)$ is a continuous function on $[a,b]$. Then $g(x)$ is a **contraction** on $[a,b]$ if there exists a constant L such that $0 < L < 1$ for which for any x and y in $[a,b]$:

$$|g(x) - g(y)| \geq L|x-y|. \tag{3.18}$$

The equation (3.18) is referred to as a Lipschitz condition and the constant L is the *Lipschitz constant*.

Indeed, if the function $g(x)$ is a contraction, i.e., if it satisfies the Lipschitz condition (3.18), we can expect the iterative Newton' method converges. This is established in the Contraction Mapping Theorem.

Theorem 3.8.2. ***(Contraction Mapping)*** *Assume that $g(x)$ is a continuous function on $[a,b]$. Also, suppose that $g(x)$ satisfies the Lipschitz condition (3.18), and that $g([a,b]) \subseteq [a,b]$. Then $g(x)$ has a unique fixed point $c \in [a,b]$. Also, the sequence $\{x_n\}$ defined in (3.17) converges to c as $n \to \infty$ for any $x_0 \in [a,b]$.*

Proof. By the Brower's Theorem, we know that $g(x)$ has at least one fixed point. So, to prove the uniqueness, we assume that there are two fixed points c_1 and c_2. We will prove that these two points must be identical. We know that

$$|c_1 - c_2| = |g(c_1) - g(c_2)| \leq L|c_1 - c_2| \text{ and } 0 < L < 1,$$

consequently, c_1 must be equal to c_2.

Finally, we need to prove that the succession described in (3.17) converge to c, for any $x_0 \in [a,b]$. note that

$$|x_{n+1} - c| = |g(x_n) - g(c)| \leq L|x_n - c| \leq \ldots \leq L^{n+1}|x_0 - c|.$$

Since $0 < L < 1$, we have that $|x_{n+1} - c| \to 0$, as $n \to \infty$. The succession converges to the fixed point of $g(x)$, independently of the starting point x_0. $\square$

Non-identically Distributed Random Variables

In previous section we compute the price vector at equilibrium when V_j and V_{ji} are independent random variables exponentially distributed with parameter $\lambda > 0$. In this section, we assume that buyers and sellers are statistically independent but not identically distributed. We analyse the case where random variables are exponentially distributed with different parameters.

In the exponential distribution, the parameter λ represents the occurrence of a rare event. In our housing model, we interpret this parameter parameter as the occurrence of selling a house. Naturally, this parameter changes according to the seller, for example: a construction company decides to sell a house every month in a year, i.e. $\lambda = 1$. On the other hand, a professor decide to sell a house one time in five years, $\lambda = 60$. So, the mean of the distribution of variable V_s, $1/\lambda$, can be understanding as the number of times that seller s decides to sell her house. In buyers case, first note that each buyer has a vector of parameters $\hat{\lambda}_i = (\lambda_{ji})_{j \in S}$, each parameter represents the occurrence of buying a house from seller j. For example, if $\lambda_{ji} = 10$, this means that every 10 months buyer i buys a house from seller s_j. However, we assume that each buyer can have at most one indivisible good, and, more important, we do not analyse a dynamic problem. Hence, we interpret $1/\lambda_j$ as the expected time of sale her house, for each $s_j \in S$, and $1/\lambda_{ji}$ as the interest of buying the house of s_j, for all $i \in B$.

Consider $S = \{s_1, s_2, \ldots, s_m\}$ and $B = \{1, 2, \ldots, n\}$. Given that $\hat{v}_i = (v_{1i}, v_{2i}, \ldots, v_{mi})$ is a random vector, we suppose that random variables v_{ji} and v_j are independent. We assume that variables v_j are exponentially distributed with parameter $\lambda_j > 0$. Thus, the mean and the variance of V_j are denoted by $\mu_j = 1/\lambda_j$ and $\sigma_j^2 = 1/\lambda_j^2$. Variables v_{ji} are exponentially distributed with parameter λ_{ji}, for all $j = 1, \ldots, m$ and $i = 1, \ldots, n$. We write μ_{ji} and σ_{j^2} to describe the mean and the variance of V_{ji}. So, the joint distribution of $(V_{\tau i}, V_\tau, V_{ji})$ is

$$f_{V_\tau V_{\tau i} V_{ji}}(v_\tau, v_{\tau i}, v_{ji}) = \lambda_\tau \lambda_{\tau i} \lambda_{ji} e^{-\lambda_\tau v_\tau - \lambda_{\tau i} v_{\tau i} - \lambda_{ji} v_{ji}}.$$

Applying Observation 3.4.1 for the random variable $X_{j,\tau,i}$, we have that

$$\begin{aligned} f_{X_{j,\tau,i}}(x) &= \int_0^\infty \int_{\alpha v_\tau + x}^\infty \lambda_\tau \lambda_{\tau i} \lambda_{ji} e^{-\lambda_\tau v_\tau - \lambda_{\tau i} v_{\tau i} - \lambda_{ji}(v_{\tau i} - \alpha v_\tau - x)} d v_{\tau i} d_{v_\tau} \\ &= \frac{\lambda_\tau \lambda_{\tau i} \lambda_{ji} e^{-\lambda_{\tau i} x}}{(\lambda_{\tau i} + \lambda_{ji})(\lambda_\tau + \alpha \lambda_{\tau i})}. \end{aligned}$$

Then

$$\begin{aligned} Pr[X_{j,\tau,i} \le -p_j] &= Pr[-X_{j,\tau,i} \ge p_j] \\ &= \int_{p_j}^{\infty} \frac{\lambda_\tau \lambda_{\tau i} \lambda_{ji} e^{-\lambda_{\tau i} x}}{(\lambda_{\tau i} + \lambda_{ji})(\lambda_\tau + \alpha\lambda_{\tau i})} dx \\ &= \lambda_{ji} e^{-\lambda_{ji} p_j} \prod_{\tau \ne j} \frac{\lambda_\tau e^{-\lambda_{\tau i} p_j}}{(\lambda_\tau + \alpha\lambda_{\tau i})}. \end{aligned}$$

Since we assume that random variables are independent and exponentially distributed with different parameters, the expected utility $E[\bar{u}_{s_j}]$ is given by

$$E[\bar{u}_{s_j}] = (p_j - v_j)\lambda_{ji} e^{-\lambda_{ji} p_j} \prod_{\tau \ne j} \frac{\lambda_\tau e^{-\lambda_{\tau i} p_j}}{(\lambda_\tau + \alpha\lambda_{\tau i})}.$$

The first order condition is

$$\begin{aligned} 0 &= \frac{\partial E[\bar{u}_{s_j}]}{\partial p_j} \\ &= -\frac{\lambda_{ji}^{m-1} \prod_{\tau \ne j} \lambda_\tau e^{-p_j \sum_{\tau=1}^{m} \lambda_{\tau i}} \left(-1 + (p_j - v_j) \sum_{\tau=1}^{m} \lambda_{\tau i} \right)}{\prod_{\tau \ne j} (\lambda_{\tau i} + \lambda_{ji}) \prod_{\tau \ne j} (\lambda_\tau + \alpha\lambda_{\tau i})}. \end{aligned}$$

The best response of s_j when others set a linear price is

$$p_j^* = \frac{1}{\lambda_{ji} + \sum_{\tau \ne j} \lambda_{\tau i}} + v_j.$$

The second order condition evaluated in p_j^* is

$$\begin{aligned} \left. \frac{\partial^2 E[\bar{u}_{s_j}]}{\partial p_j^2} \right|_{p_j = p_j^*} &= \\ \left. \frac{\lambda_{ji}^{m-1} \prod_{\tau \ne j} \lambda_\tau \sum_{\tau=1}^{m} \lambda_{\tau i} \left(-2 + (p_j - v_j) \sum_{\tau=1}^{m} \lambda_{\tau i} \right) e^{-p_j \sum_{\tau=1}^{m} \lambda_{\tau i}}}{\prod_{\tau \ne j} (\lambda_{\tau i} + \lambda_{ji}) \prod_{\tau \ne j} (\lambda_\tau + \alpha\lambda_{\tau i})} \right|_{p_j = p_j^*} &= \\ -\frac{\lambda_{ji}^{m-1} \prod_{\tau \ne j} \lambda_\tau \sum_{\tau \ne j} \lambda_{\tau i} e^{-1 - v_j \sum_{\tau=1}^{m} \lambda_{\tau i}}}{\prod_{\tau \ne j} (\lambda_{\tau i} + \lambda_{ji}) \prod_{\tau \ne j} (\lambda_\tau + \alpha\lambda_{\tau i})} &< 0. \end{aligned}$$

Therefore p_j^* is the equilibrium price.

Theorem 3.8.3. *Suppose that V_j and V_{ji} are independent and exponentially distributed with parameters λ_j and λ_{ji}, respectively, for all $s_j \in S$ and $i \in B$. Let p_j^* be the unique price at the symmetric equilibrium. Then*

- *The relation between p_j^* and v_j is positive,*
- *The relation between p_j^* and $\lambda_{\tau i}$ is negative for all $\tau \neq j$.*
- *The relation between p_j^* and $\mu_{\tau i}$ is positive $\tau = 1, 2, \ldots, m$.*

Proof. Note that

$$p_j^* = \frac{1}{\lambda_{ji} + (m-1)\lambda_{\tau i}} + v_j.$$

The respecting derivatives are:

$$\frac{\partial p_j^*}{\partial v_j} = 1.$$

The relation with her own valuation is positive.

$$\frac{\partial p_j^*}{\partial \lambda_{\tau i}} = -\frac{1}{(\lambda_{ji} + \sum_{\tau \neq j} \lambda_{\tau i})^2} < 0.$$

Finally

$$\frac{\partial p_j^*}{\partial \lambda_{ji}} = -\frac{1}{(\lambda_{ji} + \sum_{\tau \neq j} \lambda_{\tau i})^2} < 0.$$

On the other hand, we can compute the relation between the price at equilibrium and the mean $\mu_{\tau i}$ for all $s_\tau \in S$. First, note that

$$p_j^* = \frac{1}{\mu_{ji}^{-1} + \sum_{\tau \neq j} \mu_{\tau i}^{-1}} + v_j.$$

Then

$$\frac{\partial p_j^*}{\partial \mu_{ji}} = \frac{1}{\mu_{ji}^2 \left(\frac{1}{\mu_{ji}} + \sum_{\tau \neq j} \frac{1}{\mu_{\tau i}} \right)^2} > 0,$$

and

$$\frac{\partial p_j^*}{\partial \mu_{\tau i}} = \frac{1}{\mu_{\tau i}^2\left(\frac{1}{\mu_{ji}}+\sum_{\tau\neq j}\frac{1}{\mu_{\tau i}}\right)^2} > 0.$$

□

Overlapping Valuations

To analyse overlapping valuations, we assume that each variable V_j and V_{ji} has a minimum value r_j and r_{ji}, respectively. Then, their probability distributions are

$$f_{V_j}(v_j) = \begin{cases} \lambda_j e^{-\lambda_j(v_j-r_j)} & \text{if } v_j > r_j, \\ 0 & \text{Otherwise.} \end{cases} \quad \textit{and} \quad f_{V_{ji}}(v_{ji}) = \begin{cases} \lambda_{ji} e^{-\lambda_{ji}(v_{ji}-r_{ji})} & \text{if } v_{ji} > r_{ji}, \\ 0 & \text{Otherwise.} \end{cases}$$

The probability distribution of $X_{j,\tau,i}$ is given by

$$f_{X_{j,\tau,i}}(x) = \frac{\lambda_{ji}\lambda_\tau\lambda_{\tau i} e^{\lambda_{ji}(r_{ji}-r_\tau)-\lambda_{\tau i}(r_\tau+\alpha r_\tau - r_{\tau i}+x)}}{(\lambda_{ji}+\lambda_{\tau i})(\alpha\lambda_{\tau i}+\lambda_\tau)}.$$

Consequently, the expected utility is

$$E[\overline{u}_{s_j}] = (p_j - v_j)e^{-\lambda_{ji}(p_j-r_{ji})}\prod_{\tau\neq j}\frac{\lambda_{ji}\lambda_\tau e^{\lambda_{ji}(r_{ji}-r_\tau)-\lambda_{\tau i}(p_j+r_\tau+\alpha r_\tau-r_{\tau i})}}{(\lambda_{ji}+\lambda_{\tau i})(\alpha\lambda_{\tau i}+\lambda_\tau)}.$$

The first order condition is

$$\begin{aligned} 0 &= \frac{\partial E[\overline{u}_{s_j}]}{\partial p_j} \\ &= \frac{\lambda_{ji}^{m-1}\prod_{\tau\neq j}\lambda_\tau e^{\lambda_{ji}(p_j+r_{ji}-\sum_{\tau\neq j} r_\tau)-\sum_{\tau\neq j}\lambda_{\tau i}(p_j+r_\tau+\alpha r_\tau - r_{\tau i})}(1+\lambda_{ji}(v_j-p_j)+\sum_{\tau\neq j}\lambda_{\tau i}(v_j-p_j))}{\prod_{\tau\neq j}(\lambda_{ji}+\lambda_{\tau i})(\alpha\lambda_{\tau i}+\lambda_\tau)}. \end{aligned}$$

we get that

$$p_j^* = \frac{1}{\sum_{j=1}^{m}\lambda_{ji}} + v_j.$$

Verifying the second order condition, we have that

$$\begin{aligned} \left.\frac{\partial^2 E[\overline{u}_{s_j}]}{\partial p_j^2}\right|_{p_j=p_j^*} &= \lambda_{ji}^{m-1}\prod_{\tau\neq j}\lambda_\tau \left.\frac{e^{-\sum_{\tau\neq j}\lambda_{\tau i}(p_j+r_{\tau i}+\alpha r_\tau - r_{\tau i})-\lambda_{ji}(p_j - mrji+\sum_{\tau\neq j} r_\tau)}}{\prod_{\tau\neq j}(\lambda_{ji}+\lambda_{\tau i})(\alpha\lambda_{\tau i}+\lambda_\tau)}\right|_{p_j=p_j^*} \\ &= -\lambda_{ji}^{m-1}\prod_{\tau\neq j}\lambda_\tau\frac{e^{-1-\sum_{\tau\neq j}\lambda_{\tau i}(r_\tau+\alpha r_\tau - r_{\tau i}+v_j)-\lambda_{ji}(-mr_{ji}+v_j+\sum_{\tau\neq j} r_\tau)}}{\prod_{\tau\neq j}(\lambda_{ji}+\lambda_{\tau i})(\alpha\lambda_{\tau i}+\lambda_\tau)} < 0. \end{aligned}$$

Therefore, when we assume overlapping valuations, the unique price set at equilibrium is

$$p_j^* = \frac{1}{\sum_j^m \lambda_{ji}} + v_j.$$

This equilibrium price coincides with the equilibrium price vector when we use that random variables are independent and exponentially distributed with different parameters.

3.9 Appendix E2

Uniform Case

To characterize the price vector at equilibrium we assume that random variables are independent and uniformly distributed. Without loss of generality, we assume that variables V_τ and $V_{\tau i}$ are independent and uniformly distributed over $[0,1]$. Thus, its probability distribution is:

$$f_{v_j}(x) = f_{\hat{V}_i}(\bar{x}) = \begin{cases} 1 & \text{if } x \in [0,1]/\bar{x} \in [0,1]^n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.19}$$

We compute the probability distribution of the random variable

$$X_{j,\tau,i} \equiv V_{\tau i} - \alpha V_\tau - V_{ji},$$

assuming that $\alpha > 1$, i.e. sellers do not set a price lower than its valuation. To do that, we follow the methodology described in Observation 3.4.1.

Vector $g_{j,\tau,i}$ is a linear transformation of the vector

$$\begin{pmatrix} V_{\tau i} \\ V_\tau \\ V_{ji} \end{pmatrix},$$

where each probability is uniformly distributed over $[0,1]$, we have that their joint distribution is

$$f_{V_{\tau i},V_\tau,V_{ji}}(x,y,z) = \frac{1}{\alpha} \text{ for all } (x,y,z) \in [0,1] \times [-\alpha,0] \times [0,1] = \Delta.$$

We know that αV_τ is uniformly distributed over $[0,\alpha]$ and V_{ji} is uniformly distributed over $[0,1]$. Then $-1-\alpha \le X_{j,\tau,i} \le 1$, consequently

$$\begin{aligned} f_{X_{j,\tau,i}}(x) &= \int\int_\Delta 1 dv_{\tau i} dv_\tau \\ &= \begin{cases} \int_0^1 \int_0^{\frac{x}{\alpha}} \frac{1}{\alpha} dv_{\tau i} dv_\tau & 0 \le x \le 1, \\ \int_0^1 \int_{\frac{x}{\alpha}-1-\alpha}^{1} \frac{1}{\alpha} dv_{\tau i} dv_\tau & -1-\alpha \le x \le 0. \end{cases} \\ &= \begin{cases} \frac{x}{\alpha^2} & 0 \le x \le 1, \\ \frac{\alpha+\alpha^2-x}{\alpha^2} & -1-\alpha \le x \le 0. \end{cases} \end{aligned}$$

When other sellers have a linear behaviour, we calculate the best response of each seller analysing the following cases:.

A. $|S| = 1$ and $|B| = n$, with $n \in \mathbb{N}$, and

B. $|S| = m$ and $|B| = n$, with $n, m \in \mathbb{N}$ and $2 < m \leq n$.

Both cases satisfy the second condition of a regular sequence of markets, required for the veracity of Theorem 3.3.1.

Case A. $|\mathbf{S}| = \mathbf{1}$, and $|\mathbf{B}| = \mathbf{n}$

Consider $S = \{s_1\}$ and $B = \{1, 2, \ldots, n\}$. Seller s_1 sets the price of her good strategically, she wants to maximize her expected utility. Moreover, since the TTCo algorithm ends at its first step in large thick markets, the payoff function of seller s_1 is:

$$u_{s_1}(\omega, s; v_1) = \begin{cases} p_1 & \text{if } v_{1i} - p_1 > 0 \text{ for some } i \in B, \\ v_1 & \text{otherwise.} \end{cases}$$

Then, the expected utility function for seller s_1 is

$$E[u_{s_j}(\omega, s; v_1)] = p_1 Pr[v_{1i} > p_1] + v_1 Pr[v_{11} \leq p_1],$$

for some $i \in B$. Also, we assume that v_{1i} is a random variable uniformly distributed over $[0, 1]$, for all $i \in B$. This implies that

$$\begin{aligned} E[u_{s_j}(\omega, s : v_1)] &= p_1 Pr[v_{1i} > p_1] + v_1 Pr[v_{1i} \leq p_1] \\ &= p_1(1 - p_1) + v_1 p_1. \end{aligned}$$

The first order condition is

$$\begin{aligned} 0 &= \frac{\partial E[u_{s_j}]}{\partial p_1} \\ &= 1 - 2p_1 + v_1. \end{aligned}$$

The best response is

$$p_1^* = \frac{1 + v_1}{2}.$$

The second derivative evaluated in p_1^*, we have that

$$\left.\frac{\partial E[u_{s_1}]}{\partial p_1}\right|_{p_1=p_1^*} = -2.$$

Therefore, the price that maximizes $E[u_{s_1}]$ is p_1^*.

Case B. $\mathbf{2 \leq |S| = m \leq |B| = n}$

Let there be $S = \{s_1, s_2, \dots, s_m\}$ and $B = \{1, 2, \dots, n\}$. Given that $\hat{v}_i = (v_{1i}, v_{2i}, \dots, v_{mi})$ is a random vector, we suppose that the random variables v_{ji} and v_j are independent and uniformly distributed over $[0,1]$ for $j = 1, \dots, m$ and $i = 1, \dots, n$. As before, we maximize the expected utility of s_j through the transformation $\overline{u_{s_j}}$ of her payoff function:

$$\overline{u}_{s_j}(\omega, s; v_1) = \begin{cases} p_j - v_j & \text{in } A^i_{j0} \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases} \tag{3.20}$$

We consider that s_j assumes that seller s_τ sets a price $p_\tau = \alpha v_j$, for all $\tau \neq j$. So, the expected utility function is

$$E[u_{s_j}] = (p_j - v_j) Prob[X_{j,\tau,i} \leq -p_j]^{m-1}(1 - Pr[v_{ji} \leq p_j]). \tag{3.21}$$

where $Prob[X_{j,\tau,i} \leq -p_j] = \frac{5}{2} + \frac{1}{2\alpha^2} + \frac{2}{\alpha} + \alpha - p_j - \frac{p_j}{\alpha} - \frac{p_j^2}{2\alpha^2}$. Since V_{ji} is uniformly distributed over $[0,1]$, we have that $F_{V_{ji}}(p_j) = p_j$. Replacing it in (3.21), the expected utility is

$$E[\overline{u}_{s_j}] = (p_j - v_j)\left(\frac{5}{2} + \frac{1}{2\alpha^2} + \frac{2}{\alpha} + \alpha - p_j - \frac{p_j}{\alpha} - \frac{p_j^2}{2\alpha^2}\right)(1 - pj).$$

Clearly, above expression induces an equation of degree at least $2m$, so, the uniform distribution is not useful to do comparative statistics.

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Chapter 4

The subsidized housing problem in Paris

4.1 Introduction

The French tradition in subsidized housing aims at ”mixité sociale”, namely promoting social diversity in all districts. In Paris, according to the Atelier Parisien d'Urbanisme, 70% of the households are eligible to one of the many existing programs. Institutions promoting social housing are diverse; the main City Hall, City Halls of all districts, ministries, the Societé National des Chemins de Fer (national railways), La Poste (national mail service), ... The diversity of involved institutions is financially profitable to the programs. They apply, however, very different eligibility criteria and the conflict between them to allocate a common pool of subsidized housing is solved by the institutions in Committees. The assignment process, thus, is not transparent, which raises criticisms over its discretionality.

In contrast, the anglo saxon tradition is focused on low income households, which allows a systematic approach to allocate subsidized apartments. Consider the case of Toronto[1]. The households register on the internet filling a form where they describe their family situation: number of members, age of each member, income, working status, health condition, ... Every two months the pool of free apartments and the score of each household are updated, which determines the ranking of all eligible households for each apartment[2]. Then, parallel serial dictatorships are performed. Is it possible to

[1] http://www1.toronto.ca/wps/portal

[2] http://www.torontohousing.ca/

adopt such systematic method in the context of Paris subsidized housing? Is there a mechanism that copes with ”mixité sociale” and deals with different scoring schemes fairly and efficiently?

Our problem casts similarities with the one of affirmative action in the school choice problem. One stream of papers interprets affirmative action as ”leveling the playing field”, like in Kojima (2010) and Hafalir, Yanmez and Yildirim (2011). Students from less privileged ethnic groups, independently of their effort, have lower chance to incorporate the schools which are academically most demanding. It is fair to give priority to these students with criteria less demanding than those required to students belonging to privileged groups. Kominers and Sönmez (2013) generalize Hafalir, Yanmez and Yildirim (2011) to allow for slot-specific priorities in a model with contracts. The consequence on welfare is that assignments with affirmative action should be Pareto superior for minority students to assignments without affirmative action. We are closest to the ones for which affirmative action is an objective per se, formalized by the existence of quotas of agents to be fulfilled like in Abdulkadiroglu (2003), Ehlers (2010) and Ehlers, Hafalir, Yanmez and Yildirim (2011). Affirmative action might hurt agents, individually, who do not take into account the externalities generated by affirmative action. The consequence on welfare is that the Pareto criteria are only used to compare assignments where affirmative action is implemented. Unlike these papers, however, we do not consider the existence of lower and upper bound for each category. In a different approach, Echenique and Yenmez (2013) and Erdil and Kumano (2012) study the diversity as a policy goal focused on schools priorities.

We model the subsidized housing problem in Paris as a three sided market with households, institutions and apartments. Households are assigned apartments through programs operated by institutions and might qualify to different programs. What matters for them is the apartment they are assigned to, not the institution that promote them, that is why preferences of households are only defined over the set of apartments. Our approach requires formalizing the following. First, institutions have priorities defined over: 1. apartments, and 2. households. Priorities over apartments are motivated by the amount of financing to the real estates (the more an institution spends in a building, the stronger its interest in using it), or because it is closest to its headquarters. Priorities over households are generated by a scoring method which can be different from one institution to another. Second, apartments have priorities over institutions: the more an institution has spent in financing a building, the highest its priority.

The literature of three sided markets focuses on the existence of a stable matching, where a matching is a set of disjoint families with three agents, commonly of the form (man, woman, dog). A matching is stable if there is no blocking family that is preferred by all its members to their current families in the matching. Chuang (2007) considers a model where each agent has two preference lists over the other two sets. Since a stable three sided matching does not always exists, he introduces indifferences and defines a hierarchy of stabilities. Under these structure he proves that a stable matching always exists but is NP-complete regardless the definition of stability. On the other hand, Biró and McDermind (2010) consider a model with cyclic preferences, i.e. men only care about women, women only care about dogs and dogs only care about men. They prove that to find a three sided stable matching is a NP-complete problem.

To deal with the subsidized housing problem, we introduce the Nested Deferred Acceptance (NDA) algorithm. This assignment procedure nests two Deferred Acceptance algorithms. During the first one, each household asks for her most preferred apartment, i.e. the demand of each household is elicited. Then, we run the nested DA between institutions and those apartments that are demanded by a household of its type. Institutions choose a set of apartments that maximizes its priority and does not exceed its vector of quotas. If more than one institution is interested in assigning an apartment, the apartment priority breaks the tie. After that, each apartment is temporarily assigned to the household with the highest priority among the households with the same type that demand it. The rejected households ask for their next preferred apartment and the procedure continues until no household is rejected.

However, the NDA alone fails to cope with mixité sociale and fairness for the same type. We identify that these problems arise due to interrupters. As Kesten (2010), we say that some institutions make interruptions and define two types of interrupters, one for each problem. The ”mixité sociale” condition is not reached given that households can have more than one type. Thus, some institutions assign an apartment to a household that should receive the same apartment from a different institution so as the ”mixité condition” to hold. On the other hand, the NDA assignment is not fair for households of the same type because there are institutions that tentatively hold an apartment during some steps, but this apartment is not included in their final assignment. So, this can induce justified envy for the same type because the institutions that assign such apartments do not consider again households rejected from these apartments.

The NDA with Interrupters (NDAI) improves the NDA by dropping apartments from the priority of the interrupter institutions. Assuming a the market that admits at least one fair for the same type assignment that satisfies mixité sociale, our main result shows that the output given by the NDAI is fair for households of the same type, Pareto undominated by an assignment fair for households of the same type and satisfies mixité sociale. Even more, telling the truth is a dominant strategy in the induced revelation game for households. On the opposite, the Efficiency Adjusted Deferred Acceptance Mechanism (EADAM) by Kesten is manipulable because students are interrupters and make offer during the DA, while our interrupters receive offers during the NDA.

Previous result assumes the compatibility between mixité sociale and fairness for the same type. However, this is not always true. Analysing the relation between these two concepts, we find a condition to guarantee the existence of an assignment that satisfies the mixité condition. Moreover, we explain the behavior of the NDAI mechanism in markets where mixité social and fairness for the same type are not compatible.

The present work is organized as follows. Section 2 introduces formally the subsidized housing model. Section 3 presents the Nested DA and Section 4 the interrupters discussion. In Section 5 we present or main result for the NDAI mechanism. Section 6 analyses the existence issues related with mixité sociale and fairness for the same type.

4.2 Model

A subsidized housing market with mixité is $(I, H\tau, D, Q, A, \delta, P, \succ, \pi)$ where:

1. $I = \{1, 2, ..., N\}$ is the finite set of institutions, a generic institution is i;

2. $H = \{h_1, ..., h_H\}$ is the finite set of Households, a generic household is h;

3. $\tau : H \to 2^I$ is the type function, where the types of household h are $\tau(h)$; $H_i = \{h \in H \mid i \in \tau(h)\}$.

4. $D = \{d_1, ..., d_D\}$ is the finite set of districts, there are D districts, a generic district is d;

5. $Q = ((q_i^{d_j})_{i=1}^N)_{j=1}^D$ is the vector of quotas, where q_i^d is the quota of households of type i in district d, a generic quota is q;

6. $A = \{a_1, ..., a_A\}$ is the finite set of apartments, a generic apartment is a;

7. $\delta : A \to D$ is the district function, where the district of apartment a is $\delta(a)$; the number of apartments in district d is $\#d$, $\sum_{i=1}^{N} q_i^d = \#d$ for all districts d;

8. $P = (P_{h_1}, ..., P_{h_H})$ is the vector of households' preferences, P_h is the strict preferences of household $h \in H$ over $A \cup \{h\}$, aP_ha' means that households h prefers a to a', an apartment a is acceptable for household h if $a \succ_h h$. Let $P_h : a_{1_h}, a_{2_h}, \ldots a_{A_h}$, and consider R_h be the antisymmetric preference list where aP_hb and bP_ha if and only if $a = b$.

9. $\succ = (\succ^i)_{i \in I}$ is the vector of institutions' priorities

$$\succ^i = (\succ_A^i, (\succ_{a_1}^i, \ldots, \succ_{a_A}^i)),$$

we assume that priorities are lexicographic in:

   9.a $\succ_A^i$ the priority of institution i over apartments A, $a_r \succ_A^i a_s$ means that department a_r is prioritized over apartment a_s by institution i; and

   9.b $\succ_a^i$ the strict priority of apartment $a \in A$ over H_i, it is generating by the score of households, $h \succ_a^i h'$ means that household h has priority over household h' at apartment a; an household h is acceptable for apartment a if $h \succ_a^i a$, else h is unacceptable;

   9.c We suppose that for all acceptable apartments a for institution i, at least one household is acceptable.

10. π_a is a priority of institutions over apartment a.

To design an assignment procedure for the subsidized housing problem, we consider that this problem is: 1. a many-to-one matching problem between apartments and institutions, and 2. a ono-to-one matching problem between households and pairs composed by one apartment and one institution. Such assignment is formalized in the following definition.

An **assignment** $\mu = (\theta, \varphi)$ is a duple such that:

i. $\theta : A \cup I \to 2^A \cup I \cup \{\varnothing\}$ where

  i.a $\theta(a) \in I \cup \{\varnothing\}$,

  i.b $\theta(i) \in 2^A$,

  i.c $\theta(a) \in \theta(i)$ if and only if $\theta(a) = i$.

ii. $\varphi : A \times I \cup H \to (A \times I) \cup H \cup \{\varnothing\}$, where

  ii.a $\varphi(h) \in A \times I \cup \{\varnothing\}$,

  ii.b $\varphi(a,i) \in H \cup \{\varnothing\}$,

  ii.c $\varphi(h) = (a,i) \Leftrightarrow \varphi((a,i)) = h$. The corresponding projections are $\varphi_A(h) = a$ and $\varphi_I(h) = i$,

iii. $\theta(a) = i$ if and only if $\varphi(h) = (a,i)$.

Conditions i. a, b and c refer to the many-to-one matching problem between apartments and institutions. Conditions ii. a, b and c refers to the one-to-one matching problem between institutions and pairs composed by one apartment and one institution. Condition 3 says that a household cannot be assigned a pair of one apartment and one institution apartment if the apartment is not assigned to the institution.

The match of $k \in H$ is $\varphi(k) \in (A \times I) \cup \{\varnothing\}$, k is unmatched if $\varphi(k) = \varnothing$. The match of $i \in I$ is $\theta(\in 2^A$, i is unmatched if $\theta(i) = \varnothing$.

Let $(I,H,\tau,D,Q,A,\delta,P,\succ,\pi)$ a subsidized housing market with "mixité sociale", an assignment representation is

$$\mu = \left\{ \begin{array}{cccccc} h_3 & h_5 & h_1 & h_2 & h_4 & \varphi^{-1}[\varnothing] \\ \varphi(h_3) & \varphi(h_5) & \varphi(h_1) & \varphi(h_2) & \varphi(h_4) & \varnothing \end{array} \right\}$$

$$= \left\{ \begin{array}{c} h_3 \;\; h_5 \;\; h_1 \;\; \varnothing \;\; h_2 \;\; h_4 \;\; \varnothing \\ a_2 \;\; a_4 \;\; a_3 \;\; \varnothing \;\; \varnothing \;\; \varnothing \;\; a_1 \\ \underbrace{i_2 \;\; i_2}_{\theta(i_2)} \;\; \underbrace{i_3}_{\theta(i_3)} \;\; \underbrace{i_1}_{\theta(i_1)} \;\; \underbrace{\varnothing \;\; \varnothing \;\; \varnothing}_{\theta^{-1}[\varnothing]} \end{array} \right\}.$$

An assignment μ is **individually rational** if

i. for all $h \in H$ either $\varphi_A(h) P_h h$ or $\varphi(h) = \varnothing$, and

ii. for all $i \in I$, $a \succ^i_A \varnothing$ for all $a \in \theta(i)$ and $h \succ^i_a a$ for all household h such that $\varphi(h) = (a,i)$ and $i \in \tau(h)$.

We assume that the set of *individually rational* matchings is not empty.

For any individually rational assignment μ, let $H^d_i(\mu)$ be the set of households of type i assigned at district d, formally $H^d_i(\mu) = \{h \in H \mid \varphi_I(h) = i \text{ and } \delta(\varphi_A(h)) = d\}$. An assignment μ **respects mixité** whenever $\#H^d_i(\mu) = q^d_i$ for all $i \in I$ and $d \in D$.

An assignment μ is **non-wasteful** if no household justifiably claims an empty apartment, i.e. there is no i, h and a such that:

i. $aP_h\varphi_A(h)$,

ii. $\theta(a) = \varnothing$,

iii. $a \succ^i_A \varnothing$ and $h \succ^i_a \varnothing$.

Household h has **justified-envy** over household h' at individually rational assignment μ if

i. $\varphi_A(h')P_h\varphi_A(h)$ and

ii. $h \succ^i_{\varphi_A(h')} h'$ for some institution $i \in I$.

Institution i has **justified-envy** over institution i' at individually rational assignment μ if there exists $a \in \theta(i')$ such that

i. $a \succ^i_A a'$, for some $a' \in \theta(i)$, and

ii. $i\pi_a i'$.

An assignment μ is **fair** if it is individually rational, non wasteful and there is no justified envy. A matching μ is fair for households of the same type if it is individually rational, non-wasteful, there is no justified envy over institutions and there is no justified envy for households of the same type, i.e. when $i = \varphi_I(h)$.

An assignment μ is **Pareto efficient** if there is no matching μ' such that all households prefer μ' to μ, with strict inequality for at least one household. An assignment μ' **Pareto dominates** other assignment μ if $\mu'(h)R_h(h)$ for each $h \in H$, and $\mu'(h')P_{h'}(h')$ for at least one $h' \in H$.

A mechanism χ associates a profile of preference list with an assignment μ. Let R_h be the true preference list of each household h. The set of all possible preference lists of household h is denoted by $\Re_h$. A profile of preference list is a vector $R' = (R'_{h_1}, R'_{h_2}, \ldots, R'_{h_H}) \in \Re_{h_1} \times \Re_{h_2} \times \cdots \times \Re_{h_H} = \Re$. As usual, R_{-h} is the profile of all preference list except R_h. A mechanism is **strategy proof** if telling the truth is a dominant strategy, i.e.

$$\varphi_A[R_h, R^*_{-h}](h)R_h\varphi_A[R'_h, R^*_{-h}](h) \text{ for all } R'_h \in \Re_h \text{ and } R^*_{-h} \in \Re_{-h}.$$

4.3 The Nested Deferred Acceptance Mechanism

In this section we present a mechanism that deals with the subsidized housing problem. We introduce the Nested Deferred Acceptance (NDA) to find an assignment $\mu = (\theta, \varphi)$. The idea behind this assignment procedure is to compute simultaneously a many-to-one matching, θ, and a one-to-one matching, φ. To do that, the NDA mechanism nests two deferred acceptance (DA) algorithms: in the first one, each household asks for her most preferred apartment, i.e. the demand of each household is elicited. Then, we run the nested DA between institutions and those apartments which are demanded by a household of its type. Formally the NDA proceed as follows:

Initialization

For all households $h \in H$, let $A_h^t := A$. and $t := 1$.

A. Eliciting the demand of households

All unassigned households h ask for the most preferred apartment in A_h^t, denoted D_h^t, while matched households h' iterate their demand to their match, $D_{h'}^t = \varphi_A^{t-1}(h)$.

For all $i \in I$ and $a \in A$ define the set of households that demand apartment a and i is one of her possible types:

$$H_a^{it} = \{h \in H \mid (D_h^t = \{a\}) \text{ and } i \in \tau(h)\}.$$

For all $i \in I$ define the set of apartments to which i can be assigned to an household at an individually rational matching:

$$A_i^t = \{a \in A \mid (\exists h \in H_a^{it}\}), (a \succ_A^i \varnothing) \text{ and } (h \succ_a^i \varnothing)\}.$$

B. Assignment of apartments to institutions

B.1 All institutions i demand apartments in

$$Ch_i(A_i^t, \succ_A^i) = \{s \in 2^{A_i^t} \mid \#s \leq q_i \text{ and } s \text{ is maximal for } \succ_A^i \text{ in } A_i^t\}.$$

Moreover

$$I_a^t = \{i \in I \mid a \in Ch_i(A_i, \succ_A^i)\}.$$

B.2 For all apartments a which received a demand from some institution, the apartment a is tentatively assigned to the institution highest ranked in π_a, i.e. $a \in \theta^t(\max_{\pi_a} I_a^t)$.

C. Assignment of households to institutions and apartments

Each institution i assigns an apartment $a \in \theta^t(i)$ to the household with the highest priority, that is to say $\varphi^t(a,i) = \max_{\succ_a^{ix}} H_a^{it}$.

For all unassigned household, h, let $A_h^{t+1} := A_h^t \setminus \max_{P_h} A_h^t$, $t := t+1$, go to A. Else stop, if each household has been rejected from all the apartments in her preference list or is matched.

The tentative matching is the outcome assignment. Each household tentatively matched with a pair composed by one apartment and one institution at the last step is assigned that apartment by the corresponding institution in the pair. Other households are assigned the null object $\varnothing$.

We denote by $\mu^{NDA} = (\theta^{NDA}, \varphi^{NDA})$ the assignment given by the NDA. Note, that the NDA algorithm has a finite number of steps because each DA ends in polynomial time. In the following example we show how the NDA mechanism works.

Example 4.3.1. Consider $A = \{a_1, a_2\}$, $H = \{h_1, h_2, h_3\}$, $I = \{i_1, i_2\}$. Assume only one district, $\delta(a_1) = \delta(a_2) = 1$, households type function is given by $\tau^{-1}(1) = \{1,2\}$ and $\tau^{-1}(2) = \{h_2, h_3\}$. The quota for both institutions is equal to 1. Households preferences are

$$P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} \\ a_1 & a_2 & a_2 \\ a_2 & a_1 & a_1 \end{pmatrix}.$$

The priorities of institutions and apartments are the following

$$\succ^1 = \begin{pmatrix} \succ_A^1 & \succ_{a_1}^1 & \succ_{a_2}^1 \\ a_1 & h_2 & h_2 \\ a_2 & h_1 & h_1 \end{pmatrix}, \quad \succ^2 = \begin{pmatrix} \succ_A^2 & \succ_{a_1}^2 & \succ_{a_2}^2 \\ a_2 & h_3 & h_2 \\ a_1 & h_1 & h_3 \end{pmatrix} \text{ and } \pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} \\ 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Running the NDA, the demand elicited by households at Step 1 is the following

- Apartments demanded by type 1 households are: $H_{a_1}^1 = \{h_1\}$ and $H_{a_2}^1 = \{h_2\}$,

- Apartments demanded by type 2 households are: $H^2_{a_1} = \varnothing$ and $H^2_{a_2} = \{h_2, h_3\}$.

Consequently $A^1_1 = \{a_1, a_2\}$ and $A^1_2 = \{a_2\}$. According to the vector of quotas and the vector of priorities $\succ_A$, the set of apartments that each institution chooses is

$$Ch^1_1(A^1_1, \succ^1_A) = \{a_1\} \text{ and } Ch^1_2(A^1_2, \succ^2_A) = \{a_2\}.$$

Therefore, we get tentatively the triads

$$(h_1, a_1, 1) \text{ and } (h_2, a_2, 2).$$

We have that h_3 was rejected from apartment h_2. Then, the demand elicited by households at Step 2 is

- Apartments demanded by type 1 households are: $H^1_{a_1} = \{h_1\}$ and $H^1_{a_2} = \{h_2\}$,
- Apartments demanded by type 2 households are: $H^2_{a_1} = \{h_3\}$ and $H^2_{a_2} = \{h_2\}$.

Consequently $A^1_1 = \{a_1, a_2\}$ and $A^1_2 = \{a_1, a_2\}$. Note that institution $a_1 \succ^1_A a_2$ and $a_2 \succ^1_A a_2$; moreover, the quota of each institution is equal to one. Then, each institution chooses the following set of apartments

$$Ch^1_1(A^1_1, \succ^1_A) = \{a_1\} \text{ and } Ch^1_2(A^1_2, \succ^2_A) = \{a_2\}.$$

Therefore, we get tentatively the triads

$$(h_1, a_1, 1) \text{ and } (h_2, a_2, 2).$$

We have that household 3 has been rejected from all her acceptable apartments Thus, the algorithm stops, and final assignment is

$$\mu^{NDA} = \begin{pmatrix} h_1 & h_2 & h_3 \\ a_1 & a_2 & \varnothing \\ 1 & 2 & \varnothing \end{pmatrix}.$$

4.4 Interrupters

In this section we show that the assignment produced by the NDA algorithm fails to cope with the mixité condition and fairness for the same type. The following example show that these problems arise because some institutions turn out to be interrupters as defined by Kesten (2010). We identified two types of interrupters.

Example 4.4.1. (There is justified envy for households of the same type). Let $I = \{i_1, i_2\}$, $A = \{a_1, a_2, a_3\}$ and $H = \{h_1, h_2, h_3, h_4\}$, where households type function is given by $\tau^{-1}(1) = \{h_1, h_2\}$ and $\tau^{-1}(2) = \{h_3, h_4\}$. The vector of quotas is $q = (2, 1)$. The priorities for the institutions are

$$\succ^1 = \begin{pmatrix} \succ^1_A & \succ^1_{a_1} & \succ^1_{a_2} & \succ^1_{a_3} \\ a_1 & h_1 & h_1 & h_1 \\ a_2 & h_2 & h_2 & h_2 \\ a_3 & & & \end{pmatrix}, \succ^2 = \begin{pmatrix} \succ^2_A & \succ^2_{a_1} & \succ^2_{a_2} & \succ^2_{a_3} \\ a_2 & h_3 & h_3 & h_3 \\ a_1 & h_4 & h_4 & h_4 \\ a_3 & & & \end{pmatrix}.$$

Households preferences and apartments priorities are

$$P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} & P_{h_4} \\ a_1 & a_2 & a_1 & a_1 \\ a_2 & a_1 & a_2 & a_2 \\ a_3 & a_3 & a_3 & a_3 \end{pmatrix}, \pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} & \pi_{a_3} \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

At Step 1 of the NDA algorithm, we have that:

- the type 1 households that demand an apartment are $H^1_{a_1} = \{h_1\}$ and $H^1_{a_2} = \{h_2\}$,
- the type 2 households that demand an apartment are $H^2_{a_1} = \{h_3, h_4\}$ and $H^2_{a_2} = \varnothing$.

Under the priority π_{a_1}, institution 2 assigns apartment a_1. Then, we tentatively get the triads

$$(h_3, a_1, 2),\ (h_2, a_2, 1).$$

During Step 2, we have that

- the type 1 households that demand an apartment are $H^1_{a_2} = \{h_1, h_2\}$,
- the type 2 households that demand an apartment are $H^2_{a_1} = \{h_3\}$ and $H^2_{a_2} = \{h_4\}$.

The priority of apartment a_2 implies that institution 2 assigns apartment a_2. However, its quota is 1 and apartment a_2 is preferred to a_1 under $\succ^2_A$. Therefore, we get temporarily the triad

$$(h_4, a_2, 2).$$

That is to say, the triads $(h_3, a_1, 2)$ and $(h_2, a_2, 1)$ are dissolved at the end of Step 2.
Since each household makes an offer to those apartments such that she has not been rejected, at Step 3, we have that:

- the type 1 households that demand an apartment are $H^1_{a_1} = \{h_2\}$ and $H^1_{a_3} = \{h_1\}$,
- the type 2 households that demand an apartment are $H^2_{a_2} = \{h_3, h_4\}$.

According to the quotas, institutions 1 and 2 choose the set $\{a_1, a_2\}$ and a_3, respectively. Since $h_3 \succ^2_{a_3} h_4$, the tentative assignment is

$$\mu^3 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 \\ a_3 & a_1 & a_2 & \varnothing \\ 1 & 1 & 2 & \varnothing \end{pmatrix}.$$

Thus, household h_4 is rejected from a_3. During the final Step 4, h_4 asks for apartment a_3. However, $1\pi_{a_3}2$ and $h_1 \in H^1_{a_3}$. Therefore, h_4 does not get this apartment and the algorithm stops. Therefore, the final assignment is

$$\mu^{NDA} = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 \\ a_3 & a_1 & a_2 & \varnothing \\ 1 & 1 & 2 & \varnothing \end{pmatrix}.$$

Note that $\varphi^{NDA}_A(h_2) = a_1 P_{h_1} a_3 = \varphi^{NDA}_A(h_1)$, $h_1 \succ^1_{a_1} h_2$ where $1 = \varphi^{NDA}_I(h_1) = \varphi^{NDA}_I(h_2)$. Therefore, household h_1 justifiably claims the apartment a_1 to household h_2, i.e. there is justify envy for households of the same type.

Previous example illustrates the interruption of type 1. The NDA does not satisfy fairness for the same type because institution 2 tentatively assigns a_1, so household h_1 and institution 1 are displaced from it at Step 1. However, institution 2 does not assign a_1 at the end of the mechanism due to $a_2 \succ_A^2 a_1$ and $q_2 = 1$, i.e. a_1 is displaced from $\theta(2)$ at Step 2. We define formally type 1 interrupters.

Given a problem to which the NDA is applied, we say that i is a **type 1 interrupter for** a if it exists

1. Steps t to $t+n$ such that $a \in \theta^{t'}(i)$ for all $t' \in \{t, t+1, \ldots, t+n\}$ but $a \notin \theta^{t'}(i)$ for all $t' > t+n$, and

2. institution $j \neq i$ such that $a \in Ch_j(A_j^l, \succ_A^j)$ but $a \notin \theta^l(j)$ for some $l \in \{t, t+1, \ldots, t+n\}$.

Now, we show an example where the NDA algorithm does not output an assignment that respects mixité sociale.

Example 4.4.2. (Mixité Condition) Consider $I = \{1, 2\}$, $A = \{a, b, c\}$, $H = \{h_1, h_2, h_3\}$ where the households function type is given by $\tau^{-1}(1) = \{h_1, h_2, h_3\}$ and $\tau^{-1}(2) = \{h_2\}$. The vector of quotas is $q = (q^1, q^2) = (2, 1)$. The of households preferences is

$$P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} \\ a & a & a \\ b & b & b \\ c & c & c \end{pmatrix}.$$

Institutions priorities over households is

$$\succ^1 = \begin{pmatrix} \succ_A^1 & \succ_a^1 & \succ_b^1 & \succ_c^1 \\ a & h_1 & h_1 & h_1 \\ b & h_2 & h_2 & h_2 \\ c & h_3 & h_3 & h_3 \end{pmatrix} \text{ and } \succ^2 = \begin{pmatrix} \succ_A^2 & \succ_a^2 & \succ_b^2 & \succ_c^2 \\ a & h_2 & h_2 & h_2 \\ b & & & \\ c & & & \end{pmatrix} -$$

Priorities over apartments and apartments are

$$\pi = \begin{pmatrix} \pi_a & \pi_b & \pi_c \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}.$$

Running the algorithm, at Step 1 we have that:

- type 1 households that demand an apartment are $H_a^1 = \{h_1, h_2, h_3\}$,

- type 2 households that demand an apartment are $H_{a^2}^2 = \{h_2\}$.

So, following the priorities, we tentatively get the triad

$$(h_1, a, 1).$$

At Step 2, the elicited demand by households implies that

- type 1 apartments that demand an apartment are $H_a^1 = \{h_1\}$, $H_b^1 = \{h_2, h_3\}$,

- type 2 apartments that demand an apartment are $H_b^2 = \{h_2\}$.

The priority vector π determines that

$$(h_1, a, 1),\ (h_2, b, 1).$$

At Step 3, household h_3 asks for apartment a_3. So, $a_1, a_2, a_3 \in A_1^3$. Since a_1, a_2 are preferred to a_3 under $\succ_A^1$, and $1\pi_{a_2}2$, we have that institution 1 assigns apartments a_1, a_2 to households $h_1, h-2$, respectively. Consequently, h_3 is rejected from a_3 and the algorithm stops. The assignment produce by the NDA algorithm is

$$\mu = \begin{pmatrix} h_1 & h_2 & h_3 \\ a & b & \varnothing \\ 1 & 1 & \varnothing \end{pmatrix}.$$

On the other hand, the assignment

$$\mu' = \begin{pmatrix} h_1 & h_3 & h_2 \\ a & c & b \\ 1 & 1 & 2 \end{pmatrix}$$

is a feasible assignment that respects mixité sociale condition. In other words, the market described in previous example satisfies the over-demand condition. However, the output of the NDA mechanism does not satisfy the mixité condition because $\#\theta(2) = 0 < q_2$. This interruption is caused because institution 1 assigns the apartment a_2 to household h_2, where h_2 has the possibility to receive the same apartment by institution 2. whose quota is not fulfilled. We define formally type 2 interrupters.

Given a problem to which NDA is applied, we say that i is a **type 2 interrupter** of the NDA algorithm if

1. $\varphi^{NDA}(h) = (a, i)$,
2. $\#\tau(h) \geq 2$, i.e., there exists $j \in \tau(h) - \{i\}$,
3. $i\pi_a j$,and
4. $\#\theta^{NDA}(j) < q_j$.

The triad (h, a, i) is a **type 2 interrupting triad** for the triad (h, a, j).

Observation 4.4.1. Note that assignment μ' satisfies mixité condition and does not have type 2 interrupters. However, institution 1 justifiably claims apartment a_2 to institution 2.

4.5 Nested Deferred Acceptance with Interrupters

We have shown that the NDA assignment is not always fair for households of the same type and it does not always respects the mixité condition. Moreover, these problems are caused given the existence of type 1 and 2 interrupters. Following the Efficiency Adjusted Deferred Acceptance Mechanism (Kesten, 2010), we modify the NDA introducing a second stage where we search for all the interrupter institutions. Then, these institutions delete the apartment where they caused the interruption from their priority defined over apartments. That is to say, we define a delete operation on priorities $\succ_A^i$.

Let Λ be the set of all possible priorities $\succ_A^i$, for all $i \in I$. The **delete operation** over Λ is the function $\setminus : \Lambda \times A \to \Lambda$ such that $\setminus(\succ, a)$, or simply $\succ \setminus a$, is the priority that declares apartment a unacceptable for i only if $a \succ \varnothing$. In other words, the priority $\succ \setminus a$ drops a from $\succ$ and holds the original order in $\succ$. Otherwise, $\succ \setminus a = \succ$. Note that Kesten defines this operation over students preferences, the agents who makes the proposal, because he identifies that students causes the loss of efficiency during the Deferred Acceptance mechanism. In our case, institutions, the agents who "receive" the proposal, cause a loss in fairness and non-wastefulness because the mixité condition is not satisfied.

Now, we introduce the Nested Deferred Acceptance with Interrupters (NDAI). Each step if this mechanism has two stage: in the first we run the NDA algorithm, and the second stage actualizes the priorities of all interrupters. Formally, the NDAI proceed as follows.

Initialization

Counter of iterations over interrupter institutions, $x := 0$.

STEP 0. This step is divided in the following stages:

Stage 0.1 NDA Phase. Let $\succ_A^0 = (\succ_A^i)_{i\in I}$. Run the NDA algorithm using the profile of priorities and preferences $(\succ_A^0, ((\succ_a^i)_{a\in A})_{i\in I}, P)$.

Stage 0.2 Priorities Actualization. Find the last step of the NDA phase in which an interrupter is rejected from the apartment which she is an interrupter. For each interrupting triad (h,a,i), do $\succ_A^{i1} = \succ_A^{i0} \setminus a$, and $\succ_A^{j1} = \succ_A^{j0}$ if j is not an interrupter. If there is no interrupters, the algorithm stops.

STEP x. The stages are the following.

Stage x.1 NDA Phase. Run the NDA algorithm with the profile of priorities and preferences $(\succ_A^x, ((\succ_a^i)_{a\in A})_{i\in I}, P)$.

Stage x.2 Priorities Actualization. Find all the interrupting triads (of any type) in the last step of the NDA phase in Round $x-1$. For each interrupting triad (h,a,i), do $\succ_A^{ix+1} = \succ_A^{ix} \setminus a$, and $\succ_A^{jx+1} = \succ_A^{jx}$ if j is not an interrupter. If there is no interrupters, the algorithm stops.

The output of the previous mechanism is denoted by $\mu^{NDAI}[H, A, P, I, \succ, \pi_A, q]$. We know that the NDA phase is solvable in polynomial time and there are at most $\#I$ interrupters in each stage $x.2$. Also, priorities $\succ_A^i$ have a finite length. Therefore, the NDAI is solvable in polynomial, i.e. the number of iterations x is finite. This implies the existence of an iteration x^* such that there is no interrupters in its corresponding NDA phase (Stage $x^*.1$)

Before to present our main result about the NDAI mechanism, it is important to note that μ^{NDAI} is not fair because there is justified envy between institutions. For an example see Observation 4.4.1, where the assignment μ' is equal to μ^{NDAI}. The justified envy between institutions is a direct consequence of the definition of type 2 interrupter. For this mechanism it is not possible to get mixité sociale and fairness for institutions.

Also, if there is a unique institution in the market, it is easy to see that all the apartments are assigned by this institution i. Moreover, each pair (i,a) has associated the priority $\succ_a^i$. Hence, we only have to solved a one-to-one matching problem between pairs (i,a) and households. In other words, the NDAI coincides with the Deferred Acceptance mechanism when there is a unique institution. Thus, the NDAI can be understood as a generalization of the DA.

We present our main results about the NDAI mechanism in the following theorem.

Theorem 4.5.1. *Consider a subsidized housing market* $\aleph = (I,H,\tau,D,Q,A,\delta,P,\succ,\pi)$. *We have the following cases*

1. *Suppose the existence of at least one assignment that satisfies mixité sociale under* $\aleph$.
   A. *If mixité sociale and fairness for the same type are compatible, then the assignment* μ^{NDAI} *satisfies both properties. Moreover, there is no fair for the same type assignment that satisfies mixité sociale that Pareto dominates* μ^{NDAI}.
   B. *Otherwise, the assignment* μ^{NDAI} *only satisfies the mixité condition. Moreover, there is no assignment that satisfies mixité that Pareto dominates* μ^{NDAI}.
2. *If there does not exist an assignment that satisfies mixité sociale under* $\aleph$, *then* μ^{NDAI} *is fair for the same type.*
3. *The NDAI mechanism is strategy-proof.*

Proof. Consider x^* the NDAI last iteration where there is no interrupters.

We know that the corresponding Stage $x^*.1$ is solvable in polynomial time, i.e., the corresponding NDA phase ends at Step t^*. By construction, we have that

- $\theta^{NDAI}(i) = Ch_i(A_i^{t^*}, \succ_A^i | q_i, \pi_A)$ for all $i \in I$, then $\theta^{NDAI}(a) = i$ if and only if $a \in \theta^{NDAI}(i)$,
- The function $\varphi^{NDAI}(\cdot)$ is defined as $\varphi^{NDAI}(h) = (a,i)$ if and only if $h = \varphi^{NDAI}(a,i) = \max_{\succ_a^{ix^*}} H_a^{it^*}$ for all $(a,i) \in Ch_i(A_i^{t^*}, \succ_A^i | q_i, \pi_A) \times \{i\}$. So, $i \in \tau(h)$.

Therefore, $\mu^{NDAI} = (\theta^{NDAI}, \varphi^{NDAI})$ where $\theta^{NDAI} : I \cup A \to I \cup 2^A \cup \{\varnothing\}$ and $\varphi^{NDAI} : H \cup I \times A \to H \cup (I \times A) \cup \{\varnothing\}$. Moreover, $\varphi_I(h) \in \tau(h)$ for all $h \in H$. That is to say μ^{NDAI} is feasible.

Mixité Condition. We have to prove that $\#H_i^d(\mu^{NDAI}) = q_i^d$ for all $i \in I$.

By definition, we know that $\theta^{NDAI}(i) = Ch_i(A_i^{t^*}, \succ_A^i | q_i, \pi_A)$, then institution i chooses its highest ranked apartments such that its quota in each district is not violated. So, $\#H_i^d(\mu^{NDAI}) \leq q_i^d$.

Now, we need to prove that $\#H_i^d(\mu^{NDAI}) \geq q_i^d$. We proceed by contradiction assuming that $\#H_i^d(\mu^{NDAI}) < q_i^d$. Consequently, institution i does not fulfilled its quota, $\#\theta(i) < q_i$, and we have that $\sum_{i \in I} \#\theta^{NDAI}(i) < \sum_{i \in I} q_i = \#A$. This implies the existence of at least one apartment a that remains unassigned, $\theta^{NDAI}(a) = \varnothing$. Moreover, we assume the existence of at least one assignment that satisfies the mixité. Such assignment ensures the existence of a household h such that $\varphi^{NDAI}(h) = \varnothing$ and $\tau(h) = i$ because we consider that μ^{NDAI} does not satisfy the mixité condition. Since P_h is a strict preference list of all apartments, we have that $h \in H_a^{it}$ and $a \in A_i^t$, for some Step t, at iteration x^*. We have two cases:

- **First case**, institution i is not an interrupter of apartment a, that is to say, $a \succ_A^{ix^*} \varnothing$. Thus, we have the following subcases:

I.A Institution i does not choose this apartment at Step t, i.e. $a \notin Ch_i(A_i^t, \succ_A^i | q_i, \pi_A)$. Moreover, we know that $\#\theta(i) < q_i$ and $a \in A_i^t$. Consequently, there must exists an institution $j \neq i$ such that apartment a belongs to $Ch_j(A_j^t, \succ_A^j | q_j, \pi_A)$. So, some household $h' = \max_{\succ_a^j} H_a^{jt}$ is tentatively assigned to a by institution j. However, apartment a remains unassigned at the end of the NDAI algorithm, this means that the triad (h', a, j) was deleted in some later Step. We conclude that institution j is a type 1interrupter for apartment a, which contradicts the election of x^*.

I.B Institution i chooses this apartment at Step t, i.e. $a \in Ch_i(A_i^t, \succ_A^i |q_i, \pi_A)$, we have two possibilities: $h = \max_{\succ_a^i} H_a^{it}$ or not. Let h' the household assigned to a under i. Since $\theta^{NDAI}(a) = \varnothing$ and $\#\theta^{NDAI}(i) < q_i$, we conclude that h' was displaced in some later step by an institution j with a higher priority under π_a. At the end of the algorithm, j does not assigned apartment a. That is to say, the institution j is a type 2 interrupter, which contradicts the election of x^*.

- **Second case**, institution i is an interrupter of apartment a, that is to say $\varnothing \succ_A^{ix^*} a$. So, we analyze each type of interruption in the following subcases:

II.A Institution i is a type 1 interrupter for a in some iteration x'. By definition, institution i tentatively assigns a to some household h' at Step t, but the apartment a is not assigned by i at the end of the NDAI algorithm. Then there exists institution $i' \neq i$ interested in apartment a in $\{t, t+1, \ldots, t+n\}$ and $i\pi_a i'$ because $a \notin Ch_{i'}^{t'}(A_{i'}^t; \succ_A^{i'}, q_{i'})$ On the other hand, institution i rejects apartment a because its quota its fulfilled or there exists an institution $j \neq i$ with a higher priority than i under π_a. The first case is not possible because we assume that $\#\theta(i) < q_i$. Then, we have that $j\pi_a i \pi_a i'$. Since $\theta(a) = \varnothing$ we conclude that j is a type 1 interrupter for apartment a and the interruptions occurs after the interruption made by institution i at iteration x'. Therefore, the last interruption does not occur at Step t of the stage $x'.1$. This is a contradiction with the election of x' and t.

II.B Institution i is a type 2 interrupter for a in some iteration $x < x^*$. Then there exists an institution j and a household h such that the quota of j is not fulfilled, i has a higher priority than j under π_a, both institutions belong to the type set of h and institution i assigns the apartment a to h. Since the apartment a remains unassigned, the apartment a was displaced from j because its quota is fulfilled with a higher ranked apartment than a under $\succ_A^j$or by the existence of an interrupter. The first case is not possible because j needs the apartment a to fulfilled its quota. So, deleting a from all the interrupters of (a, h, j), we conclude that a must belong to $\theta(j)$. Otherwise, we will have that x^* is not the last iteration.

We get a contradiction in any case. Hence, $\#\theta^{NDAI}(i) \geq q_i$. Therefore, the mixité condition is satisfied $\#\theta^{NDAI}(i) = q_i$ for all $i \in I$.

Non-wastefulness. We proceed by contradiction. Suppose the existence of a triad (h,a,i) such that $aP_h\varphi_A^{NDAI}(h)$, $\theta^{NDAI}(a)=\varnothing$ and the institution i can assign the apartment a to the household h, i.e. $\#\theta^{NDAI}(i)<q_i$, $h\succ_a^i\varnothing$ and $i\in\tau(h)$. As before, let x^* be the last iteration where no institution is an interrupter. So, at stage $x^*.1$, in some Step t of the NDA phase we have that

$$h\in H_a^{it} \text{ and } a\in A_i^t.$$

Moreover, we know that $\#\theta^{NDAI}(i)<q_i$ and $Ch_i(A_i^t,\succ_A^i)=\{s\in 2^{A_i^t}\mid \#s\leq q_i$ and s is maximal for $\succ_A^i$ in A_i^t. Thus, $I_a^t\neq\varnothing$, particularly $i\in I_a^t$.So, consider $i'\in I_a^t$ such that $a\in Ch_{i'}(A_{i'}^t,\pi^{i'})$.

On the other hand, $\theta^{NDAI}(a)=\varnothing$,then a is displaced from $\theta^{NDAIt}(i')$ by some i'', but she does not assign it. This means that i'' is a type 1 interrupter, which contradicts the election of x^*. Therefore, μ^{NDAI} is non-wasteful.

There is no justified envy. Let x^* be the last iteration of the NDAI algorithm. Consider households h, h' such that

$$\varphi_I^{NDAI}(h)=\varphi_I^{NDAI}(h')=i \text{ and } a=\varphi_A^{NDAI}(h')P_h\varphi_A^{NDAI}(h).$$

So, apartment a is acceptable for both households.

Consequently

$$h\in H_a^{it} \text{ and } h'\in H_a^{it'}$$

for some Steps t, t' during the NDA phase at Stage $x^*.1$. Since there are not interrupters at iteration x^* and $a\in\theta^{NDAI}(i)$, we have that

$$a\in Ch_i^s(A_i^s,\succ_A^i) \text{ for all } s\geq\min\{t,t'\}.$$

Moreover, $h'=\max_{\succ_a^i}H_a^{is}$ for all $s\geq t'$ because $a=\varphi_A^{NDAI}(h')$. We analyze the following cases:

Case I. Consider $t'\leq t$. We know that $h'=\max_{\succ_a^i}H_a^{is}$ for all $s\geq t'$ and $h\in H_a^{it}$. Then h is not assigned to a because $h'\succ_a^i h$.

Case II. Suppose that $t<t'$. Since $h\in H_a^{it}$ and $a\in\theta(i)$, we have two possibilities: household h is the household in H_a^{it} with the highest priority under $\succ_a^i$ or not. In any case, institution i assigns the apartment a to some household h'' at Step t. By definition, $h''=\max_{\succ_a^i}H_a^{it}$, then $h''\succeq_a^i h$. However, $h'=\max_{\succ_a^i}H_a^{it'}$ at Step t' and $\varphi(h')=(a,i)$. That is to say, h'' is displaced from a by an household with a higher priority under $\succ_a^i$. By transitivity, we conclude that $h'\succ_a^i h$.

In any case, $h' \succ_a^i h$. Therefore households does not have justified envy under μ^{NDAI}. So, the assignment μ^{NDAI} is fair for households of the the same type.

Pareto Undominated. Now, we prove that there is no fair for households the same type assignment that Pareto dominates assignment μ^{NDAI}.

We proceed by contradiction. Let μ be a fair for households of the same type assignment that Pareto dominates the assignment μ^{NDAI}. Thus, $\varphi_A(h) = \varphi_A^{NDAI}(h)$ for all $h \in H$, but exists at least one household h^* such that strictly improves under φ, that is to say $\varphi_A(h^*) P_{h^*} \varphi_A^{NDAI}(h^*)$.

Now, let h_α be the household that is assigned to the apartment α under the NDAI algorithm. So, $\varphi_A^{NDAI}(h_\alpha) = \alpha$ for all $\alpha \in A$. Particularly, we have that

$$\varphi_A^{NDAI}(h_a) = a = \varphi_A(h^*)$$

Moreover, denote by $\varphi_I(h^*) = i$, $\varphi_I^{NDAI}(h^*) = i^*$, $\varphi_I(h_a) = j$ and $\varphi_I^{NDAI}(h_a) = j^*$, the corresponding institution that assigns an apartment to the households h, h_a.

We define and denote the set of households that improve (and remain equal) under the assignment μ as follows:

$$H^+ = \{h \in H : \varphi_A(h) P_h \varphi_A^{NDAI}(h)\} \text{ and } H^= = \{h \in H : \varphi_A(h) = \varphi_A^{NDAI}(h)\}.$$

It is easy to note that $H^+ \cap H^= = \varnothing$ and $H^+ \cap H^= = H$.

In the spirit of the decomposition lemma, consider $h \in H^+$ and $b = \varphi_A(h)$. We know that exists the household h_b because μ^{NDAI} is non-wasteful and respects the mixité condition. Since μ is a fair for households of the same type assignment, we have that $\varphi_A(h_b) \neq \varphi_A^{NDAI}(h_b)$. So, $h_b \notin H^=$, this implies that h_b must belong to H^+. In words, the assignment μ re-allocates all the apartments in $\varphi_A^{NDAI}(H^+)$ between the elements in H^+.

We know that $h^* \in H^*$. Then there exists household $h_a \in H^*$ such that we have the following cycle:

$$a P_{h^*} \varphi_A^{NDAI}(h^*),\ \varphi_A^{NDAI}(h^*) P_{h_a} a = \varphi_A^{NDAI}(h_a) \text{ and } \varphi_A(h_a) = \varphi_A^{NDAI}(h^*) = a^*.$$

This cycle arises during the last iteration of the NDAI algorithm because there exist households h'_a and h'_β that are tentatively assigned to (a, j') and $(\varphi_A^{NDAI}(h^*), i')$ at Steps t_a, t_β, respectively, but are rejected in later steps. We analyse the next cases

Case I. Suppose that $i = i^* = j = j^*$. Since μ is fair for households of the same type, previous cycle implies that assignment μ^{NDAI} is not fair for households of the same type.

Case II. Otherwise. Without loss of generality, consider that $t_a < t_\beta$. We conclude that institution j' is a type 1 interrupter of (h^*, a, i). This is a contradiction because there is no interrupters at iteration x^*. Analogously for larger cycles.

In any case we get a contradiction by assuming that μ^{NDAI} is Pareto dominated. □

To prove the strategy-proofness, we need the following concepts.

Definition 20. Consider that R_h is the true preference list of each household h. A **dropping strategy** is a preference list R'_h such that

1. R'_h preserves the order of the true preference list. Thus, $aR'_h b \implies aR_h b$, and

2. All unacceptable apartments under R_h are also unacceptable under R'_h. So, $\varnothing R_h a \Longrightarrow \varnothing R'_h a$, for all a, $b \in A$.

For each apartment a, let R^a_h be the dropping strategy where all the apartments preferred to a under R_h are declared unacceptable, i.e. if $bP_h a$ then $\varnothing R^a_h b$. As Kojima and Pathak (2009), we have that dropping strategies are exhaustive, that is to say, every possible apartment for a household h can be gotten through a dropping strategy, when other preference list remain unchanged.

Lemma 4.5.1. *(Dropping strategies are exhaustive) Fix a household h. Suppose that h reports $\bar{R}_h$ and other households preference profile is $\bar{R}_{-h}$. Also, consider that $\varphi[\bar{R}](h) = (a,i)$. Then $\varphi_A[R^a_h \bar{R}_{-h}](h) = a = \varphi_A[\bar{R}](h)$, for all $\bar{R}_h \in \Re_h$.*

Proof. We proceed by contradiction, suppose that $\varphi_a[R^a_h, \bar{R}_{-h}](h) \neq a$. Consider x^* the last iteration of the mechanism with the profile $(R^a_h, \bar{R}_{-h})$. By assumption, household h was rejected from a at some Step $t \geq 1$ during the NDA phase at Stage $x^*.1$. Consequently, there exists a household h' such that $\varphi[R^a_h, \bar{R}_{-h}](h') = (a, j)$. We have the following cases:

Case I. If $j = i$, the allocation $\varphi[R^a_h, \bar{R}_{-h}](h') = (a, j)$ implies that $h' \succ^i_a h$. On the other hand, we know that each household of type i makes an offer to apartment a during the last iteration because there is no interrupter institutions. Since $\varphi_a[\bar{R}](h) = a$, we conclude that $h \succ_a^i h'$. We get a contradiction.

Case II. If $j \neq i$, we know that h was rejected from a at some Step $t_h^a \geq 1$, then $j\pi_a i$ because $\varphi[R_h^a, \bar{R}_{-h}](h') = (a, j)$ and $a \in \theta[\bar{R}](i)$, institution i prefers assign a by the mixité condition. In other words, is not possible that i does not assign a to h by quota restrictions. On the other hand, the output of the $NDAI[R_h^a, \bar{R}_{-h}]$ procedure also satisfies the mixité condition. Consequently there exists an apartment b such that

$$b \neq \theta[R_h^a, \bar{R}_{-h}](j) \text{ and } b \in \theta[\bar{R}](j),$$

in words, the apartment a displaced the apartment b from institution j. Moreover, other households do not change their preference lists, then $a, b \in A_j^t$ and $\#A_j^t > q_j$ at some Step t during the last NDA phase under the profile $(R_h^a, \bar{R}_{-h})$. We have that $a \succ_A^j b$ because $\varphi[R_h^a, \bar{R}_{-h}](h') = (a, j)$. However, $a \notin \theta[\bar{R}](j)$ although $h' \in H_a^{jt_{h'}}$ at some step $t_{h'}$. Since there is no interrupter institutions at last $NDA[\bar{R}]$ iteration, we conclude that $i\pi_a j$. This is a contradiction with $j\pi_a i$.

In any case we get a contradiction by assuming that $\varphi_a[R_h^a, \bar{R}_{-h}](h) \neq a$.

Strategy Proofness. Let $R'_{-h} \in \Re_{-h}$. and $\varphi_A^{NDAI}[R_h, R'_{-h}](h) = a_{s_h}$ Since dropping strategies are exhaustive, we will show that h cannot get an apartment with a higher rank that a_{s_h}, i.e., there is no strategy R'_h such that $\varphi_A^{NDAI}[R'_h, R'_{-h}](h) = a_{r_h}$ for all $r = 1, 2, \ldots, s-1$. We proceed by induction over s, the rank of $\varphi_A^{NDAI}[R_h, R'_{-h}](h)$.

Induction Base. For $s = 1$, we have that $\varphi_A^{NDAI}[R_h, R'_{-h}](h) = a_{s_h}$ is the most preferred apartment of h. Therefore, h cannot get a better apartment.

If $s = 2$, during the NDA phase at Stage $x^*.1$ we have that household h was rejected from her most preferred apartment because. 1) there exists household h' such that $\varphi^{NDAI}(h') = (a_{1_h}, i)$ and $h' \succ_a^i h$, 2) apartment a_{1_h} is assigned by an institution $j\pi_{a_{1_h}} i$, for all $i \in \tau(h)$ or 3) (h, a_{1_h}, i) is a type 1 interrupter triad for some $i \in \tau(h)$.Therefore, h cannot get her most preferred apartment.

If $s = 3$, it is clear that h cannot get her most preferred apartments by the same reasons exposed in previous case.. Now, we show that apartment a_{2_h} is not possible for h when R'_{-h} is fixed. We proceed by contradiction, suppose that a_{2_h} is possible. First, we know that $\varphi_A^{NDAI}[R_h, R'_{-h}](h) \neq a_{2_h}$, there exists h' such that $\varphi_A^{NDAI}[R_h, R'_{-h}](h') = a_{2_h}$. This implies that h' does not get a_{1_h} by one of the same reasons that h. Since in the last iteration there is no interrupter institutions and other households report the same preference list, we have that $h \in H_{a_{2_h}}^{it}$, $h' \in H_{a_{2_h}}^{jt'}$ at some Steps t, t' respectively. Then a_{2_h} is assigned to h' because: $j\pi_{a_{2_h}} i$ or $h' \succ_{a_{2_h}}^{i} h$.

On the other hand, if a_{2_h} is possible for h, the exhaustiveness of dropping strategies implies that $\varphi_A^{NDAI}[R_h^{a_{2_h}}, R'_{-h}](h) = a_{2_h}$. Hence, there exist Steps t, t' at Stage $x^*.1$ of the $NDA[R_h^{a_{2_h}}, R'_{-h}]$ such that $h \in H_{a_{2_h}}^{it}$, $h' \in H_{a_{2_h}}^{jt'}$. Then a_{2_h} is assigned to h because: $i\pi_{a_{2_h}} j$ or $h \succ_{a_{2_h}}^{i} h'$. We get a contradiction in any case.

Hypothesis of Induction. If $\varphi_A^{NDAI}[R_h, R'_{-h}](h) = a_{n_h}$, then apartments a_{r_h} are not possible for h, for all $r \in \{1, 2, \ldots, n-1\}$.

Induction Step. For $s = n+1$, we have to prove that apartments a_{r_h} are not possible for h, for all $r \in \{1, 2, \ldots, n\}$. We proceed by contradiction, i.e., suppose that apartment $a_{r_h^*}$ is possible for some $r^* \in \{1, 2, \ldots, n\}$. By the induction hypothesis, we have that apartments a_{r_h} are not possible for h for all $r \in \{1, 2, ..., r^* - 1\}$. Analogously to the case $s = 3$, we get a contradiction because $\varphi_A^{NDAI}[R_h, R'_{-h}](h) \neq a_{r_h^*}$.

Then, household h cannot get an apartment preferred to $\varphi_A[R_h, R'_{-h}](h)$, for all fixed profile R'_{-h}. Therefore, the NDAI mechanism is strategy-proof.

Suppose that an assignment that satisfies mixité sociale does not exist. During the last iteration of the NDAI, we have that $\#Ch_i^t < q_i$ for all step t of the NDA phase. Consequently, the assignment μ^{NDAI} is fair for the same type because in the last step there are not interrupters, and each institution assigns an apartment according to $\{Ch_i^t\}_{t\mathbb{N}}$.

Suppose that mixité sociale and fairness for the same type are not compatible We proceed analogously to the proof of mixité sociale. Justify envy arises because the algorithm is forced by type 2 interrupters to reach mixité sociale even if the household in the interrupting triad does not get the same apartment. Moreover, if we assume that μ^{NDAI} is Pareto dominated by a mixité assignment, then we find interrupters during the last iteration of the NDAI algorithm.

Therefore, the interrupters of the NDAI algorithm can determine the compatibility between fairness for the same type and mixité sociale. $\square$

Previous theorem establishes that the assignment μ^{NDAI} is fair for the same type and satisfies the mixité condition in markets where these two concepts are compatible. However, we can not generalize the existence of such assignments for all markets. Even more, there are markets where an assignment that satisfies mixité does not exist. The following section focuses on these problems.

4.6 Existence issues

It is important to recall that a fair for the same type assignment that respects the mixité condition does not always exists.

Example 4.6.1. Consider a subsidized market such that $H = \{h_1, h_2\}$, $I = \{i_1, i_2\}$ and $A = \{a_1, a_2\}$. The type function is described by $\tau^{-1}(i_1) = \tau^{-1}(i_2) = \{h_1, h_2\}$. Households preferences and institutions priorities are

$$P = \begin{pmatrix} P_{h_1} & P_{h_2} \\ a_2 & a_2 \\ a_1 & a_1 \end{pmatrix} \text{ and } \succ = \begin{pmatrix} \succ^{i_1}_{A} & \succ^{i_1}_{a_1} & \succ^{i_1}_{a_2} & \succ^{i_2}_{A} & \succ^{i_2}_{a_1} & \succ^{i_2}_{a_2} \\ a_2 & h_1 & h_1 & a_2 & h_1 & h_1 \\ a_1 & h_2 & & a_1 & h_2 & \end{pmatrix}.$$

The priorities of apartments over institutions are $\pi_{a_1} = \pi_{a_2} : i_1, i_2$.

Note that the only individually rational assignments that satisfies the mixité condition are

$$\mu = \begin{pmatrix} h_1 & h_2 \\ a_1 & a_2 \\ i_1 & i_2 \end{pmatrix} \text{ and } \mu' = \begin{pmatrix} h_1 & h_2 \\ a_1 & a_2 \\ i_2 & i_1 \end{pmatrix}.$$

In both cases, we have that

$$a_2 P_{h_1} a_1 \text{ and } h_1 \succ^{i_j}_{a_2} h_2, \text{ for all } j = 1, 2.$$

So, household h_1 has justify envy over the household of the same type h_2.

4.6.1 Analysis of Mixité

Mixité sociale is not guaranteed

An assignment that respects mixité does not always exists.

Example 4.6.2. Let $H = \{h_1, h_2, h_3\}$, $A = \{a_1, a_2, a_3\}$ and $I = \{1, 2\}$. The type function is described as follows $\tau^{-1}(1) = \{h_1, h_2, h_3\}$ and $\tau^{-1}(2) = \{h_1\}$. The quotas are $q^1 = 2$ and $q^2 = 1$. The institutions priorities are

$$\succ_1 = \begin{pmatrix} \succ_A^1 & \succ_{a_1}^1 & \succ_{a_2}^1 & \succ_{a_3}^1 \\ a_1 & h_1 & h_1 & h_1 \\ a_2 & h_2 & & \\ a_3 & h_3 & & \end{pmatrix} \text{ and } \succ_2 \begin{pmatrix} \succ_A^1 & \succ_{a_1}^1 & \succ_{a_2}^1 & \succ_{a_3}^1 \\ a_1 & h_1 & h_1 & h_1 \\ a_2 & & & \\ a_3 & & & \end{pmatrix}.$$

Since h_1 is the unique household of type 2, in order to satisfy the mixité condition, we have that $\varphi_I(h_1)$ for all assignment μ. So, h_2, h_3 must be assigned through institution 1. This condition generates a non-individually rational assignment because $\varnothing \succ_{a_j} h_2, h_3$ for all $j \in \{1, 2\}$. Therefore, in this market it does not exist an IR assignment that respects the mixité condition.

The NDAI and the existence of mixité sociale

The NDAI algorithm is also useful to identity the compatibility between fairness for the same type and mixité sociale. According to Theorem 4.5.1 we have that

1. Mixité sociale and fairness for the same type are compatible if during the last iteration of the NDAI algorithm there is no interrupters,
2. Suppose that a mixité sociale assignment exists. Fairness for the same type and mixité sociale are not compatible if during the last iteration of the NDAI there are type 2 interrupters, and
3. An assignment that satisfies mixité sociale does not exist if some institution never reaches its quota in the last iteration.

A condition for the existence of mixité sociale

A **hyper-graph** $\hat{Y}$ is an ordered pair $\hat{Y} = (V[\hat{Y}], HE[\hat{Y}])$ where $V[\hat{Y}]$ is the set of nodes; and $HE[\hat{Y}]$ is the set of hyper-edges $\hat{e} \subseteq V[\hat{Y}]$.

Considering a subsidized housing market $\aleph = (I, H, \tau, D, Q, A, \delta, P, \succ, \pi)$, let $H_a^i = \{h \in H \mid aP_h\varnothing$ and $i \in$

$\tau(h)\}$ be the households of type i that prefers the apartment a to remain unassigned. With this information, we can infer a demand matrix.

Definition 21. A **demand matrix** is a matrix $D = (\Delta_{\alpha\beta})$ such that $\Delta_{\alpha\beta} = H^{\alpha}_{\beta}$ where $\alpha \in I$ and $\beta \in A$.

Since $I \cap A = \varnothing$ and institutions are not interested in other institutions, the demand matrix is in fact the matrix of adjacency[3] of a bipartite hyper-graph. Formally, the **bipartite hyper-graph induced by $\aleph$** is a hyper-graph $Y = (V[Y], HE[Y])$ where the set of nodes is $V[Y] = I \cup A$, and the set of hyper-edges $HE[Y]$ is a subset of $\cup_{i\in I} 2^{\{i\}\times A}$. We say that $\hat{e} \in HE[Y]$ is a hyper-edge if it satisfies the following two conditions:

1. $\hat{e} \subseteq \{i\} \times A$, so $|\hat{e} \cap I| = 1$, and
2. $H^i_a \neq \varnothing$ for all $(i,a) \in \hat{e}$.

Condition (1) says that hyper-edges must not contain more than one institution. Condition (2) is about the relation between an institution and an apartment; that is to say, the institution i is interested in the apartment a, both nodes in the same hyper-edge, if some household of type i demands the apartment a.

A subset $C \subseteq HE[Y]$ is a **matching** if no pair of hyper-edges in C has nodes in common. A node $v \in V[Y]$ is **covered** by the matching C if v is element of a pair in some hyper-edge of C. An institution i fills its quota whenever it is covered by a hyper-edge $\hat{e}$ with cardinality q_i. A matching is **perfect** if it covers all the nodes in the bipartite hyper-graph Y. Consider $S \subseteq V[Y]$, the **neighbourhood** of S is the set $N_Y(S) = \{v \in V[Y] \mid \text{if there exist } \hat{e} \in HE[Y] \text{ that covers } v \text{ and an element } s \in S, s \neq v\}$. We say that x is a **neighbour of** y if there exists a hyper-edge $\hat{e}$ that covers x and y. Since we do not consider the existence of loops[4], a single node is not its own neighbour. Therefore, S is not necessarily a subset of it neighbourhood.

Example 4.6.3. Consider the hyper-graph Y with nodes $V[Y] = \{i, j\} \cup \{a_1, a_2, a_3, a_4, a_5\}$ and hyper-edges $HE[Y] = \{\hat{e}_1, \hat{e}_2\}$ where $\hat{e}_1 = \{(i,a_1),(i,a_2),(i,a_3)\}$ and $\hat{e}_2 = \{(j,a_4),(j,a_5)\}$. Now, let $S_1 =$

[3] Consider x, y nodes in $V[\hat{Y}]$, the **matrix of adjacency** is a matrix (a_{xy}) such that $a_{xy} \neq \varnothing$ if and only if there is a hyper-edge that contains x and y.

[4] For a node $v \in V[Y]$, a **loop** is the hyper-edge $\hat{e} = \{(v,v)\}$.

$\{i,a_2\}$ and $S_2=\{i,j\}$. The corresponding neighbourhoods are

$$N_Y(S_1)=\{i,a_1,a_2,a_3\} \text{ and } N_Y[S_2]=\{a_1,a_2,\dots,a_5\}.$$

Note that $S_1 \subseteq N_Y(S_1)$ and $S_2 \nsubseteq N_Y(S_2)$.

The following example shows the construction of a bipartite hyper-graph.

Example 4.6.4. Consider the market $H=\{h_1,h_2,h_3,h_4,h_5\}$, $I=\{1,2,3\}$ and $A=\{a_1,a_2,a_3,a_4\}$. The type function is described by $\tau^{-1}(1)=\{h_1,h_2\}$, $\tau^{-1}(2)=\{h_3,h_4\}$ and $\tau^{-1}(3)=\{h_3,h_5\}$. Households preferences are the following

$$P=\begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} & P_{h_4} & P_{h_5} \\ a_1 & a_2 & a_4 & a_3 & a_1 \\ a_2 & a_3 & a_2 & a_4 & a_3 \\ a_3 & a_4 & a_1 & a_1 & a_4 \end{pmatrix}.$$

The corresponding demand matrix is:

$$D=\begin{pmatrix} & a_1 & a_2 & a_3 & a_4 \\ 1 & h_1 & h_1,h_2 & h_1,h_2 & h_2 \\ 2 & h_3,h_4 & h_2 & h_3 & h_3,h_4 \\ 3 & h_3,h_5 & \varnothing & h_3,h_5 & h_3,h_5 \end{pmatrix}.$$

Consequently, an hyper-graph is $V=I\cup A$ and $HE=\{\hat{e}_1,\hat{e}_2,\hat{e}_3,\hat{e}_4\}$ where

$$\begin{aligned} \hat{e}_1 &= \{(1,a_1),(1,a_4)\} \\ \hat{e}_2 &= \{(1,a_1),(1,a_2),(1,a_3),(1,a_4)\} \\ \hat{e}_3 &= \{(2,a_1),(2,a_2),(2,a_4)\} \\ \hat{e}_4 &= \{(3,a_1),(3,a_3)\}. \end{aligned}$$

Previous hyper-graph is represented in the Figure 4.1.

Observation 4.6.1. In the previous example, all hyper-edges have a_1 as a common node and only a_1. Therefore there is no matching with cardinality more than 1. So, each hyper-edge is a matching.

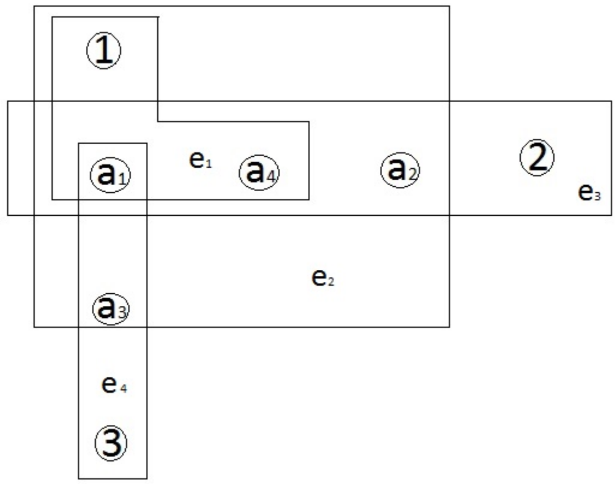

Figure 4.1: The institutions and apartments are represented by a node. The hyper-edges are illustrated as a box with its corresponding nodes.

Now, finding an assignment that satisfies the mixité condition is equivalent to finding a matching 1. that covers all the nodes in $V[Y] \cap I$, and 2. such that the cardinality of the hyper-edge $\hat{e}$ that covers each i is equal to $q_i + 1$. Our objective is finding a condition that guarantees the existence of a matching with cardinality I.

Recall that a bipartite hyper-graph includes hyper-edges of all cardinalities. Then, perfect matchings in a bipartite hypergraph cover all the institutions, but cannot satisfy the mixité condition. In order to avoid this problem, we focus our attention on a specific bipartite hyper-graph.

Definition 22. Considering a subsidized housing problem $(I, H, \tau, D, Q, A, \delta, P, \succ, \pi)$, a **subsidized housing hyper-graph** Δ is an ordered pair $(V[\Delta], E[\Delta])$ such that $V[\Delta] = I \cup A$ is the set of nodes, and the set of hyper-edges is $E[\Delta] = \{\hat{e} \subseteq \cup_{i \in I} 2^{\{i\} \times A} \mid \#(\hat{e} \cap A) = q_i \text{ if } i \in \hat{e}\}$.

Remark 1. The subsidized hyper-graph does not always exist. For example, consider a market with only one institution, three households and two apartments. If an apartment is not acceptable for all the households, all the hyper-edges in the induced hyper-graph cover the institution and the acceptable apartment. So, there are no hyper-edges such that $\#(\hat{e} \cap A) = 2$, that is to say, the subsidized hyper-graph does not exist. Moreover, the mixité condition is never reached.

We need some condition over the size of the market to guarantee the existence of hyper-edges that cover the quota of each institution.

A subsidized housing market satisfies the **hyper-edge condition** if for all apartments $b \neq a$, there exists a household $h_a^i \in H_a^i$ such that $h_a^i \notin H_b^i$.

Proposition 4.6.1. *If a subsidized housing market satisfies the hyper-edge condition, then the subsidized hyper-graph exists.*

Proof. For each institution i, we have to prove the existence of at least one hyper-edge $\hat{e}$ such that $i \in \hat{e}$ and $\#(\hat{e} \cap A) = q_i$.
Consider an apartment a and an institution i. Since the market satisfies the hyper-edge condition, there exists a household $h_a^i \in H_a^i$ such that $h_a^i \notin H_b^i$ for all $b \neq a$. In other words, $H_a^i \cap H_b^i = \varnothing$ for all $a \neq b$, there exists $h_a \in H_a^i$ such that $h_a \in H_a^i$ and $h_a \notin H_b^i$ for all apartment $b \neq a$. Moreover, it is clear that $H_a^i \neq \varnothing$. Consequently, all the apartments are demanded by the institution i. Therefore, there exist $\frac{|A|!}{(|A|-q_i)!q_!}$ hyper-edges $\hat{e}$ that covers institution i and $\#(\hat{e} \cap A) = q_i$. That is to say, the subsidized hyper-graph exists. $\square$

For simple graphs, the Hall's theorem[5] establishes a necessary and sufficient condition to guarantee the existence of a perfect matching in simple bipartite graphs. The generalization of the Hall's Theorem for hyper-graphs remains as an open question. However, there exist sufficient, but not necessary, conditions that determine the existence of a perfect matching in a hyper-graph.

A **transversal** is a subset $T \subseteq V$ with the property that $E \cap T \neq \varnothing$ for all $E \subseteq HE$. We denote by $\eta[Y]$ the maximum cardinality of a matching in Y, and $\tau[Y]$ the maximum cardinality of a transversal of Y.

Remark 2. Let $r \geq 2$. A bipartite hyper-graph satisfies the **Haxell's condition** if $|\hat{e} \cap I| = 1$ and $|\hat{e} \cap A| \leq r - 1$ for all $\hat{e} \in HE$.

[5]**Hall's Theorem**. Consider a bipartite hyper-graph Y such that $|\hat{e}| = 1$ for all $\hat{e} \in E[Y]$. A perfect matching exists if and only if $\#S \leq N_Y(S)$ for all $S \subseteq N[Y]$.

It is important to note that subsidized hyper-graphs satisfies the Haxell's condition by definition: an hyper-edge $\hat{e}$ is a subset of $\{i\} \times A$ for some $i \in I$ and its cardinality is equal to q_i. So, $|\hat{e} \cap I| = 1$ and $|\hat{e} \cap A| \leq q^* + 1 - 1$ for all $\hat{e} \in HE$, where q^* is the highest quota.

Theorem 4.6.1. *If $Y = (V, HE)$ is a subsidized hyper-graph that satisfies the hyper-edge condition and $\tau(Y_C) > (2r-3)(|C|-1)$ for every $C \subseteq I$, then a perfect matching exists.*

Proof. This is the application of Theorem 3 in Haxell (1995) for subsidized hyper-graphs. □

Previous theorem ensures the existence of a permutation of apartments such that each institution i is assigned q_i apartments. However, we have not solved the subsidized housing problem. We need to ensure the existence of enough households to be assigned one apartment from one and only one institution.

A subsidized market satisfies the **labelled condition** if there exists a household $h_a^i \in H_a^i$ such that $h_a^i \notin H_b^j$ for all apartments $b \in A$ and $j \neq i$.

Therefore, to guarantee the existence of at least one assignment that satisfies mixité, a subsidized housing market must satisfy the following condition.

A subsidized housing market satisfies **Mixité Existence Condition** if there exists $h_a^i \in H_a^i$ such that

1. $h_a^i \notin H_b^i$ for all apartment $b \neq a$.,
2. $h_a^i \notin H_a^j$ for all $a \in A$ and $j \in I \setminus \{i\}$, and
3. $\tau(Y_C) > (2q^* - 3)(|C| - 1)$ for every $C \subseteq I$, where Y in the correspondent subsidized hyper-graph.

The subsidized housing market described in Example 4.3.1 does not satisfy the Mixité Existence Condition because $\tau^1(1) = \{h_2\}$ and $h_2 \in \tau^{-1}(1)$. That is to say $H_a^1 \cap H_a^2 \neq \varnothing$.

4.6.2 Analysis of Fairness

We can ensure the existence of at least one fair for the same type assignment.

Proposition 4.6.2. *There always exist at least one fair for the same type assignment.*

Proof. Randomly, we can choose a matching θ between institutions and apartments such that no apartment remains unassigned. Then, as in a School Choice problem, we run the following Deferred Acceptance algorithm:

For each household h, consider $A_h = \{a \in A : \theta^{-1}(a) \in \tau(h)\}$.

Step 1. Each household demands her most preferred apartment in A_h. Among the households that demand each apartment a, the pairs $(\theta^{-1}(a), a)$ tentatively accept the highest priority household under $\succ_a^{\theta^{-1}(a)}$. The pair rejects the other households.

Step t. Each household that was rejected in the previous step demands her most preferred apartment in A_h that has not rejected her. Among the households that demand each apartment a and the households tentatively accepted in the previous step, the pairs $(\theta^{-1}(a), a)$ tentatively accept the highest priority household under $\succ_a^{\theta^{-1}(a)}$. The pair rejects the other households.

The assignment produced by previous procedure in denote by μ^{SDA}. For some household h, suppose the existence of a household h' such that

$$\varphi_A(h') P_h \varphi_A(h) \text{ and } \varphi_I(h') \in \tau(h).$$

By construction of the algorithm, the households h and h' demand the apartment $\varphi_A(h')$ in some steps t, $t' \in \mathbb{N}$. Without loss of generality, we assume that $t < t'$. Consequently, the institution $\theta^{-1}(a)$ assigns the apartment a to the highest household under $\succ_a^{\theta^{-1}(a)}$. Thus, we conclude that

$$h' \succ_a^{\theta^{-1}} h.$$

Therefore, μ^{SDA} is a fair for the same type assignment. $\square$

The existence of interrupters induces loss in fairness.

Proposition 4.6.3. *If there are not interrupters, the μ^{NDAI} is fair.*

Proof. We proceed by contradiction. Consider the existence of a blocking pair triad (h,a,i) such that

1. $aP_h\varphi_A^{NDA}(h)$,
2. $a \succsim_A^i b$, form some $b \in \theta^{NDA}(i)$,
3. $i\pi_a(\theta^{NDA})^{-1}(a)$, or $i = (\theta^{NDA})^{-1}(a)$,
4. $h \succ_a^i \varnothing$, or $h \succ_a^i \varphi^{-1}(a,i)$.

Case I. If $a = b$, we have that $a \in \theta(i)$. By construction of the NDAI, we know that

$$(\varphi^t(a,i))^{-1} = \max_{\succ_a^i} H_a^{it}$$

for all $t \in \{1,\ldots,t_{last}\}$. Moreover, the apartment a is acceptable for household h and $i \in \tau(h)$, so there exists step t^* such that $h \in H_a^{it^*}$. Since $\varphi(h) \neq (i,a)$, we conclude that

$$\varphi^{-1}(a,i) \succ_a^i h.$$

Since there is no Type 1 interrupters, previous paragraph induces a contradiction because $h \succ_a^i \varphi^{-1}(a,i)$.

Case II. If $a \neq b$, since $aP_h\varnothing$ and $i \in \tau(h)$, there exists step t^* such that

$$h \in H_a^{it^*} \text{ and } a \in A_i^{t*}.$$

Moreover, $a \in \theta(j)$, for some institution $j \neq i$, because $a \notin \theta(i)$. By construction of the NDAI algorithm, in some step $t' \geq t^*$ we have that

$$a \in Ch_i(A_i^{t'},\succ_A^i) \cap Ch_j(A_j,\succ_A^j).$$

This implies that

$$j\pi_a i,$$

which is a contradiction because we assume that $i\pi_a\theta^{-1}(a) = j$ and there are not type 2 interrupters. In any case the contradiction arises because we assume the existence of a blocking triad. Therefore, the assignment μ^{NDAI} is fair. □

Conversely statement is not always true, that is to say, there are interrupters even if the μ^{NDAI} is fair.

Example 4.6.5. Let $H=\{h_1,h_2\}$, $I=\{1,2\}$ and $A=\{a_1,a_2\}$. Households preferences are

$$P=\begin{pmatrix} P_{h_1} & P_{h_2} \\ a_1 & a_1 \\ a_2 & a_2 \end{pmatrix}.$$

The type function is described by $\tau^{-1}(1)=\{h_1,h_2\}$ and $\tau^{-1}(2)=\{h_2\}$. The institutions quotas are $q_1=q_2=1$, and priorities institutions are

$$\succ^1=\begin{pmatrix} \succ^1_A & \succ^1_{a_1} & \succ^1_{a_2} \\ a_2 & h_1 & h_2 \\ a_1 & h_2 & h_1 \end{pmatrix} \text{ and } \succ^2=\begin{pmatrix} \succ^2_A & \succ^2_{a_1} & \succ^2_{a_2} \\ a_2 & h_2 & h_2 \\ a_1 & & \end{pmatrix}.$$

Finally, we assume the following apartments priorities $\pi_{a_1}=\pi_{a_2}:1,2$.

Running the NDAI algorithm, in first step of the NDA phase we have that

$$H^{11}_{a_1}=\{h_1,h_2\}, H^{21}_{a_1}=\{h_2\}, \text{ and } H^{11}_{a_2}=H^{21}_{a_2}=\varnothing.$$

Since $1\pi_{a_1}2$, we tentatively get the triad

$$(h_1,a_1,1).$$

Consequently, household h_2 is rejected from apartment a_1.

At second step, households that demand an apartment, according to its type, are

$$H^{12}_{a_1}=\{h_1\}, H^{22}_{a_1}=\varnothing, \text{ and } H^{12}_{a_2}=H^{21}_{a_2}=\{h_2\}.$$

Since $a_2\succ^1_A a_1$, household h_1 is rejected. The tentatively triad is

$$(h_2,a_2,1).$$

At step 3, after eliciting households demand, we have that

$$H^{13}_{a_1}=H^{23}_{a_1}=\varnothing, H^{13}_{a_2}=\{h_1,h_2\} \text{ and } H^{23}_{a_2}=\{h_2\}.$$

Since $1\pi_{a_2}2$ and $h_2 \succ^1_{a_2} h_1$, the final assignment in the first NDA stage is

$$\mu_1^{NDA} = \begin{pmatrix} h_1 & h_2 \\ \varnothing & a_2 \\ \varnothing & 1 \end{pmatrix}.$$

Previous assignment does not satisfy the mixité condition. Moreover, institution 2 can assign apartment a_2 to household h_2. Therefore, $(h_2, a_2, 1)$ is a type 2 interrupter triad. Consequently, we actualize institutions priorities

$$\succ_A^{21} = \succ_A^{20} \setminus a_2, \succ_A^{11} = \succ_A^{10}.$$

Running a second round of the NDA phase with priorities $\succ^1 = (\succ^{11}, \succ^{21})$, the final assignment is

$$\mu^{NDA2} = \begin{pmatrix} h_1 & h_2 \\ a_1 & a_2 \\ 1 & 2 \end{pmatrix} = \mu^{NDAI}.$$

Also, note that μ^{NDAI} is fair and satisfies the mixité condition, even in the presence of type 2 interrupters.

4.7 Concluding Remarks

We model the subsidized housing problem in Paris as a three sided market. Specifically, we assume that institutions have priorities over the total pool of subsidized apartments, and apartments have priorities over the set of institutions. However, both assumptions are not formalized in the real problem and can be not necessary. We would like to investigate if a fair for the same type assignment that respects mixité sociale exists in a simpler model that do not consider these priorities.

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