# EL COLEGIO DE MÉXICO, A.C.
# CENTRO DE ESTUDIOS ECONÓMICOS

*ASSIGNMENT PROBLEMS IN MATCHING MARKETS*

**TESIS PRESENTADA POR:**

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PARA OPTAR POR EL GRADO DE

**DOCTOR EN ECONOMÍA**

PROMOCIÓN 2010-2013

DIRECTOR DE TESIS

DR. DAVID CANTALA

MÉXICO D.F.            JUNIO 2013

# CENTRO DE ESTUDIOS ECONÓMICOS

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México, D.F., 28 de junio de 2013

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**Agradecimientos**

A lo largo de mis estudios doctorales conté con el apoyo de personas e instituciones que hicieron posible la culminación de este proceso. Mi sincero agradecimiento para todos ellos.

En primer lugar, quiero agradecer muy especialmente a David Cantala, mi director de tesis. Gracias a él comencé a interesarme, ya desde el final de la maestría, en los temas que luego desarrollé durante el doctorado. Su dedicación durante todo el proceso de elaboración de este trabajo significó un constante estímulo para encauzar mis intereses académicos, compartiendo generosamente sus conocimientos a la vez que dejándome ahondar en mis inquietudes y estimulando el intercambio con distintos investigadores. Tuve además el gran privilegio de realizar uno de los capítulos de esta tesis bajo la supervisión de Federico Echenique quien además de compartirme sus conocimientos y guiarme en la elaboración del capítulo, apoyó ampliamente mi formación como investigador. A él le agradezco las constantes discusiones que sostuvimos durante la elaboración de todos los capítulos de esta tesis y su genuino interés por colaborar con mi formación durante el doctorado. Debo darle las gracias además por el respaldo en la búsqueda para continuar mi trayectoria académica.

También tuve el privilegio de contar con dos lectores agudos. En Kaniska Dam y Alexander Elbittar encontré profesores siempre ansiosos por aportar a la discusión de distintos aspectos de mi investigación. Gracias a Horacio Sobarzo quien se incorporó en las últimas instancias de este proceso.

Durante todos mis estudios de postgrado fue indispensable la ayuda que recibí de Juan Gabriel Brida, Martín Puchet, Elvio Accinelli y Edgar J. Sánchez Carrera. Sin la ayuda de todos ellos hubiera sido mucho más difícil la culminación de este proceso. En particular quiero agradecer a Gabriel por su incansable apoyo desde ya hace mucho tiempo, sus consejos, y su entrañable amistad.

Los dos primeros capítulos de esta tesis se vieron beneficiados por los excelentes comentarios de Jordi Massó. Fue fundamental su apoyo en todos estos años. Su calidez humana y generosidad son, sin duda, un ejemplo a seguir. Tuve además la suerte de poder contar con la ayuda de Andrés Sambarino y de Serafín Frache cuando las cosas se pusieron difíciles.

En lo personal y profesional, quiero agradecer a mi gran amiga Mariana Pereira, con quien tuve la suerte y alegría de compartir la maestría y el doctorado.

Mi hermano Rodrigo Puchet fue quien me sugirió hace muchos años en Uruguay pensar en continuar mis estudios en México. Desde ese momento me siento privilegiado de contar con su amistad. Con su infinita generosidad me abrió las puertas de su casa en México, gracias a lo cual pude conocer a Silvia y Martín, Andrea, Caro, Ezra y, un poco después, a Darío. Gracias a todos ellos por hacerme sentir que en México también pueden despeinarte los vientos del Sur. En particular a Martín quiero agradecerle por todo su apoyo en el plano académico y sus consejos, siempre tan oportunos.

Buenos amigos hice en estos años en México. Gracias a ellos por su cariño. Gracias a Melina, Xime, Cris, Iliana, Gabi, Sara María, Chemor, Jorge, Carlos (mi valedor), Ivo, a todos los Puchet y a todos los Dutrénit. En Manuel y Mónica descubrí dos amigos que me acompañarán de ahora en adelante. En particular, gracias Manuel por todo lo que me has enseñado en estos años.

Mis padres, mis hermanos, mi abuela, mi tío y Mili desde Uruguay siempre estuvieron a mi lado en estos cinco años. También tuve la dicha de contar con el soporte de mi "otra" familia: muchas gracias a los Espíndola y a Maqui y a Nenela. Sin ellos esto no podría ni siquiera ser imaginado.

En la base de este trabajo y de todos estos años en México está el amor de Fabiana. Además de alegrías, este trabajo ha significado muchos sacrificios de su parte. Para ella van mis más sinceras gracias por su paciencia y sostén de todos los días, en las buenas, en las malas y en las peores. Cuando el panorama no era claro, fue fundamental saberte a mi lado y escuchar tus consejos. Gracias a tu amor hoy podemos estar cerrando este ciclo, abriendo otro, y seguir disfrutando de la felicidad de estar juntos.

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# Chapter 1

# Introduction

This thesis is a collection of three essays on assignment problems in matching markets. Each essay is self-contained and can be read in any order chosen by the reader.

The problems analyzed in this thesis lie at the intersection of game theory, social choice theory, and mechanism design. Our main concern is to study the allocation of indivisibles goods (e.g., school seats, houses, hospitals) to different agents (e.g., students, buyers, doctors), in a model where each agent has preferences over goods, and each good is endowed with a priority relation over the set of agents. As we shall see, the existence of indivisible goods arises difficulties that are no present in the standard model, in which all goods are supposed to be perfectly divisible.

In the first two essays our general goal is to investigate, in two different models, the existence of allocations that satisfy some desirable properties. The first essay focuses on a variant of the well known School Choice Model, where agents may enter and exit the market in different periods. In particular, the essay examines the problem of assigning teacher positions to teachers in the Mexican public education system. The second essay studies a situation where each agent owns an indivisible good (e.g., a house) and has a certain amount of money, and seeks to sell her good to buy another. In both essays we analyze some mechanisms to find the solutions. We will concentrate on strategy-proof mechanisms; that is, mechanisms with the property that no matter the report of the other agents, an agent can do no better than reporting her preferences truthfully. In the last essay, we adopt a different point of view. We take as given the rules of the market, and we explore the reasons that may force some agents to take actions that will produce inefficient outcomes. This last analysis complements the one of the previous chapters: instead of finding an optimal design of the market, we give a theoretical explanation of why matching markets may not behave as the standard theory predicts.

The first essay inspires from a real-life assignment problem faced by the Mexican Ministry of Public Education. I introduce a dynamic school choice problem that consists in assigning positions to overlapping generations of teachers. From one period to another, teachers can either retain their current positions or choose a preferred one. In this framework, a solution concept that conciliates the fairness criteria with the individual rationality condition is introduced. It is then proved that a fair matching always exists and that it can be reached by a modified version of the deferred acceptance algorithm of Gale and Shapley. I also show that the mechanism is dynamically strategy-proof, and respects improvements whenever the set of orders is lexicographic by tenure.

The second essay studies economies with indivisible goods and budget-constrained agents with unit-demand who act as both sellers and buyers. In prior literature on the existence of competitive equilibrium, it is assumed the indispensability of money, which in turn implies that budgets constraints are irrelevant. A new condition, Money Scarcity (MS), is introduced. The condition considers agents' budget constraints and ensures the existence of equilibrium. Moreover, two mechanisms to find a competitive equilibrium are analyzed: an extended version of Gale's top trading cycles algorithm and of the exact auction mechanism of Demange, Gale and Sotomayor. Under MS, the first mechanism is strategy-proof; the second is not. The MS condition is strengthened to guarantee the existence of Nash equilibria in the revelation game induced by the second mechanism, where the outcome coincides with the assignment found by the first mechanism; moreover, this assignment is Pareto efficient.

The last essay studies unraveling in labor markets, and in matching markets in general. Unraveling is a phenomenon by which matches are made too early. They are made at a point in time when there is too little information about the quality of a match. In particular, the essay investigates the role of impatience as a factor driving unraveling in two-sided matching markets. A two-period Bayesian game where firms and workers can contract in either period is considered. In the first period qualities are unknown and this uncertainty is resolved just prior to the second period. In order to decide whether or not to make early offers, firms compare the expected utility of each decision and use their discount factors. The number

of firms that will unravel is unknown since discounts factors are agents' private information. Moreover, there is a negative externality imposed on the rest of the market by agents who make match early. As a consequence, an agent may make an early offer because she is concerned that others are making early offers. Yet other agents make early offers because they are concerned that others worry about early offers; and so on and so forth. The end result is that any given agent is more likely to make an early offer than a late offer.

# Chapter 2

# A Dynamic School Choice Model

## 2.1 Introduction

Since David Gale and Lloyd Shapley published their famous paper "College admissions and the stability of marriage" (Gale and Shapley (12)), many authors have studied assignment problems in different contexts. Therefore, there is an extensive literature on allocation problems, which primarily considers static models. In contrast, there are many real-life applications where the assignment is made in a dynamic context. Some examples are on-campus housing for college students, in which freshmen apply to move in and graduating seniors leave (Kurino (15)), kidney exchange of patients, in which each agent arrives with an object to trade (Ünver (21)), and firms with workers whose entry and exit lead to a reassignment of fixed resources (Bloch and Cantala (4)). In this paper we study a dynamic version of the well-known school choice model. Specifically, our model assigns school positions to overlapping generations of teachers. In each period, the central authority must assign positions to teachers, taking into account each school's priority ranking and the previous matching. From one period to another, teachers can either retain their current positions or choose a preferred one. Hence, the central authority faces a dynamic allocation problem.

The original motivation for this paper is an assignment problem faced by the Mexican Min-

istry of Public Education. In May 2008 the Mexican Federal Government, through the Ministry of Public Education, signed an agreement with the National Education Workers Union called "The Alliance for the Quality of Education".[1] Part of the agreement was the creation of the National Contest for the Allocation of Teaching Positions, a mechanism to assign teachers to teaching positions. As a consequence of this agreement, teachers looking for a position in the public education system are required to sit an exam. According to each teacher's grade, the central authority ranks teachers and then assigns each a teaching position. Specifically, under the mechanism used by the central authority, all open positions (that is, positions that are not already assigned) are offered to the first teacher in the ranking. Once this first ranked teacher chooses a school, remaining open positions are offered to the second teacher, and so on. Moreover, any teacher that had been previously assigned a position may choose to retain it over the new positions that are offered, but if she chooses a new one, her previous position becomes "open". In Appendix A we present an example with an application of this mechanism. It is worth noting that the algorithm is not a sort of "You-request-my-house-I-get-your-turn" (Abdulkadiroğlu and Sönmez (1)) since a position that was assigned to a teacher in a previous period, cannot be asked by another teacher until the teacher who is assigned to it, moves to another position. Thus, the central authority applies a variant of the serial dictatorship mechanism, which takes into account that some teachers are initially assigned a position. In 2012, 134,704 teachers participated in the exam in order to obtain a position.

Cantala (5) shows that the mechanism has some major flaws (see Appendix A for an illustrative example). In particular, a teacher can profit in a period after she enters the market by misrepresenting her preferences. This implies that the mechanism is not dynamically strategy-proof: it can be manipulated by teachers. Another flaw is that the mechanism does not respect improvements made by teachers (Balinski and Sönmez, (3)), that is, a teacher may increase her order in one school's priority ranking, but be assigned to a worse position. Finally, Cantala (5) shows that the mechanism is not efficient. In this paper, we study

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[1]More information is available at http://www.concursonacionalalianza.org

the described problem within a more general framework in order to cast some light on the resource allocation problem faced by the Mexican Ministry of Public Education.

A central concept in matching theory is stability: a matching is stable if there does not exist any unmatched teacher-school pair $(i, s)$ such that $i$ prefers $s$ to the school that she is assigned to and there exists a teacher assigned to $s$ who has a lower priority at $s$ than $i$. In school choice models, this concept is usually referred to as the elimination of justified envy (Abdulkadiroğlu and Sönmez (2)) and embodies a notion of fairness. In addition to elimination of justified envy, since we cannot assign a teacher to a less preferred school than the one where she is teaching, we have to address the individual rationality condition. We present a new solution concept to accommodate these concepts.

In order to define our solution concept, we consider the claims that could exist in a matching. A teacher has a claim over a school if there exists a school that she prefers over her assignment, and she has higher priority for it than one of the assigned teachers. Note that a matching eliminates the justified envy if and only if there is no claim in the matching. Moreover, we consider two kinds of claims. If the teacher in the preferred school was not assigned to it in the previous period, we say that it is a justified claim. On the contrary, if the teacher was assigned to the school in the previous period, the claim is considered inappropriate. Observe that the last type of claim is inappropriate due to the individual rationality restriction. Finally, our solution concept is as follows. We say that a matching minimizes inappropriate claims if:

- it is individually rational, non-wasteful (whenever a teacher prefers a school to her own assignment, that school already has all its positions filled), and does not have justified claims; and

- if there are inappropriate claims, the following must hold: there is no other matching that satisfies the three previous properties and one inappropriate claim is *solved* without creating a new one.

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It is worth noting that Mexican Ministry of Public Education did not propose an explicit fairness concept and, also, that the mechanism which is used by this central authority does not satisfy the last definition.

In this context, we show that within the set of matchings that minimize inappropriate claims, there is a unique matching Pareto superior to all other matchings. In order to find it, a modified version of the deferred acceptance algorithm of Gale and Shapley is introduced. Before applying the algorithm, we modify each school's priority ranking by moving teachers who had been assigned to the school in the previous period to the top of the school's priority ranking.[2] With these new orders we define the related market in which the deferred acceptance algorithm is applied.

A new dynamic version of strategy-proofness is introduced. The classic concept in static matching problems only makes reference to the benefit in one period. Our notion of strategy-proofness is dynamic in the sense that it involves not only the period when the teacher enters the market but also all the later periods while she is in the market. In our framework, teachers reveal their preferences in the period in which they enter the market. In the following periods, they cannot modify the announced preferences. We prove that if each school's priority ranking is lexicographic by tenure, that is, if teachers who were present in the previous period have priority over new teachers, then the proposed mechanism is dynamically strategy-proof. Finally, it is shown under the same condition that the mechanism also respects improvements made by teachers. Our concept of respecting improvements involves not only the period when the teacher improves her position in the ranking (like the classic notion), but also every following period.

Our model assumes that teachers' preferences are time invariant, that is, teachers reveal their preferences entering the market and do not change from one period to the other. Although the assumption of time-invariant preferences is strong in many real-life applications, we think

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[2]The idea was originally introduced by Guillen and Kesten (13). Compte and Jehiel (7) also use the same idea.

that it fits the market studied in the paper. The assumption reflects two teachers' behavioral patterns present in this market. In the first place, we know that most teachers that work in rural areas want to settle in a big city. The reason for this behavior is that residing in a big city implies better standards of living and more educational opportunities for teacher's children.[3] Thus, even if a teacher begins to work in a small town, she wants to change the school where she is assigned to in order to be closer to the capital city of the State where she works. In the second place, once a teacher is assigned to a school in a city, changes from one school to another are spare. In this sense, descriptive statistics on Mexico City presented by de Ibarrola et al. (8), support the idea that teachers' preferences are stable. In Mexico City there are 64,000 teacher positions and in each year 5,500 new positions are created. Therefore, it is possible to change from one school to another. Nevertheless, 82.3% of teachers in Mexico City had their first position in that city, 35.3% never changed the school where they started to teach and, another 37.5% changed their appointment no more than three times during their careers. Thus, there is some evidence that teachers' preferences are stable over time in the Mexican market of public school teachers, although the assumption may not be valid in other markets.[4]

All the results about the existence of a matching that minimizes inappropriate claims and the proposed mechanism can be easily extended to the case in which the time-invariant preferences assumption does not hold. The extension is straightforward since those results refer to the static problem of our model and thus, they only use the information of the period. When preferences change over time, however, our analysis over strategy-proofness and respecting improvements does not carry over.[5]

As we mentioned, the literature on matching is mostly devoted to static matching problems (see, for example, the excellent surveys of Roth and Sotomayor (19) and, Sönmez and Ünver

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[3]There are also teachers who want to go back to their home town. We are very grateful with Manuel Gil Antón and Rodolfo Ramírez for information on this issue.

[4]It is worth noting that there is no national data about teachers' preferences.

[5]See Section 2.7 for a discussion.

(20)). Recently, some articles have presented assignment problems in dynamic contexts. Kurino (15) is closest to our model. The author introduces a model of house allocation with overlapping agents and analyzes the impact of orderings on Pareto efficiency and strategy-proofness. In this sense, it is shown that under time-invariant preferences, orders that favor existing tenants perform better, in terms of Pareto efficiency and strategy-proofness, than those that favor newcomers. The concept of an order that favors existing tenants defined by this author is similar to our concept of school's priority ranking lexicographic by tenure and in relation to the mechanism proposed in this paper, it is a spot mechanism with property-rights transfer, according to Kurino's classification. Nevertheless, there are three main differences that distinguish our work from Kurino's (15). In the first place, we consider a fairness concept. We are interested in matchings that minimize inappropriate claims because each school's priority ranking should be taken into account. In the second place, since fairness is a key concept in our analysis, instead of the top trading cycles algorithm studied by Kurino (15), we focus in the deferred acceptance algorithm of Gale and Shapley. Finally, Kurino's general results about the top trading cycles mechanism are restricted to the case in which agents live two periods. Indeed, in the general case where agents stay in the market for at least three periods, the main properties of the top trading cycles spot rule favoring existing tenants are no longer valid. In contrast, our positive results are valid in the general case.

The rest of the paper is organized as follows. In Section 2, we introduce the ingredients of our model and the main concepts. Section 3 is devoted to the existence of a solution to our problem. In the next section, the proposed mechanism is introduced. Sections 5 and 6 analyze dynamics problems that arise in the model: dynamic strategy-proofness and respecting improvements properties. In Section 7, we present the conclusions and directions for future research.

## 2.2 Preliminary definitions

### 2.2.1 The Model

We consider the allocation of teaching positions to overlapping generations of teachers. Time is discrete, starts at $t = 1$, and lasts forever. In each period, there is a set of schools denoted by $S$. Each school $s \in S$ has $q_s$ positions, and in each period, some of them can already be assigned.[6] Additionally, we have the null school, denoted by $s_0$, which will be used to assign no school to teachers; we suppose that $s_0$ is not scarce. Denote by $I^t$ the set of teachers in period $t$. Note that $I^t$ changes over time because in each period some teachers may exit the market while new teachers may enter.

Another ingredient of the model is a set of strict priority orders of all teachers, denoted by $>^t \equiv \{>_s^t\}_{s \in S}$, which includes one different order for each school. When teacher $i$ has priority over $j$ to choose a position in school $s$ in period $t$, we write $i >_s^t j$. We suppose that the relative order of teachers for each school does not change over time, that is, if $i >_s^t j$ at some $t$, then $i >_s^\tau j$ for all $\tau$ such that $i, j \in I^\tau$.[7].

Each teacher has preferences defined over a single period, and the comparisons of sequences of assignments that are considered are made period by period. Formally, we suppose that preferences over path of schools are time separable.[8] Then, each teacher $i \in I^t$ has a complete and transitive preference relation over $S \cup \{s_0\}$, denoted by $\succsim_i$, and $\succ_i$ is the induced strict preference relation over the same set. Teachers reveal their preferences in the period in which they enter. In the following periods their announced preferences remain constant. Let $\Lambda$ be the domain of admissible preference relations of each teacher. A preference profile at $t$ is an element of the Cartesian product of the set of preferences of all teachers present at $t$, that is, an element of $\Lambda^{|I^t|}$; we denote by $\succ = (\succ_i)_{i \in I^t}$ a preference profile at $t$.[9]

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[6]We will use also the notation $q_i$ to refer to the number of positions of school $s_i$.

[7]As it is common in this type of model, we assume that each school's priority ranking is responsive (see Roth and Sotomayor (19) for more details)

[8]See Kurino (15) for a discussion about this assumption.

[9]Although formal notation would be $\succ_t$, to simplify it we will not use the subindex $t$.

### 2.2.2 Matchings

A **matching** at $t$ is an assignment of teachers to schools such that every teacher is assigned one school, and no school has more teachers assigned than positions, i.e., a function $\mu_t : I^t \rightarrow S \cup \{s_0\}$ such that $\left| \mu_t^{-1}(s) \right| \leq q_s$ for each $s \in S$. To indicate that teacher $i$ is matched to school $s$ in period $t$, we write $\mu_t(i) = s$. Let $\mathcal{M}_t$ be the set of all matchings in period $t$. A **submatching** is a matching with restricted domain, i.e., a function $\nu_t : J \subset I^t \rightarrow S \cup \{s_0\}$.

In the **initial period,** we have a set of teachers (denoted by $I_E^1 \subset I^1$), each of whom is initially assigned to a school. The initial assignment can be considered as a submatching in which each teacher in the set $I_E^1$ is matched to her school. Hence, we describe the initial submatching of period 1 as a function $\nu_1 : I_E^1 \rightarrow S$ such that $\nu_1(i) = s$ if and only if $i$ is initially matched to school $s$. For any period $t \geq 2$, the initial submatching, denoted by $\nu_t$, is defined by the matching of the previous period; that is, given the matching of the previous period $\mu_{t-1}$ and sets $S, I^t$, we have $\nu_t = \mu_{t-1} \mid I_E^t$ with $I_E^t = \mu_{t-1}^{-1}(S) \cap I^t$.[10] Clearly, at each $t$ we have $\left| \nu_t^{-1}(s) \right| \leq q_s$ for all $s$. Note that $I^t \setminus I_E^t$ is the set of teachers who do not hold positions and are competing to hold one.

Given a matching $\mu_{t-1}$, sets $S, I^t$, the number of positions in each school $\{q_s\}_s$, the set of strict orders $>^t = \{>_s^t\}_s$, and the preference profile at $t \succ$, **an overlapping teacher placement problem** is represented by $M^t = \langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$. Notice that the problem $M^t$ defines the initial submatching of period $t$, since $\nu_t = \mu_{t-1} \mid I_E^t$ and $I_E^t = \mu_{t-1}^{-1}(S) \cap I^t$ if $t \geq 2$ (for $t = 1$ we have $\mu_0 \equiv \nu_1$). A solution of an overlapping teacher placement problem is a matching.

A **mechanism** is a systematic procedure that assigns a matching for each problem; that is, a function $\varphi$ such that $\varphi\left(\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle\right) \in \mathcal{M}_t$, for any problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$. We will often abbreviate notation by omitting most of the arguments and we will write $\varphi\left(I^t, \succ\right)$. We believe that this abuse does not confuse and it makes the notation more manageable.

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[10] Here $\mu_{t-1} \mid I_E^t$ means the restriction of function $\mu_{t-1}$ to the set $I_E^t$.

An **economy** is defined by the set of schools $S$ and its positions $\{q_s\}_s$, an initial submatching $\nu_1$, sequences of sets $\{I^t\}_t$, preference profiles $\{\succ\}_t = \{(\succ_i)_{i \in I^t}\}_t$, strict priority orders of all teachers for each school $\{>^t\}_t$ and finally, the mechanism, denoted by $\varphi$. Note that in the context of our model, the mechanism is included in the economy because the matching in one period links this period with the one following. Specifically, the matching in one period determines the initial submatching for the next period. Therefore, the mechanism plays the role of a transition rule between periods. Finally, note that an economy defines the problem of each period.

### 2.2.3    Acceptability

In this section we define some properties that a matching should verify. We combine two traditional concepts present in the literature. On the one hand, since there are incumbent teachers in our model, we cannot assign a teacher to a less preferred school than the one where she is teaching. Therefore, a matching should satisfy the individual rationality condition, as defined in Abdulkadiroğlu and Sönmez (1). On the other hand, we must respect the strict priority order of all teachers for each school. Hence, a matching should eliminate justified envy, as defined by Abdulkadiroğlu and Sönmez (2).

Consider a period $t$ of the model. The information in that period is given by $M^t = \langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ^t, >^t \rangle$. Then, the initial submatching of period $t$ is defined by $I_E^t = \mu_{t-1}^{-1}(S) \cap I^t$ and $\nu_t = \mu_{t-1} \mid I_E^t$. In order to present the solution concept adopted for the problem $M^t$, we first define the concepts of individual rationality and non-wastefulness:

- A matching is individually rational if no teacher prefers the null school option or the school she was initially assigned to her newly assigned school.

- A matching is non-wasteful if whenever a teacher prefers a school to her own assignment, that school already has all its positions filled.

Next, we consider the claims that could exist after the matching. We say that a teacher

has a claim over a school if she prefers that school over her own assignment and if a lower ranked teacher (in the priority order) has been assigned to that school. Moreover, two kinds of claims may occur. The formal definitions are the following.

*Definition* 1. A matching $\mu_t$ is **individually rational** if:

*i)* $\mu_t(i) \succsim_i s_0$, for all $i \in I^t$,

*ii)* $\mu_t(i) \succsim_i \nu_t(i)$, for all $i \in I_E^t$.

*Definition* 2. A matching $\mu_t$ is **non-wasteful** if whenever a teacher $i \in I^t$ exists and a school $s$, such that $s \succ_i \mu_t(i)$ then $\left|\mu_t^{-1}(s)\right| = q_s$.

*Definition* 3. Given a matching $\mu_t$, teacher $i$ has a **justified claim** over school $s$ if:

*i)* $i$ prefers $s$ to her assignment: $s \succ_i \mu_t(i)$, and

*ii)* there exists a teacher $k$ assigned to $s$ such that $i$ has priority over teacher $k$ in the school's ranking, and $k$ was not assigned to $s$ in the previous period; that is: $\exists\, k \in I^t$ such that $\mu_t(k) = s$, $i >_s^t k$, and $k \notin \nu_t^{-1}(s)$.

We say that a matching **eliminates the justified claims** if there is no justified claim in the matching.

The last definition takes into account that teacher $i$ has justified envy of the assignment of teacher $k$. We define the claim as justified because the teacher who is assigned to the preferred school, in the previous period was assigned to another school and thus, she is not an incumbent teacher. Clearly, the existence of justified claims is an undesirable situation that the solution concept should prevent. Also note that if a teacher in the preferred school was assigned to it in the previous period, the claim is inappropriate, because as an incumbent she has the right to continue in that school.

*Definition* 4. Given a matching $\mu_t$ , teacher $i$ has an **inappropriate claim** over school $s$ if:

*i)* $i$ prefers $s$ to her assignment: $s \succ_i \mu_t(i)$, and

*ii)* there exists a teacher $k$ assigned to $s$, such that $i$ has priority over teacher $k$ in the school's ranking, and $k$ was assigned to $s$ in the previous period; that is: $\exists\ k \in I^t$ such that $\mu_t(k) = s,\ \ i >_s^t k$, and $k \in \nu_t^{-1}(s)$.

We say that a matching **eliminates the inappropriate claims** if there is no inappropriate claim in the matching.

Let $\Gamma(\mu_t)$ be the set of all inappropriate claims in matching $\mu_t$, that is: $\Gamma(\mu_t) = \{(i, s) \in I^t \times S,$ such that $i$ has an inappropriate claim over $s$ in $\mu_t\}$.

The usual definition of a fair matching (also called stable) implies that there are no claims in the matching (neither justified nor inappropriate). As we note in the next example, for some markets there may not exist a fair matching.

*Definition* 5. A matching is fair if it is individually rational, non-wasteful and eliminates both justified and inappropriate claims.

*Example* 1. Consider the following problem with $S = \{s_1, s_2\}, q_1 = q_2 = 1, I^t = \{i, j\}, \nu_t = \{(i, s_1)\}$, and the following preferences (from best to worst) and orders:

$$\begin{bmatrix} \succ_i & \succ_j \\ s_1 & s_1 \\ s_2 & s_2 \end{bmatrix} \qquad \begin{bmatrix} >_1^t & >_2^t \\ j & j \\ i & i \end{bmatrix}$$

The unique individually rational matching is:

$$\mu_t = \begin{pmatrix} i & j \\ s_1 & s_2 \end{pmatrix}.$$

Note that in this matching teacher $j$ has an inappropriate claim over $s_1$. Thus, there is no fair matching in this market. ∎

The new ingredient in our model is the existence of incumbents whose rights should be considered. Thus, we need to relax the definition of a fair matching in order to adapt it to our framework. The next definition enumerates the minimal desirable properties that a matching should verify in order to be a solution to the problem $M^t$.

*Definition* 6. A matching is **acceptable** if it:

*i)* is individually rational,

*ii)* is non-wasteful, and

*iii)* eliminates the justified claims.

Let $\mathcal{C}_t \subset \mathcal{M}_t$ denote the set of all acceptable matchings.

On the one hand, inappropriate claims should not be considered from a fairness perspective since they do not proceed because of the incumbents' rights. On the other hand, when an inappropriate claim is solved without creating a new one, the new matching is Pareto superior. Roughly speaking, an inappropriate claim $(i, s)$ can be settled by reallocating teacher $i$ to a school better than $s$ or by changing the teacher who is assigned to $s$ with another teacher with priority for $s$ over $i$. In this last case, since the new matching should be individually rational, the teacher who was originally assigned to $s$ should be reallocated to a better school. If these modifications are made such that no other inappropriate claim is created, then the new matching is Pareto superior to the original matching. Thus, from an efficiency point of view, is desirable a matching that minimizes the inappropriate claims. The following example points out this remark.[11]

---

[11]The link between inappropriate claims and efficiency will be clear after we prove Lemma 2. In that Lemma we prove that if a matching is Pareto superior to another matching, then the set of inappropriate claims of the first matching is included in the set of inappropriate claims of the other matching

*Example* 2. Consider the same problem of Example 1, but with the following preference for teacher $i$: $\succ_i = (s_2, s_1)$.

The following matchings are acceptable:

$$\mu_t^1 = \begin{pmatrix} i & j \\ s_1 & s_2 \end{pmatrix} \quad \text{and} \quad \mu_t^2 = \begin{pmatrix} i & j \\ s_2 & s_1 \end{pmatrix}.$$

Note that $\Gamma(\mu_t^2) = \emptyset \subsetneq \Gamma(\mu_t^1) = \{(j, s_1)\}$. Thus, $\mu_t^2$ solves the claim present in $\mu_t^1$, without creating a new one. Also note that all agents prefer their assignment in $\mu_t^2$ at least as well as their assignment in $\mu_t^1$. Therefore, $\mu_t^2$ is Pareto superior to the other matching ∎

We refine the concept of acceptability based on the situation illustrated by the last example. We introduce a new solution concept, *minimization of inappropriate claims*, which is fair-consistent: if there is a fair matching (i.e., an acceptable matching with no claims), it is selected. In those cases where there is no fair matching (which means that all acceptable matchings have at least one inappropriate claim) the criterion is to minimize the inappropriate claims. If all acceptable matchings have the same inappropriate claims, then all of them are selected. However, if there are two acceptable matchings such that the set of inappropriate claims of one matching is strictly included in the set of inappropriate claims of the other matching (as in the last example), the solution concept chooses the first matching. In this situation, the criterion implies that the selected matchings are no Pareto dominated by others matchings. The solution concept is the following.

*Definition* 7. A matching $\mu_t$ **minimizes inappropriate claims**:

*i)* if it is acceptable, and

*ii)* there is no acceptable matching $\mu_t'$ such that $\Gamma(\mu_t') \subsetneq \Gamma(\mu_t)$.

If there is an acceptable matching without inappropriate claims (that is, a fair matching) then, by the previous definition, it minimizes inappropriate claims. Also notice that the concept does not imply a utilitarian perspective.

Although minimization of inappropriate claims is also related with improving efficiency, the link is not straightforward. Indeed, a matching that minimizes inappropriate claims could be Pareto dominated by another matching with the same property: if there are no incumbents in the market, all acceptable matchings minimize inappropriate claims, but some of these matchings are Pareto dominated by others. In fact, as we will prove in the next Section, within the set of matchings that minimize inappropriate claims, there is a unique matching Pareto superior to all other matchings.

## 2.3   Existence

In order to prove the existence of a matching that minimizes inappropriate claims, we introduce the concept of related market. We want to apply the deferred acceptance (DA) algorithm of Gale and Shapley to obtain an individually rational matching. With that purpose, we modify each school's priority ranking. In each new priority ranking, we have two groups of teachers. The first group in the new ranking is the set of teachers who had been assigned to the school in the previous period, and the second is the remaining teachers. Within each group, the order is defined by the original ranking $>_s^t$ . With these new orders, we define the related market in which the DA algorithm is applied. By Ergin (11) Proposition 1, we know that the outcome of the DA algorithm adapts to the order structure: there is no teacher such that there is a school that she prefers over her assignment, and she has priority for it over one of the assigned teachers. Next, we prove that the DA outcome is an acceptable matching in the original market. Finally, since the set $\mathcal{C}_t$ is finite and not empty, we choose one acceptable matching with the fewer number of claims; then we find a matching that minimizes inappropriate claims.

*Definition* 8. Let $M^t = \langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$ be an overlapping teacher placement problem. For each school $s \in S$ with priority ranking $>_s^t$, let's define the following order of all teachers, denoted by $O_s^t(M^t)$, in the following way: for each pair $i, j \in I^t$ and $s \in S$, if

1. $i, j \in \nu_t^{-1}(s)$ the order is defined by $>_s^t$, that is $i \ O_s^t(M^t) \ j \Leftrightarrow i >_s^t j$,

2. $i \in \nu_t^{-1}(s)$ and $j \notin \nu_t^{-1}(s)$, then $i \ O_s^t(M^t) \ j$, and

3. $i, j \in I^t \backslash \nu_t^{-1}(s)$ the order is defined by $>_s^t$, that is $i \ O_s^t(M^t) j \Leftrightarrow i >_s^t j$.

Let $O^t(M^t) = \{O_s^t(M^t)\}_{s \in S}$ be the set of all such orders indexed by the school. Then, given a problem $M^t = \langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$, the **related market** is $\langle S, \{q_s\}_s, I^t, \succ, O^t(M^t) \rangle$. We will often abbreviate notation by omitting the argument $M^t$ of the new orders $O_s^t$.[12]

Given a problem $M^t = \langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$ and the related market $\langle S, \{q_s\}_s, I^t, \succ, O^t \rangle$, we have all elements to apply the DA algorithm of Gale and Shapley (12) to the related market. The algorithm works as follows:

*Step 1.* Each teacher proposes to her top choice. Each school $s$ rejects all but the best $q_s$ teachers among those teachers who proposed to it. Those that remain are "tentatively" assigned one position at school $s$.

*In general,*

*Step k.* Each teacher who is rejected in the last step proposes to her top choice among those schools that have not yet rejected her. Each school $s$ rejects all but the best $q_s$ teachers among those teachers who have just proposed and those who were tentatively assigned to it at the last step. Those who remain are "tentatively" assigned one position at school $s$.

The algorithm terminates when no teacher proposal is rejected. Each teacher is assigned to her final tentative assignment.

---

[12]The idea of the related market in which position-specific priorities are modified was originally introduced by Guillen and Kesten (13).

When we apply the DA algorithm, since $\left|\nu_t^{-1}(s)\right| \le q_s$, if $\mu_t(k) \ne \nu_t(k)$ for some $k \in \nu_t^{-1}(s)$, then $\mu_t(k) \succ_k \nu_t(k)$. That is, using orders $O^t(M^t)$ and applying the DA algorithm, we obtain an individually rational matching.

Following Ergin (11), we present the next definition.

*Definition* 9. Given an overlapping teacher placement problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$ and the related market $\langle S, \{q_s\}_s, I^t, \succ, O^t \rangle$, we say that matching $\mu_t$ **violates the priority** of $i$ for $s$, if there is a teacher $h$ such that $\mu_t(h) = s$, $s \succ_i \mu_t(i)$ and $i \; O_s^t \; h$. The matching $\mu_t$ **adapts** to $O^t$ if it does not violate any priorities.

The relation between a matching that adapts to $O^t$ and an acceptable matching is straightforward, as we prove in the next lemma.

*Lemma* 1. Given an overlapping teacher placement problem $M^t = \langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$ and the related market $\langle S, \{q_s\}_s, I^t, \succ, O^t \rangle$, a matching is acceptable (relative to the problem $M^t$) if and only if it adapts to $O^t$ (regarding the related market) and it is non-wasteful.

*Proof.* ($\Rightarrow$) An acceptable matching is, by definition, non-wasteful. Then, suppose that $\mu_t$ is acceptable but violates the priority of $i$ for $s$. Then, there is a teacher $j$ such that $\mu_t(j) = s$, $s \succ_i \mu_t(i)$ and $i \; O_s^t \; j$. We have two cases: $i >_s^t j$ or $j >_s^t i$. The latter implies that $i$ was originally assigned to school $s$, that is $i \in \nu_t^{-1}(s)$, but this violates the individual rationality assumption. The first implies that both $i$ and $j$ were originally assigned to school $s$, since $\mu_t$ is an acceptable matching. But, once again, the latter violates the individual rationality assumption for the assignment of $i$.

($\Leftarrow$) Suppose that $\mu_t$ adapts to $O^t$ and is non-wasteful, but it is not acceptable. Then we have two cases: $\mu_t$ is not individually rational or there is a justified claim in $\mu_t$. In the first case, suppose that $i$ is such that $s = \nu_t(i) \succ_i \mu_t(i)$. Since matching $\mu_t$ is non-wasteful, we have teacher $j$, such that $j \notin \nu_t^{-1}(s)$ and $\mu_t(j) = s$. But then, $i \; O_s^t \; j$, and $\mu_t$ does not adapt

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to $O^t$. If there is a justified claim in $\mu_t$, we have two teachers $i$, $j$ and a school $s$, such that $\mu_t(j) = s \succ_i \mu_t(i)$, $i >_s^t j$ and $j \notin \nu_t^{-1}(s)$. But then $i \, O_s^t \, j$ and $\mu_t$ does not adapt to $O^t$. ∎

Therefore, the problem of finding an acceptable matching in our original framework is equivalent to finding a matching that adapts to $O^t$ and is non-wasteful in the related market.

*Proposition* 1. **Given an overlapping teacher placement problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$, there is an acceptable matching**.

*Proof.* Given the related market $\langle S, \{q_s\}_s, I^t, \succ, O^t \rangle$, we apply the DA algorithm. It is well known (see Ergin (11), Proposition 1), that the outcome of the algorithm is a matching that adapts to $O^t$. It is easy to show that the outcome is also non-wasteful. Then, by Lemma 1, we have an acceptable matching in the problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$. ∎

*Corollary* 1. **Given an overlapping teacher placement problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$, there always exists a matching that minimizes inappropriate claims.**

*Proof.* We know that $\mathcal{C}_t$ is nonempty and finite. For each matching $\mu_t \in \mathcal{C}_t$, compute $|\Gamma(\mu_t)|$. Therefore, we have a finite set of real numbers; take $\mu_t' \in \mathcal{C}_t$ such that $|\Gamma(\mu_t')| \leq |\Gamma(\mu_t)|$, for all $\mu_t \in \mathcal{C}_t$. Then, $\mu_t'$ minimizes inappropriate claims. ∎

We know that in every problem, there is one matching that minimizes inappropriate claims. One easily finds examples in which there is more than one matching with this property.

It is a classic result of matching theory that the outcome of the DA algorithm satisfies that every agent prefers her partner at this outcome at least as well as the partner of any other acceptable matching. (It is said that the matching is agent-optimal in the subset of acceptable matchings.) Then we know that DA outcome is Pareto superior to any other matching that minimizes inappropriate claims. If we proved that the outcome of the DA algorithm minimizes inappropriate claims, we would prove that it is also *the best* matching that minimizes inappropriate claims, because it is a well-known result that if preferences are strict,

there is only one acceptable matching Pareto superior to any other acceptable matching.[13] This is the purpose of the following results.

*Lemma* 2. Given an overlapping teacher placement problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$, consider the outcome of the DA algorithm, denoted by $\mu_t^{GS}$, when it is applied to the related market $\langle S, \{q_s\}_s, I^t, \succ, O^t \rangle$. Then $\mu_t^{GS}$ minimizes inappropriate claims.

(See Appendix for a proof).

Since preferences are strict, we have the following characterization theorem.

*Theorem* 1. **Given an overlapping teacher placement problem** $\langle S,\ \{q_s\}_s, I^t,\ \mu_{t-1},$ $\succ,\ >^t \rangle$**, a matching that minimizes inappropriate claims is Pareto superior to any other matching that minimizes inappropriate claims if and only if it is the outcome of the DA algorithm (applied to the related market).**

## 2.4   A Mechanism

As we have defined, an economy includes a mechanism, because the dynamics of our problem are defined by the relation between the matching of one period and the initial assignment of the following one. We know that the outcome of the DA algorithm is the *best* matching, in the sense that within the set of matchings that minimize inappropriate claims it is Pareto superior to all other matchings. Then, we have the following definition.

*Definition* 10. The **teacher proposing deferred acceptance mechanism** is the mechanism that assigns to each overlapping teacher placement problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$ the outcome of the DA algorithm when it is applied to the related market $\langle S,\ \{q_s\}_s,\ I^t,\ \succ,\ O^t \rangle$.

*Definition* 11. The **teacher proposing deferred acceptance economy** is an economy in which the mechanism is the teacher proposing deferred acceptance mechanism.

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[13]See Ergin (11), Proposition 1, and Balinski and Sönmez (3), Theorem 2.

*Definition* 12. A mechanism **minimizes inappropriate claims** if it always selects a matching that minimizes inappropriate claims. An economy **minimizes inappropriate claims** if the used mechanism minimizes inappropriate claims.

The previous sections show that if we restrict our attention to economies that minimize inappropriate claims, the best economy in terms of efficiency is the teacher proposing deferred acceptance economy. And we also know that essentially it is the unique economy with that property. Hence, we have the following proposition.

*Proposition* 2. **A mechanism minimizes inappropriate claims and is Pareto superior to any other mechanism that minimizes inappropriate claims, if and only if it is the teacher proposing deferred acceptance mechanism.**

In the next two sections, we study some dynamic properties of the proposed mechanism.

## 2.5   Dynamic Strategy-Proofness

Suppose that a new teacher enters the market to compete for a position at time $t_0$. A natural question is whether this new teacher can ever benefit by unilaterally misrepresenting her preferences. If the DA algorithm is used, it is a well-known result that she cannot benefit in period $t_0$ by manipulating her preferences (Dubins and Freedman (9); Roth (18)). But, what can be said about the following periods? Can a teacher benefit, in the following periods, by sacrificing her school in period $t_0$? After some definitions, we study this issue.

*Notation* 1. We denote by $\varphi\left[I^t, \succ\right](i)$ the school assigned in period $t$ to teacher $i$ under the mechanism $\varphi$.

*Definition* 13. Suppose an economy $S, \{q_s\}_s, \nu_1, \{I^t\}_t, \{(\succ_i)_{i \in I^t}\}_t, \{>^t\}_t, \varphi$ and a teacher $i$ who enters the market at time $t_0$. We say that the mechanism $\varphi$ is **dynamically strategy-proof** if teacher $i$ cannot ever benefit by unilaterally misrepresenting her preferences, that is: $\varphi$ is dynamically strategy-proof if $\varphi[I^t, \succ_{-i}, \succ_i](i) \succsim_i \varphi[I^t, \succ_{-i}, \succ_i'](i)$ for all $i$, $\succ_{-i}$, $\succ_i'$

and **for all** $t \geq t_0$ such that $i \in I^t$, where $\succ_{-i}$ are the preferences of teachers in the set $I^t \setminus \{i\}$.

*Remark* 1. The classic concept in static matching problems only makes reference to the benefit in one period. In our framework, the concept involves not only the period when the teacher enters the market (and reveals her preferences), but also all the later periods while she is in the market.

It is interesting to note that a mechanism can be strategy-proof (with the usual static definition) but not dynamically strategy-proof. Appendix A shows a mechanism with this property.

As we remarked in the beginning of this section, when the teacher proposing deferred acceptance mechanism is used and a new teacher enters the market, she cannot benefit in that period by misrepresenting her preferences. In this sense, the mechanism is strategy-proof in the "static" problem. Then, we can wonder if this property is also verified by the mechanism in a dynamic context. In the next example, we prove that in our dynamic model, the mechanism can be manipulated by teachers.

*Example* 3. Consider the following problem:

$I_E^t = \{j, k\} \subset I^t = \{i, j, k\}$,

$S = \{s_1, s_2, s_3\}$, $q_i = 1$, $i = 1, 2, 3$, $\nu_t = \{(j, s_2), (k, s_3)\}$, and

the following teacher preferences (from best to worst) and orders:

$$
\begin{bmatrix}
\succ_i & \succ_j & \succ_k \\
s_2 & s_3 & s_2 \\
s_3 & s_2 & s_3 \\
s_1 & s_1 & s_1
\end{bmatrix}
\qquad
\begin{bmatrix}
>_1^t & >_2^t & >_3^t \\
i & j & k \\
j & k & i \\
k & i & j
\end{bmatrix}
$$

Then the outcome of the teacher proposing deferred acceptance mechanism is:

$$
\mu_t = \begin{pmatrix} i & j & k \\ s_1 & s_2 & s_3 \end{pmatrix}
$$

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For the next period assume: $I^{t+1} = \{i, j, l\}$,

$$
\begin{bmatrix} \succ_l \\ s_2 \\ s_3 \\ s_1 \end{bmatrix}
\qquad
\begin{bmatrix} >_1^{t+1} & >_2^{t+1} & >_3^{t+1} \\ i & j & l \\ j & i & i \\ l & l & j \end{bmatrix}
$$

The matching in this period is:

$$
\mu_{t+1} = \begin{pmatrix} i & j & l \\ s_1 & s_2 & s_3 \end{pmatrix}
$$

Suppose that instead of her true preferences, teacher $i$ reveals the following preferences: $\succ_i' = (s_2, s_1, s_3)$. Then the matching generated in each period is:

$$
\mu_t' = \begin{pmatrix} i & j & k \\ s_1 & s_3 & s_2 \end{pmatrix}
\qquad
\mu_{t+1}' = \begin{pmatrix} i & j & l \\ s_2 & s_3 & s_1 \end{pmatrix}
$$

Since $\mu_{t+1}'(i) = s_2 \succ_i \mu_{t+1}(i) = s_1$, teacher $i$ can benefit by unilaterally misrepresenting her preferences. ∎

Let's examine the last example more closely. By revealing other preferences, teacher $i$ can manipulate the initial submatching of period $t+1$. When she reveals $\succ_i'$, teacher $j$ is assigned in period $t$ to school $s_3$. Then $j$ has priority over new teacher $l$ to school $s_3$ even when she is lower ranked than the new teacher. If $i$ reveals her *true* preferences, new teacher $l$ has priority over $j$ to school $s_3$, then $j$ is rejected from that school and she proposes to $s_2$, causing the rejection of $i$ from that school. It is easy to see that this case is also possible when there is a unique priority order of all teachers, that is: when $>_s^t = >^t$ for all $s$ and $t$. However, as we will prove in the next theorem, if at each school's priority ranking teachers that were present in the previous period have priority over new teachers, then the teacher proposing deferred acceptance mechanism is dynamically strategy-proof. We first define this property and then we present our positive result.

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*Definition* 14. A set of orders $\{>_s^t\}_{s \in S}$ is lexicographic by tenure if for all teachers $i, j \in I^t$, whenever $i \in I_E^t$, and $j \notin I_E^t$ then $i >_s^t j$ for all schools $s \in S$.

In an overlapping teacher placement problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$ in which the set of orders $>^t = \{>_s^t\}_{s \in S}$ is lexicographic by tenure, each order in the related market consists of three groups of teachers. The first group in the order is the set of teachers who were assigned to the school; then we have the set of teachers who were assigned to another school in the previous period. Finally, we have the new teachers. Within each group, the order is defined by the original priority ranking $>_s^t$.

*Definition* 15. An **economy is dynamically strategy-proof** if the used mechanism is dynamically strategy-proof.

*Theorem* 2. **Let** $S, \{q_s\}_s, \nu_1, \{I^t\}_t, \{(\succ_i)_{i \in I^t}\}_t, \{>^t\}_t, \varphi$ **be the teacher proposing deferred acceptance economy. If in each** $t$ **the set of orders** $\{>_s^t\}_{s \in S}$ **is lexicographic by tenure, then the economy is dynamically strategy-proof.**

(See Appendix for a proof).

## 2.6  Respecting Improvements

In this section, we study another important property of mechanisms, namely, respecting improvements. We say that a mechanism does not respect improvements made by teachers if a teacher may increase her place in one school's priority ranking, everything else remains unchanged, and yet she is punished with a less preferred assignment (Balinski and Sömnez (3)). In Appendix A, we present a mechanism that does not respect improvements. In this section, we study whether or not the teacher proposing deferred acceptance mechanism has this property.

*Definition* 16. An overlapping teacher placement problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, \tilde{>}_{s'}^t, \{>_s^t\}_{s \neq s'} \rangle$ is an **improvement** for teacher $i$ over another problem $\langle S, \{q_s\}_i, I^t, \mu_{t-1}, \succ, >_{s'}^t, \{>_s^t\}_{s \neq s'} \rangle$,

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if $i >^t_{s'} j$ implies that $i \tilde{>}^t_{s'} j$ , and for all teachers $k, h$ different from $i$, we have that $h \tilde{>}^t_{s'} k \Leftrightarrow h >^t_{s'} k$.

According to Definition 16, an improvement for a teacher is basically the original placement problem with the only difference being that the teacher possibly has a higher priority in some school's priority ranking.

*Definition* 17. A mechanism **respects improvements** if for any teacher $i$ and $\langle S, \{q_s\}_s, I^t,$ $\mu_{t-1}, \succ, \tilde{>}^t_{s'}, \{>^t_s\}_{s \neq s'} \rangle$ an improvement for that teacher over another problem $\langle S, \{q_s\}_s, I^t,$ $\mu_{t-1}, \succ, >^t_{s'}, \{>^t_s\}_{s \neq s'} \rangle$, the position assigned by the mechanism to teacher $i$ in each period since the improvement (that is, in all periods $\tau \geq t$) is, for teacher $i$, at least as good as the position assigned in each period beginning with the problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t_{s'},$ $\{>^t_s\}_{s \neq s'} \rangle$. That is, let $\mu_t$ denote the matching selected by the mechanism in the problem with $>^t_{s'}$ and $\tilde{\mu}_t$ the selected matching in the problem with $\tilde{>}^t_{s'}$. Then the mechanism **respects improvements** if $\tilde{\mu}_\tau(i) \succsim_i \mu_\tau(i)$ for all $\tau \geq t$.

*Remark* 2. The comment of Remark 1 also applies to this definition. Our concept of respecting improvements involves not only the period when the teacher improves her place in the priority ranking (as in the classic notion), but also every following period while she is in the market.

It is worth noting that there is no relation between the properties of respecting improvements and dynamic strategy-proofness. Consider the static problem; on the one hand, the mechanism described in the introduction is strategy-proof but does not respect improvements made by teachers (see Appendix A.1). On the other hand, it is straightforward to find a mechanism that respects improvements but is not strategy-proof. Now consider the dynamic problem and a mechanism that is both strategy-proof and respects improvements (in the static problem). We can wonder if there is any relation between both properties in the dynamic problem. One easily finds examples of mechanisms that satisfy only one of these properties. Hence, there is no relation between these two properties, neither in the static problem nor in the dynamic one.

In the next example, we show that the problem described in the previous section also appears with this property.

*Example* 4. Consider the same problem of Example 3 and suppose another problem with the same elements, but in which the order of school $s_3$ is: $\tilde{>}_3^t = (k, j, i)$. Denote by $M^t$ and $\bar{M}^t$ the problem of Example 3 and its modification, respectively. Then, problem $M^t$ represents an improvement for teacher $i$ over $\bar{M}^t$. The outcome of the teacher proposing deferred acceptance mechanism for each problem is ($\mu_t$ corresponds to the problem $M^t$ and $\bar{\mu}_t$ to $\bar{M}^t$):

$$\mu_t = \begin{pmatrix} i & j & k \\ s_1 & s_2 & s_3 \end{pmatrix} \qquad \bar{\mu}_t = \begin{pmatrix} i & j & l \\ s_1 & s_3 & s_2 \end{pmatrix}$$

In the next period, we have $\tilde{>}_3^{t+1} = (l, j, i)$ and the following matchings:

$$\mu_{t+1} = \begin{pmatrix} i & j & k \\ s_1 & s_2 & s_3 \end{pmatrix} \qquad \bar{\mu}_{t+1} = \begin{pmatrix} i & j & l \\ s_2 & s_3 & s_1 \end{pmatrix}$$

Note that $\bar{\mu}_{t+1}(i) \succ_i \mu_{t+1}(i)$. Then, although teacher $i$ improves her position in the ranking of school $s_3$, she is assigned in period $t + 1$ to a less preferred school. ∎

As we will prove in the next theorem, if the set of orders is lexicographic by tenure, the mechanism respects improvements.

*Definition* 18. An **economy respects improvements** if the used mechanism respects improvements.

*Theorem* 3. **Consider a teacher $i$ and $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, \tilde{>}_{s'}^t, \{>_s^t\}_{s \neq s'} \rangle$, an improvement for that teacher over another problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >_{s'}^t, \{>_s^t\}_{s \neq s'} \rangle$. Denote by $\tilde{\mu}_t$ and $\mu_t$ matchings selected by the teacher proposing deferred acceptance mechanism in each problem. Then $\tilde{\mu}_t(i) \succsim_i \mu_t(i)$. Moreover, if in each period $\tau \geq t$ the set of orders is lexicographic by tenure, then the teacher proposing deferred acceptance economy respects improvements.**

(See Appendix for a proof).

## 2.7 Concluding Remarks

We conclude with a brief discussion about efficiency. A matching $\mu_t$ is **Pareto efficient (or simply efficient)** if there is no other matching that makes all teachers present at $t$ weakly better off and at least one teacher strictly better off. A mechanism is efficient if, for any preference profile, it always selects an efficient matching. Then, one can wonder if the mechanism proposed in our model is efficient. We use a result from Ergin (11) to address this question: a cycle for a given priority structure $O^t$ is constituted of distinct schools $s$, $s' \in S$ and teachers $i, j, k \in I^t$, such that $i \ O_s^t \ j \ O_s^t \ k \ O_{s'}^t \ i$. By Theorem 1 of Ergin (11), we know that the DA mechanism is Pareto efficient if and only if the priority structure is acyclical (that is, the priority structure has no cycle). In our problem, under the assumption that in each period there are at least three teachers, each of whom was assigned to a different school in the previous period, the priority structure of the related market $O^t$ always has at least one cycle. Let $i, j, k \in I_E^t$ with $\nu_t(i) = s$, $\nu_t(j) = s'$ and $\nu_t(k) = s''$, then $i \ O_s^t \ j \ O_s^t \ k \ O_{s''}^t \ i$ or $i \ O_s^t \ k \ \ O_s^t \ j \ \ O_{s'}^t \ i$, but in each case there is a cycle. Finally, applying the mentioned theorem, we know that the proposed mechanism is not Pareto efficient. However, it is important to stress that the outcome of DA algorithm is Pareto efficient in the subset of acceptable matchings. Moreover, since within the set of matchings that minimize inappropriate claims, the DA outcome is the unique matching Pareto superior to all other matchings, we have the following result: *if in each period there are at least three teachers, each of whom was assigned to a different school in a previous period, there is no efficient mechanism that minimizes inappropriate claims.*

The last result stresses the classic tradeoff between efficiency and fairness (see Abdulkadiroğlu and Sönmez (2)). Roughly speaking, one has to choose between one of these properties. In our model, we consider fairness as more important since once a teacher is assigned to a school, she cannot be changed unless she is assigned to a preferred school. In this sense a violation of the fairness condition has consequences in future periods. There are other mechanisms that select Pareto efficient matchings. Gale's top trading cycles mechanism (described in Abdulkadiroğlu and Sönmez (1)) is one of them.

In this paper, we have developed a new framework to model a dynamic school choice problem with overlapping generations of teachers. In each period, the central authority must assign teachers to teaching positions. Two elements must be considered in the assignments: the schools' priority rankings and previous assignments. From one period to another, teachers are allowed either to retain their current position, or to choose a preferred one (if available). Hence, the central authority faces a dynamic allocation problem.

The dynamics of our model are defined by the mechanism. The matching in one period links this period with the following one because it determines the initial submatching for the next period. In this framework, we introduced a new solution concept which is very natural in our context. We have proved that a matching that minimizes inappropriate claims always exists and that it can be reached by a modified version of the deferred acceptance algorithm of Gale and Shapley. In particular, the algorithm is applied to a related market in which each school's priority ranking is modified to obtain an individually rational matching. In relation to the properties of the mechanism, we proved that if the set of orders is lexicographic by tenure, it is dynamically strategy-proof and respects improvements made by teachers.

The mechanism proposed in this paper can be easily implemented in the real-life market since it is based on the DA algorithm which is widely used in practice.[14] Also, the mechanism implies an improvement upon the mechanism which is actually applied. Indeed, the teacher proposing deferred acceptance mechanism fixes all the problems that the used mechanism has. As we noted in the Introduction, the mechanism used by the central authority in the problem studied in this paper is an individually rational variant of the serial dictatorship. The outcome of this mechanism may create a situation in which a new teacher has justified envy in the resulting matching. Indeed, it may happen that a new teacher $i$ prefers the school $\mu_t(j)$ over her assignment and that $i$ has priority over $j$ and $j$ is not an incumbent teacher. This unfair situation is avoided under the mechanism that we propose. Moreover, the outcome of the teacher proposing deferred acceptance mechanism is Pareto superior to any other matching that minimizes inappropriate claims. These two properties additionally to

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[14]See Roth (16) for a list of different markets that use the DA algorithm.

dynamic strategy-proofness and respecting improvements, promote the proposed mechanism as an excellent substitute to the mechanism actually used. Although we have to convince authorities about the benefits of implementing the teacher proposing deferred acceptance mechanism, the properties that this mechanism has is a promising beginning.

Although the time-invariant preferences assumption fits the analyzed case, it should be reconsidered if one wants to study other markets. In this sense, it is reasonable to suppose that agents' preferences may evolve by learning and an option which was initially one of the top choices of the agent, in a later period it may lose ranking positions. The same comment applies to the schools' priority rankings: poorly performing teachers may be ranked below their initial ranking position. An important restriction is the individual rationality: a teacher cannot be assigned to a less preferred school than the one she is teaching and in this sense, she cannot be considered as a new teacher.

The mechanism we propose can be extended to the case where teachers' preferences may change over time. Indeed, suppose that preferences evolve and in each period, the clearing-house asks teachers for their preferences. In this framework, the mechanism introduced in this paper allows for exchanges that improve efficiency. In particular, suppose that teacher $i$ who was assigned to school $s_1$, with time she prefers $s_2$, while $j$ who is teaching at $s_2$, now prefers $s_1$. By Theorem 1, we know that the outcome of the mechanism is Pareto superior to any other matching that minimizes inappropriate claims. Then, if teachers $i$ and $j$ are allowed to express their new preferences, they may exchange their positions.

However, the concept of strategy-proofness should be reconsidered. First note that if one wants to encompass time-evolving preferences, under the definition of dynamic strategy-proofness stated in the paper, there is no dynamically strategy-proof mechanism that always selects an individually rational and acceptable matching. In this sense, consider the following example.

*Example* 5. Suppose there are no incumbent teachers in the market, two new teachers ($i$ and

$j$) and two schools. Consider the following preferences and orders:

$$\begin{bmatrix} \succ_i^t & \succ_j^t \\ s_1 & s_2 \\ s_2 & s_1 \end{bmatrix} \qquad \begin{bmatrix} >_1^t & >_2^t \\ i & i \\ j & j \end{bmatrix}$$

The matching selected by the mechanism is:

$$\mu_t = \begin{pmatrix} i & j \\ s_1 & s_2 \end{pmatrix}.$$

Suppose that in period $t+1$, teacher $i$ changes her preferences and prefers school $s_2$ to $s_1$, while teacher $j$ has the same preferences. The unique individually rational matching in $t+1$ is $\mu_t$. But, if $i$ declares in period $t$ school $s_2$ as preferred to $s_1$, then she is assigned to that school in both periods. Therefore, the mechanism is not dynamically strategy-proof, because teacher $i$ benefits in period $t+1$ if she reports other preferences ■

In the last example, we assume that teachers know their future preferences. The study of the case where each agent only knows her current preferences, requires a deeper analysis. Some articles that investigate matching models with incomplete information are: Chakraborty et al. (6), Ehlers and Massó (10) and Roth (17). Although the study of a model with incomplete information goes beyond the scope of the present work, Example 3 shows that also in that framework, strategy-proofness requires orders to be lexicographic by tenure. Otherwise, the mechanism can be manipulated by teachers via the initial submatching of the next period. [15]

Finally, it would be interesting to study the performance of the proposed mechanism in large markets. In this sense, the model of Kojima and Pathak (14) can be used to analyze the properties of the teacher proposing deferred acceptance mechanism in large markets.

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[15]These manipulations are mainly made by lying about the relative ranking positions of the schools not assigned to the teacher. Note also that overstating the preference for a school to which the teacher is assigned, is not profitable since the individually rational restriction.

## 2.8 Appendix

### A.1 The weaknesses of the mechanism used by Mexican Ministry of Public Education.

Suppose there are four schools $S = \{s_1, s_2, s_3, s_4\}$, each one with only one position and four teachers present in the market at time $t : I^t = \{i, j, k, l\}$. Assume that teachers $k$ and $l$ were assigned in a previous period to schools $s_3$ and $s_4$, respectively. Teacher preferences (from best to worst) and the ranking are (where $\succ_h$ are preferences of teacher $h$):

$$\begin{bmatrix} \succ_i & \succ_j & \succ_k & \succ_l \\ s_3 & s_4 & s_1 & s_2 \\ s_1 & s_2 & s_3 & s_3 \\ s_2 & s_3 & s_2 & s_4 \\ s_4 & s_1 & s_4 & s_1 \end{bmatrix} \quad \begin{bmatrix} >^t \\ i \\ k \\ l \\ j \end{bmatrix}$$

That is, teacher $i$'s most preferred school is $s_3$, her second choice is $s_1$, and so on. We also have that the first teacher in the ranking is $i$, the second $k$, the third $l$, and the last $j$. If we use the mechanism described in the introduction, the matching in this market is (the school below each teacher is her assigned school):

$$\mu_t = \begin{pmatrix} i & j & k & l \\ s_1 & s_4 & s_3 & s_2 \end{pmatrix}$$

Assume that in the next period, teachers $k$ and $l$ exit the market and two new teachers enter. Then we have $I^{t+1} = \{i, j, m, n\}$, $>^{t+1} = (m, i, n, j)$. The preferences of new teachers are $\succ_m = (s_1, s_3, s_4, s_2)$ and $\succ_n = (s_2, s_4, s_1, s_3)$. Then, the outcome of the mechanism is:

$$\mu_{t+1} = \begin{pmatrix} i & j & m & n \\ s_1 & s_4 & s_3 & s_2 \end{pmatrix}$$

Next we will show how a teacher can benefit by manipulating her preferences. Suppose that instead of her true preferences, teacher $i$ reveals the following preferences: $\succ_i' = (s_3, s_2, s_1, s_4)$.

Hence, the outcome of the mechanism in each period is:

$$\mu'_t = \begin{pmatrix} i & j & k & l \\ s_2 & s_4 & s_1 & s_3 \end{pmatrix} \quad \text{and} \quad \mu'_{t+1} = \begin{pmatrix} i & j & m & n \\ s_3 & s_4 & s_1 & s_2 \end{pmatrix}$$

Note that $\mu'_{t+1}(i) \succ_i \mu_{t+1}(i)$, and then teacher $i$ benefits in period $t + 1$ by misrepresenting her preferences. Hence, the mechanism is not dynamically strategy-proof. The second flaw we will illustrate is that the mechanism does not respect improvements made by teachers. Suppose that teacher $i$, instead of being the first in the ranking $>^t$, has a worse performance and she is the second in the ranking. Specifically, assume that at period $t$ the ranking of teachers is: $\tilde{>}^t = (k, i, l, j)$. Then the outcome of the mechanism is:

$$\tilde{\mu}_t = \begin{pmatrix} i & j & k & l \\ s_3 & s_4 & s_1 & s_2 \end{pmatrix}$$

Therefore, it is better for teacher $i$ to have a lower order in the ranking, because if she increases her position in the 'priority order,' like in $>^t$, she will be punished with a worse position.■

## A.2 Proof of Lemma 2

*Proof.* If $\Gamma(\mu_t^{GS}) = \varnothing$, the proof is complete. Otherwise, we already know that $\mu_t^{GS}$ is acceptable. Suppose that it does not minimize inappropriate claims; then, we have another acceptable matching $\mu_t$, such that $\Gamma(\mu_t) \subsetneq \Gamma(\mu_t^{GS})$. Since $\mu_t^{GS}$ is Pareto superior to $\mu_t$ : $\mu_t^{GS}(i) \succsim_i \mu_t(i) \ \forall i$ and there is a teacher $h$ such that $\mu_t^{GS}(h) \succ_h \mu_t(h)$. We claim that in this case $\Gamma(\mu_t^{GS}) \subset \Gamma(\mu_t)$, but this contradicts the last relation. Suppose there is a pair $(i, s) \in I^t \times S$, such that $(i, s) \in \Gamma(\mu_t^{GS})$ but $(i, s) \notin \Gamma(\mu_t)$. Then we have a teacher $j$, such that $\mu_t^{GS}(j) = s \succ_i \mu_t^{GS}(i) = s'$, $i >_s^t j$ and $j \in \nu_t^{-1}(s)$. As $(i, s) \notin \Gamma(\mu_t)$, we have two cases: $\mu_t(i) \succsim_i s$ or $j \notin \mu_t^{-1}(s)$. The first case implies $\mu_t(i) \succsim_i s \succ_i \mu_t^{GS}(i)$, but it is not possible since $\mu_t^{GS}$ is Pareto superior to all acceptable matchings. In the second case, it must be $\mu_t^{GS}(j) = s = \nu_t(j) \succ_j \mu_t(j)$, but then $\mu_t$ is not individually rational. Finally, we prove that $\Gamma(\mu_t^{GS}) \subset \Gamma(\mu_t)$. ■

## A.3 Proof of Theorem 2

*Proof.* The proof of the theorem is built on the following observation.

Let $I_{t'}^t$ be the set of teachers at period t who joined the market at period $t' \leq t$. If the priority structure is lexicographic by tenure, the outcome of the teacher proposing deferred acceptance mechanism becomes the same as the outcome of the following procedure:

**Step 0**: Teachers in $I_0^t$ propose to schools under DA (using the strict priority $>_s^t$ of each school $s$) and a matching is produced. The capacities of the schools are updated by subtracting the positions that are filled.

**Step $t' \leq t$**: Teachers in $I_{t'}^t$ propose to schools (with their unfilled capacities) under DA (using the strict priority $>_s^t$ of each school $s$) and a matching is produced. The capacities of school are updated by subtracting the positions that are filled.

To prove the equivalence between the previous procedure and the teacher proposing deferred acceptance mechanism, fix a period $t$, a school $s$ and two teachers $i$ and $j$. We have two cases to study: both teachers joined the market at the same period or one of them, say $i$, joined before $j$. In the first case, suppose $i >_s^t j$. Then, the strict priority order of school $s$ used by each mechanism can be different if and only if, teacher $j$ was assigned to school $s$ at the previous period and $i$ was assigned to another school. In the second case we have a similar situation, teacher $i$ has priority for all school over $j$ in the alternative mechanism while in the teacher proposing deferred acceptance mechanism $j$ has priority over $i$ if and only if $j$ was assigned to $s$ in $t-1$ and $i$ was not. In both cases we will prove that teacher $i$ does not propose to school $s$ when the teacher proposing deferred acceptance mechanism is used. Since this mechanism selects an individually rational matching, it is enough to prove that $\mu_{t-1}(i) \succ_i s$. Suppose, to the contrary, that $s \succ_i \mu_{t-1}(i)$ and be $l$ the period when teacher $j$ was assigned to school $s$. Then we have $\mu_l(j) = s \succ_i \mu_l(i)$ and also $i >_s^l j$ since orders are lexicographic by tenure; but this implies that the matching is not acceptable, a contradiction.

Then, once priorities are assumed to be lexicographic by tenure, at period $t$ a teacher in $I^t_{t'}$ only competes with other teachers in the same set. Teachers in $I^t_{t'}$, by changing their reported preferences, cannot influence, in the described procedure, which positions at schools remain after Step $t' - 1$. To complete the proof just note that when a teacher competes with other teachers that enter in the same period that she enters, strategy-proofness is a direct consequence of the classic results from Dubins and Freedman (9) and Roth (18).

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## A.4 Proof of Theorem 3

*Proof.* For the **first period**, the result is similar to the one presented in Balinski and Sömnez (3), Theorem 5. For the sake of completeness, we include the adapted proof. Fix a problem $M^t = \left\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t_{s'}, \{>^t_s\}_{s \neq s'} \right\rangle$ and another problem $\tilde{M}^t = \langle\, S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, \tilde{>}^t_{s'}, \{>^t_s\}_{s \neq s'} \rangle$, that represents an improvement for teacher $i$. Let $\mu_t$ denote the matching selected by the teacher proposing deferred acceptance mechanism in the problem $M^t$ and $\tilde{\mu}_t$ the one selected in the problem $\tilde{M}^t$.

We shall prove that $\tilde{\mu}_t(i) \succsim_i \mu_t(i)$. Suppose that $\mu_t(i) \succ_i \tilde{\mu}_t(i)$ and let $\mu_t(i) \equiv s_1$. Denote by $\tilde{O}^t$ the set of strict priority orders in the related market for the problem $\tilde{M}^t$. In the related market of each problem, we know that $O^t_s = \tilde{O}^t_s$ for all $s \neq s'$, $j\, O^t_{s'}\, k \iff j\, \tilde{O}^t_{s'}\, k$ (with $j, k \neq i$), if $i\, O^t_{s'}\, k$ then $i\, \tilde{O}^t_{s'}\, k$ and if $j\, \tilde{O}^t_{s'}\, i$ then $j\, O^t_{s'} i$.

First, note that $\mu_t$ does not adapt to $\tilde{O}^t$ because we know that $\tilde{\mu}_t$ is Pareto superior to any other matching that minimizes inappropriate claims. Suppose that teacher $i$ announces the following preferences $\succ'_i$: $s_1 \succ'_i s_0 \succ'_i s$ for all $s \neq s_1$ and consider the following problem $\langle\, \succ_{j \neq i}, \succ'_i, \tilde{>}^t_{s'}, \{>^t_s\}_{s \neq s'} \rangle$, that is, problem $\tilde{M}^t$ which represents an improvement for teacher $i$ but with the preferences defined before. We will prove that $\mu_t$ is a matching that minimizes inappropriate claims in the problem $\langle\, \succ_{j \neq i}, \succ'_i, \tilde{>}^t_{s'}, \{>^t_s\}_{s \neq s'} \rangle$.

**Claim 1**: $\mu_t$ is individually rational. As $\mu_t$ is individually rational in the problem $M^t$,

then for any teacher $j \neq i$ that reveals the same preference in both problems we have $\mu_t(j) \succsim_j \mu_{t-1}(j)$ if $j \in I_E^t$ or $\mu_t(j) \succsim_j s_0$ if $j$ is a new teacher, and teacher $i$ is assigned to her top school.

**Claim 2**: $\mu_t$ adapts to $\tilde{O}^t$. Matching $\mu_t$ cannot violate the priority of a teacher different to $i$, otherwise the same priority would be violated by $\mu_t$ in the problem $M^t$. Indeed, suppose there is a teacher $j \neq i$ and a school $s$ such that $s \succ_j \mu_t(j)$ and $j \, \tilde{O}_s^t \, h$ with $\mu_t(h) = s$. But then $s \succ_j \mu_t(j)$ and $j \, O_s^t \, h$ with $\mu_t(h) = s$ and $\mu_t$ is not an acceptable matching in the problem $M^t$. Clearly, matching $\mu_t$ cannot violate any priority of $i$ because she is assigned to her top school.

**Claim 3**: The school assigned to teacher $i$ by the teacher proposing deferred acceptance mechanism in the problem $\langle \succ_{j \neq i}, \succ_i', \tilde{>}_{s'}^t, \{>_s^t\}_{s \neq s'} \rangle$ is $s_1$. To prove this claim just note that the DA algorithm satisfies that every teacher prefers his assigned school at this outcome at least as well as the school that she is assigned in any other matching that minimizes inappropriate claims and in $\mu_t$, which minimizes inappropriate claims in this problem, teacher $i$ is assigned to her top school $s_1$.

**Claim 4**: In the problem $\tilde{M}^t$ teacher $i$ can benefit by misrepresenting her preferences. Indeed, if $i$ announces her true preferences, she is assigned to school $\tilde{\mu}_t(i)$, but when she announces preferences $\succ_i'$, she is assigned to $\mu_t(i) \succ_i \tilde{\mu}_t(i)$. Then the teacher proposing deferred acceptance mechanism is not strategy-proof, a contradiction.

Since the observation made in the last proof, the proof for the following periods is straightforward. An improvement for a teacher in $I_{t'}^t$ does not influence, in the alternative procedure, which positions at schools remain after Step $t' - 1$. Then, given that in each period a teacher only competes with teachers that enter the market in the same period that she entered, if the teacher improves her place in one school's priority ranking, the teacher proposing deferred acceptance mechanism will not assign her to a less preferred school. ∎

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# Chapter 3

# Housing Markets with Endogenous Budget Constraints

## 3.1 Introduction

This paper is aimed at studying economies with indivisible goods and budget-constrained agents with unit-demand who act as both sellers and buyers. Important examples of markets with indivisible goods include housing and automobile markets, the market for video games, and collectors' markets (e.g., fine art, stamps). The single unit-demand assumption best applies to a housing market where agents own a house and have a certain amount of money, and seek to sell their house to buy another. In this market, budget constraints are endogenous because each agent's total resources depend on the selling price of her house. Thus, agents endowed with an indivisible good, endogenous budget constraints and unit-demands, are important features of housing markets.

Recently, Milgrom (15) presented a model which extends the Assignment Game of Shapley and Shubik (20) in three important ways: agents may buy or sell multiple types of goods simultaneously, they may trade any integer number of units of each type of good and they may buy some goods and sell other. However, as the author asserts, the assignment message

space used in his model does not allow buyers to express directly a budget limit. Additionally, Milgrom argues that the case of indivisible goods is relevant because: "Even when goods are perfectly divisible, contracts are often denominated and traded in whole numbers of units, so the ability to respect integer constraints may be useful even in those applications."[16]

The contribution of the paper is twofold. First, we explore the existence of competitive equilibrium, complementing the analysis of Quinzii (17). Second, we extend and study two classical mechanisms, the exact auction mechanism of Demange, Gale and Sotomayor (6) and the top trading cycles *with price setting*.

### 3.1.1 Existence

In the literature on economies with indivisible goods, a widely-used assumption is the indispensability of money (IM). This condition, originally stated by Quinzii (17), says that agents should retain some of their initial amount of money in order to attain a utility level higher than their initial one. According to Quinzii (17), under the IM condition, competitive equilibrium exists. At equilibrium, an agent either trades and, under the IM assumption, keeps some of her endowment in money, which implies that her budget constraint does not bind; or she does not trade, in which case money restriction is irrelevant. Thus, under the IM condition, budget constraints never bind and then, previous studies that adopt the condition do not consider effective budget constraints.

Let us examine the IM condition more closely. In the first place, it implies, as Wako (24) asserts, that there is no active exchange unless some agents have divisible goods (money). However, in some markets without money, for example the housing market of Shapley and Scarf (19), there exists a competitive equilibrium which can be found by the top trading cycles algorithm. In the second place, as we just mentioned, under that assumption each agent's budget constraint does not bind, which is far from reality, in particular in housing markets.

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[16]Milgrom (15), page 96.

Two main problems arise when the IM does not hold. First, the existence of competitive equilibrium is not guaranteed. Second, even if a competitive equilibrium exists, there is no adequate mechanism to find it. The Exact Auction Mechanism of Demange, Gale and Sotomayor (6), which finds an equilibrium in the classic Assignment Game, when it is applied to some of the economies considered in this article, prices rise indefinitely and the algorithm does not stop.[17]

The difference in valuations of goods is the yardstick for evaluating if there is a lot or little money in the economy. If for any agent, the amount of money she owns is larger than any of her differences, there is abundance of money. Scarcity, on the contrary, is evaluated at the aggregate level. Roughly, money is scarce if the summation of money in the economy is smaller than any difference in valuations, for any agent.

At equilibrium prices, no good is overdemanded. When agents are rich enough, the mechanism of Demange, Gale and Sotomayor (6) always produces an equilibrium by increasing prices of goods which are minimally overdemanded, and holding others constant. The idea is simply to make agents indifferent between an increasing numbers of goods. Once indifferences are sufficiently widespread in the economy, all agents can be assigned one of her top choices, meaning that an equilibrium exists. The procedure does not extend when agents face budget constraint since rising prices might lead some demand to disappear.

When money is suitably scarce, prices cannot raise sufficiently 1) to make agents indifferent- so that we keep a lot of heterogeneity in the economy- 2) to make an agent switch from her top affordable choice for a very cheap but less preferred good. Thus, the main source of richness is the endowment of good, and the assignment produced by TTC is an equilibrium assignment.

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[17]Since the model we study differs from the Assignment Game analyzed by Demange, Gale and Sotomayor (6), their mechanism is not the one used in this paper. However, since we apply the algorithm stated by these authors, we abuse notation by referring as Demange, Gale and Sotomayor mechanism the application of their algorithm in our model.

The central conditions analyzed in this paper are the Money Scarcity (MS) and its Strong (SMS) and Weak (WMS) versions. The key components of these concepts include one related to the scarcity of money and other related to the number of steps in the top trading cycles algorithm. Formally, if the top trading cycles algorithm consists of $k$ steps, the WMS condition says that the difference between each agent's valuation for two goods should be greater than or equal to the sum of the initial amount of money held by the rest of agents, plus $k - 1$. Thus, the condition requires that each agent should consider goods as heterogeneous, and is independent of the agent's endowment in money and also of the initial distribution of money. The MS condition is slightly stronger: it takes into account the total amount of money, including the agent's endowment. Finally, the strong version replaces the number of steps in the top trading cycles algorithm with the total number of agents. Whereas the IM can be viewed as an assumption of abundance of money in the economy, these conditions assume the opposite.

The WMS condition ensures the existence of competitive equilibrium: an allocation of goods to agents and a price vector such that all goods are assigned to one and only one agent who demands it, and each agent satisfies her budget constraint. The intuitive understanding behind this result is the following. The assignment of goods is found by the top trading cycles algorithm, which defines a partition of the set of agents; each subset consists of agents that exit the market in the same step of the algorithm (although they might belong to different trading cycles). In order to find a competitive equilibrium, prices are set according to the step at which goods leave the market: goods that leave first have a higher price.[18] But each agent's endowment in money should also be considered. Then, all goods that leave the market in the last step are assigned a price of zero; for goods that leave in step $t$, the assigned price is the highest endowment in money of agents that leave the market in step $t + 1$, plus the price assigned in that step, plus one. This method implies that goods with the highest price are those that leave the market in the first step. The WMS condition guarantees that this procedure defines a competitive equilibrium since no agent has incentives to sell her

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[18]The idea is adopted from Shapley and Scarf (19)

good and buy another with lower price and, although an agent may prefer another good over the one she is assigned to, she cannot afford it. Thus, Quinzii's result on the existence of competitive equilibrium is extended to cases where the IM does not hold and utility functions are quasi-linear.

### 3.1.2 Mechanisms

Two mechanisms are considered to find a competitive equilibrium which are extended versions of the top trading cycles algorithm, and of the Exact Auction Mechanism of Demange, Gale and Sotomayor (6). When agents' endowment in money are common knowledge, the extended version of the top trading cycles is strategy-proof under the MS condition; this result means that truthful revelation is a weakly dominant strategy for each agent in the preference revelation game induced by this mechanism. Then, we show that the mechanism of Demange, Gale and Sotomayor does not satisfy that property, and a set of Nash equilibria for the revelation game induced by this mechanism is found. Using the SMS condition we obtain this result, specifying a link between the mechanisms, since the assignment of goods in a Nash equilibrium coincides with the assignment found by the extended version of the top trading cycles mechanism, and it is thus Pareto efficient.

Finally, we identify certain conditions under which the mechanism of Demange, Gale and Sotomayor finds a competitive equilibrium. It is shown that under the MS condition, this mechanism always finds a competitive equilibrium in a market with three agents, but may fail in a more general setting. Then, we posit a stronger condition for each agent's valuations in order to show the convergence of the mechanism to a competitive equilibrium in the general case.

An important field of application for these mechanisms is on-campus housing at universities. In this problem, a number of dormitory rooms has to be allocated to a group of students; there are existing tenants (students who are occupants of a room), new applicants, and vacant rooms. Abdulkadiroğlu and Sönmez (1) first studied this problem. They introduce

a class of mechanisms to find an assignment: the top trading cycles mechanisms. The literature assumes that rents are exogenously given (see Miyagawa (16), Kesten (11) and, Sönmez and Ünver (21)).[19] Suppose the following situation. Incoming freshmen arrive and each one is assigned to an available room (according to the assignment policies of the central clearinghouse). Then, some of these students or some existing tenants may want to exchange their rooms. We introduce a mechanism which, based on the cardinal utility of each student, finds the reassignment of rooms to students and the rent of each room. Moreover, the price vector and the new assignment is a competitive equilibrium at which no student pays a higher rent than in the initial assignment. The mechanism is strategy-proof, and thus there is no point in misrepresenting preferences.

### 3.1.3 Related literature

Shapley and Shubik (20) introduce the Assignment Game under the assumptions that there is a bipartition in the set of agents, sellers and buyers, and that no agent has ever problems to "pay the bill". The authors analyze, in the first place, the core of the economy; that is, those allocations of goods such that no coalition of agents can find another allocation which is preferred by all its members, and, in the second place, the set of competitive equilibria. They prove that the core is nonempty and its equivalence with the set of competitive equilibria. In the same line, Shapley and Scarf (19) present a model without money in which each agent owns one indivisible good and wants to exchange it for another. They show that the market has nonempty core and at least one core allocation can be decentralized as a competitive equilibrium (and thus, there is a competitive equilibrium of the model). This last result is proved using an algorithm called the top trading cycles, due to David Gale.

Quinzii (17) presents a model that embeds the two previous models. In her article there are two types of agents, those that have an endowment consisting in one indivisible good and a

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[19]Miyagawa (16), for instance, studies the house allocation problem and, although monetary transfers are allowed, the price of each object is fixed in advance.

certain amount of money and those that only have money. She proves that the core of the market is not empty. To show the equivalence of the core and the set of competitive equilibria, Quinzii (17) supposes that the IM condition holds.[20] The condition can be understood as abundance of money: if agents initially have a lot of money, they would not give up all of their money to buy any object. Since Quinzii (17) many articles have worked under this assumption, for instance: Lars-Gunnar (13), van der Laan et al. (23), Beviá et al. (4), Yang (25), Saitoh (18) and Hwang and Shih (10).

Wako (24) allows for scarcity of money and studies the strong core of the economy; that is, those allocations such that no coalition of agents can find another allocation at which no member is worse off and at least one of them is strictly better off. He shows that the strong core always coincides with the set of competitive equilibria. However, if the IM assumption does not hold, both the strong core and the set of competitive equilibria can be strictly included in the core and also can disappear in some cases. Thus, the assumption is crucial to guarantee the existence of a competitive equilibrium.

A closely related work is Demange, Gale and Sotomayor (6), which studies the Assignment Game of Shapley and Shubik (20). The paper introduces a dynamic auction mechanism, named the Exact Auction Mechanism, to find a competitive equilibrium. It is shown that prices obtained by the mechanism are the minimum equilibrium prices. Also, it is well-known that the mechanism is strategy-proof (Demange and Gale (5)). This paper discusses on the extension of this mechanism to Quinzii's model in which budget-constrained agents act as both sellers and buyers.

The model that we study is a special case of the one presented by Quinzii (17). In the first place, we assume that all agents initially own one indivisible good and a certain amount of money. In Quinzii's model there are also agents that only have money and do not own any good. In the second place, we assume quasi-linear utility functions while Quinzii allows for general utility functions, although continuous and non decreasing with respect to the quantity

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[20]The same results, but using different arguments, are showed by Gale (7)

of money. Finally, we restrict our attention to the integer case. The main difference from Quinzii (17) is that the IM is not assumed.

The rest of the paper is laid out as follows. The next section presents an example of the problems addressed in the paper. Section 3 sets up the model and introduces the main concepts. Section 4 describes the two mechanisms that are analyzed and presents the main condition that guarantees the existence of a competitive equilibrium. The following section studies some strategic properties of the two mechanisms introduced before. Section 6 concludes.

## 3.2  An illustrative example

This section presents an example that illustrates the main problems studied in the rest of the paper. Suppose an economy with four agents: $\{a_1, a_2, a_3, a_4\}$, and four indivisible goods: $\{g_1, g_2, g_3, g_4\}$. Each agent initially owns one good and a certain amount of money. The valuation of each agent for each good and her initial amount of money is given by the following matrix (the element in the *ith* row and the *jth* column, denoted by $\alpha_{ij}$, is the valuation of agent $i$ for good $j$ and the last column is the initial amount of money owned by each agent).

|       | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $\omega$ |
|-------|-------|-------|-------|-------|----------|
| $a_1$ | 1     | 10    | 19    | 28    | 4        |
| $a_2$ | 1     | 11    | 21    | 31    | 3        |
| $a_3$ | 1     | 34    | 23    | 12    | 2        |
| $a_4$ | 25    | 37    | 13    | 1     | 1        |

Assume that agent $a_i$ initially owns good $g_i$ and her utility by holding an amount of money $m$ and good $g_j$ is, $\alpha_{ij} + m$. Suppose that each agent has no use for more than one indivisible good. This example is included in the model analyzed by Quinzii (17). However, note that the example does not satisfy the IM condition. Indeed, agent $a_1$ by holding good $g_2$ and without saving money enjoys an utility level higher than her initial one (which is 5).

Let us try to find a competitive equilibrium applying the mechanism of Demange, Gale and Sotomayor (6) (DGS). The mechanism uses the concept of minimal overdemanded set. A set of goods is overdemanded if the number of agents demanding goods only in the set is greater than the number of goods. In each step a price vector is announced and each agent's demand set (the set of affordable goods that maximize agent's utility) is computed. If it is possible to assign each good to an agent who demands it, the mechanism already finds a competitive equilibrium and stops. Otherwise, a minimal overdemanded set (an overdemanded set with the property that none of its proper subset is overdemanded) is located and the price of each good in this set is raised by one unit. Then, each agent's demand set is computed. If there are no budget constraints (or, if the IM condition holds) the model reduces to the Assignment Game and, as Demange, Gale and Sotomayor (6) show, the mechanism stops and finds a competitive equilibrium.

Next figure shows the demand set of every agent (denoted by $D_i(p_1, p_2, p_3, p_4)$) in each step of the mechanism.

|       | $D_i(0,0,0,0)$ | $D_i(0,1,0,0)$ | $D_i(0,2,0,0)$ | ... | $D_i(0,7,5,5)$ |
|-------|---------------|---------------|---------------|-----|---------------|
| $a_1$ | 4             | 4             | 4             |     | 1             |
| $a_2$ | 4             | 4             | 4             |     | 4             |
| $a_3$ | 2             | 2             | 2             |     | 2             |
| $a_4$ | 2             | 2             | 1             |     | 1             |

|       | $D_i(1,7,5,5)$ | $D_i(1,7,5,6)$ | $D_i(1,8,5,6)$ | $D_i(1,8,6,6)$ | ... |
|-------|---------------|---------------|---------------|---------------|-----|
| $a_1$ | 4             | 3             | 3             | 1             |     |
| $a_2$ | 4             | 4             | 4             | 4             |     |
| $a_3$ | 2             | 2             | 3             | 2             |     |
| $a_4$ | 1             | 2             | 1             | 1             |     |

Note that when the mechanism reaches the price vector $p = (0, 7, 5, 5)$, it enters in a loop and prices rise continuously. Thus, the mechanism fails to find a competitive equilibrium.

The previous example raises some important questions. First, it is not clear why the mechanism does not stop. It could be that there is no competitive equilibrium in the economy. Or, if there is a competitive equilibrium, it could be that the mechanism is not the adequate to find it. An additional difficulty is that we are interested in competitive equilibria with integer prices because our aim is to study the performance of the DGS mechanism.

## 3.3   The model

### 3.3.1   Agents and goods

We consider an economy with $n$ agents, $n$ indivisibles goods and a perfectly divisible good or money. Each agent initially owns a certain amount of money and one and only one indivisible good.

Let $A$ be the set of agents and $G$ the set of goods. We identify a generic agent with $a_i$ or $i$ and a generic good with $g_j$ or $j$. Each agent $i \in A = \{1, 2, ..., n\}$ initially holds good $i$ and has an amount of money $\omega_i \geq 0$. Thus, the endowment of agent $i$ is $e_i = (\omega_i, g_i)$.

We suppose that each agent has no use for more than one indivisible good and that utility is quasi-linear in money. Agent's preferences can be represented by a utility function $u_i$ defined on $\mathbf{R}_+ \times G$.[21] The valuation of agent $i$ for good $j$ is $\alpha_{ij}$. We assume that all valuations are integers and that no agent has the same valuation for two different goods; that is, for each agent $i$ it holds $\alpha_{ij} \neq \alpha_{ih}$, for all $j \neq h$. If $\alpha_{ij} > \alpha_{ih}$ we say that good $j$ is preferred over $h$ by agent $i$.

For each agent $i$, $v_i$ is her vector of valuations and $V = (\alpha_{ij})_{(i,j) \in A \times G}$ is the matrix of valuations of all agents. The notation $V_{-i}$ is used to denote the set of valuations of all agents different from $i$. Thus, the utility that agent $i$ has by holding a bundle $x = (m, g_j)$ composed with an amount of money $m$ and good $g_j$ is $u_i(x) = m + \alpha_{ij}$. The initial level of utility of

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[21]By $\mathbf{R}_+$ we denote the set of nonnegative real numbers.

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each agent $i$ is $u_i(e_i)$.

An economy is represented by $E = \{(e_i)_{i=1,...,n}, V\}$.

### 3.3.2 Allocations and prices

In what follows we recall some basic definitions.

An **allocation** is a vector in $(R_+ \times \{g_1, ..., g_n\})^n$. An allocation is **feasible** if there exists a permutation $\sigma$ of the set of indices $\{1, 2, ..., n\}$ such that the allocation is of the form $((m_1, g_{\sigma(1)}), (m_2, g_{\sigma(2)}), ..., (m_n, g_{\sigma(n)}))$ and $\sum_i m_i \leq \sum_i \omega_i$.

A **coalition** is a nonempty subset $S \subset A$. An allocation $\{(m_i, g_j)\}_{i,j \in \{1,2,...,n\}}$ is feasible for a coalition $S$ if $\sum_{i \in S} m_i \leq \sum_{i \in S} \omega_i$ and there is a permutation $\sigma_S$ of $S$ such that for all $i \in S$, $(m_i, g_j) = (m_i, g_{\sigma_S(i)})$.

The **core** of the economy is the set of feasible allocations such that no coalition $S$ can find an allocation feasible for $S$ which is strictly preferred by all its members. The **strong core** of the economy consists of those allocations such that no coalition $S$ can find an allocation feasible for $S$ at which no member is worse off and at least one of them is strictly better off. We observe that the strong core is a subset of the core.

A **price vector** is a $n-$vector with non negatives entries, denoted by $p = (p_1, ..., p_n)$. We assume that all prices are integers.

For each price vector $p$, $B_i(p)$ is the **budget set** of agent $i$; that is, is the set of all affordable goods. Thus,

$$B_i(p) = \{j \in G : \omega_i + p_i - p_j \geq 0\}.$$

For each price vector $p$ we define the **demand set** of agent $i$ by:

$$D_i(p) = \{j \in B_i(p) : \alpha_{ij} - p_j \geq \alpha_{ih} - p_h, \text{ for all } h \in B_i(p)\}.$$

That is, a good is an element of an agent demand set if its price satisfies the agent budget's constraint and maximizes her utility. Note that $D_i(p)$ is a finite and nonempty set for every price vector.

*Definition* 1. A **competitive equilibrium** is a pair consisting of a price vector $p$ and a feasible allocation $(m_i, \sigma(i))_{i=1,\dots,n}$ such that for all agent $i$, $\sigma(i) \in D_i(p)$.

The concept of overdemanded set is introduced in Demange, Gale and Sotomayor (6). The authors define a mechanism which finds a competitive equilibrium in the Assignment Game. In this mechanism (that ends at the minimum equilibrium price vector), prices are increased based on information about sets of goods that are overdemanded. A set of goods is overdemanded if the number of agents demanding goods only in the set is greater than the number of goods. It is well-known that a necessary condition for reaching a competitive equilibrium is that there be no overdemanded set (Hall (9)). However, by using information only related to overdemanded sets, it is not guaranteed that, in general, the mechanism finds a competitive equilibrium.[22] Demange, Gale and Sotomayor (6) solved the problem by increasing prices in a minimal overdemanded set; that is, a set with the property that none of its proper subset is overdemanded. We finish this section stating these concepts.

For every $H \subset G$ and price vector $p$, the set of agents demanding items only in $H$ is denoted by $O(H, p)$, formally: $O(H, p) = \{i \in A : D_i(p) \subseteq H\}$.

A subset of items $H$ is an **overdemanded set** at prices $p$ if $|O(H, p)| > |H|$.

An overdemanded set at prices $p$ with the property that none of its proper subsets is overdemanded, is called a **minimal overdemanded set**.

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[22]See an example in Anderson et al. (3).

## 3.4 Competitive equilibrium

In this section, two mechanisms are introduced and the existence of competitive equilibrium is addressed. We analyze the IM assumption introduced by Quinzii (17) and observe that under this condition budget constraints are irrelevant. However, when the condition does not hold we show, by means of an example, that a competitive equilibrium may not exist. Finally, the main result of this section is presented.

### 3.4.1 Mechanisms

We start by adapting to our setting the DGS mechanism (originally stated by Demange, Gale and Sotomayor (6)). In the second place, we describe the top trading cycles mechanism with price setting, which is used in the proof of the existence theorem.

#### 3.4.1.1 The Demange, Gale and Sotomayor mechanism

The mechanism runs as follows.

Let $p^t$ denote the price vector in step $t$ of the algorithm. Set $p^0 = 0$.

**Step 1**:

1. Input $p^0$.

2. Compute the demand set $D_i(p^0)$ for every agent $i \in A$.

3. If there are no overdemanded sets, stop. Otherwise, choose a minimal overdemanded set $S^0$.

4. Update prices such that: $p_i^1 = p_i^0 + 1$ for all $i \in S^0$ and $p_i^1 = p_i^0$ for all $i \notin S^0$.

*In general,*

**Step t**:

1. Input $p^{t-1}$.

2. Compute the demand set $D_i(p^{t-1})$ for every agent $i \in A$.

3. If there are no overdemanded sets, stop. Otherwise, choose a minimal overdemanded set $S^t$.

4. Update prices such that: $p_i^t = p_i^{t-1} + 1$ for all $i \in S^t$ and $p_i^t = p_i^{t-1}$ for all $i \notin S^t$.

In every step, the mechanism computes each agent's demand set. If it is possible to assign each good to an agent who demands it, the mechanism already finds a competitive equilibrium and stops. Otherwise, the auctioneer locates a minimal overdemanded set (which exists by Hall's theorem) and raises the price of each good in this set by one unit.[23] Then, each agent's demand set is computed. If there are no budget constraints, Demange, Gale and Sotomayor (6) show that in any market of the Assignment Game there is a competitive equilibrium for some prices and the mechanism stops; moreover, the mechanism is strategy-proof (Demange and Gale (5)).

### 3.4.1.2 The top trading cycles with price setting

The second mechanism, called the top trading cycles with price setting (TTCPS), consists of two phases. The first one is the top trading cycles (TTC) algorithm defined by Shapley and Scarf (19). In the second phase prices are set. The mechanism runs as follows.

**Phase 1: Top trading cycles algorithm**

A subset $C \subseteq A$ is a top trading cycle if all members of $C$ can be indexed in a cyclic order: $C = \{c_1, c_2, ..., c_s = c_1\}$ in such a way that each agent $c_j$ has in her demand set the $c_{j+1}$'st (modulo s) good. Take all prices equal zero and consider a directed graph in which we have a node for each agent and a directed edge from $i$ to $j$ if and only if agent $i$ demands the good of agent $j$. Since for each agent we assume that $\alpha_{ij} \neq \alpha_{ih}$ for all $j \neq h$, all nodes have outdegree equal to 1, and thus, there is at least one cycle and no two cycles intersect. Then

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[23]See Gale (8) for details about this result.

we partition the set $A$ into a sequence of one or more disjoint sets: $A = C^1 \cup C^2 \cup ... \cup C^k$ by taking $C^1$ to be all top trading cycles for $A$, then taking $C^2$ to be all top trading cycles for $A - C^1$, then taking $C^3$ to be all top trading cycles for $A - (C^1 \cup C^2)$, and so on until $A$ has been exhausted. We call this partition the **TTC partition**. Note that $k$, the number of elements of the partition, is also the number of steps in the TTC algorithm.[24] This procedure defines an assignment of goods in the economy $E$, called the **TTC assignment** and denoted by $\sigma^{TTC}(E)$.

**Phase 2: Price setting**

Consider the TTC partition, $A = C^1 \cup C^2 \cup ... \cup C^k$. We define the following price vector:

For all $i \in C^k$, set $p_i = 0$. Denote by $\omega^k$ the highest endowment of money of agents in the set $C^k$; that is, $\omega^k = \max_{i \in C^k}\{\omega_i\}$.

For all $i \in C^{k-1}$, set $p_i = \omega^k + 1$. Denote by $\omega^{k-1}$ the highest endowment of money of agents in the set $C^{k-1}$; that is, $\omega^{k-1} = \max_{i \in C^{k-1}}\{\omega_i\}$.

For all $i \in C^{k-2}$, set $p_i = \omega^k + \omega^{k-1} + 2$. Denote by $\omega^{k-2}$ the highest endowment of money of agents in the set $C^{k-2}$; that is, $\omega^{k-2} = \max_{i \in C^{k-2}}\{\omega_i\}$.

And so on until:

For all $i \in C^1$, set $p_i = \omega^k + \omega^{k-1} + ... + \omega^2 + k - 1$.

We denote by $p^{TTC}$ the obtained price vector. If no agent has endowment in money, the price setting procedure reduces to the one stated by Shapley and Scarf (19).

Finally, we have defined an assignment of goods $\sigma^{TTC}(E)$ and a price vector $p^{TTC}$.

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[24]Note that a top trading cycle may consist of a single agent and that $k \leq n$.

### 3.4.2 Existence

Quinzii (17) proves that any economy has nonempty core and that the core is equivalent to the set of competitive equilibria under the IM condition. This condition states that each no agent can enjoy a utility level higher than her initial utility level, without saving some money.

*Definition* 2. An economy $E = \{(e_i)_{i=1,\dots,n}, V\}$ satisfies the **indispensability of money** (IM) condition if $\alpha_{ii} + \omega_i \geq \alpha_{ij}$ for all agent $i$ and good $j$.

It is worth noting that under the IM condition, budget constraints do not bind. Indeed, suppose that for a price vector $p$ the good which maximizes the utility of agent $i$ is not affordable for the agent. Denote by $j \in G$ that good. Clearly, this good is not the one initially held by the agent. We have: $\alpha_{ij} - p_j \geq \alpha_{ih} - p_h$ for all $h$ and $\omega_i + p_i - p_j < 0$, which is equivalent to $p_j - p_i > \omega_i$. To the extent that this is true for all good $h$, we have also that relation for good $i$: $\alpha_{ij} - p_j \geq \alpha_{ii} - p_i$, therefore: $\alpha_{ij} - \alpha_{ii} \geq p_j - p_i$. But since good $j$ is not reachable for $i$ we have $\alpha_{ij} - \alpha_{ii} \geq p_j - p_i > \omega_i$. However, the IM condition implies that for all $i, j$, $\omega_i \geq \alpha_{ij} - \alpha_{ii}$. Therefore, under this condition budget constraints are irrelevant.

Thus, in order to study the effects of agents' budget constraints we may not assume the IM condition. But, as we show in the next example, if this condition does not hold, the existence of a competitive equilibrium is not guaranteed. Wako (24) presents an example with this result in which all agents are indifferent between two goods. Clearly, the existence of indifferences in agents' valuations increases the opportunities to a coalition to weakly improve upon the allocations. We present a new example in which there are no indifferences and however, the set of competitive equilibria is empty.

*Example* 1. Consider $A = \{a_1, a_2, a_3\}$ and $G = \{g_1, g_2, g_3\}$ and the following valuations and initial amounts of money:

|       | $g_1$ | $g_2$ | $g_3$ | $\omega$ |
|-------|-------|-------|-------|----------|
| $a_1$ | 5     | 1     | 8     | 1        |
| $a_2$ | 1     | 5     | 8     | 2        |
| $a_3$ | 5     | 4     | 3     | 3        |

Since Wako (24) we know that the strong core always coincides with the set of competitive equilibria. Note that there are only two allocations that may be in the strong core: $X_1 = ((1, g_1), (0, g_3), (5, g_2))$ and $X_2 = ((0, g_3), (2, g_2), (4, g_1))$. In the first case, coalition $\{a_1, a_3\}$ can weakly improve allocation $X_1$ with the allocation $((0, g_3), (4, g_1))$. In the second case, coalition $\{a_2, a_3\}$ can weakly improve $X_2$ with the allocation $((0, g_3), (5, g_2))$. Thus, the strong core of the economy is empty and there is no competitive equilibrium. ∎

The example raises the question of which condition, different from the IM condition, guarantees the existence of a competitive equilibrium. We have noted that the IM condition can be viewed as abundance of money. It is quite surprising that the key components of the condition that ensures the existence of equilibrium are one related to the scarcity of money and the other related to the number of steps in the TTC algorithm.

*Definition* 3. Consider an economy $E = \{(e_i)_{i=1,\dots,n}, V\}$. Suppose that the TTC algorithm consists of $k$ steps. The economy satisfies the **weak version of the money scarcity condition** (WMS) if for any agent $i$ and different goods $h, l$ we have $|\alpha_{ih} - \alpha_{il}| \geq \sum_{j \in A \setminus \{i\}} \omega_j + k - 1$.

Some comments about the condition are in order. It requires that each agent should consider goods as heterogeneous. In the second place, the condition is independent of each agent's endowment in money and also of the initial distribution of money. Finally, while the WMS condition indicates a specific lower bound of the difference between any two valuations, the IM condition states an upper bound. Indeed, suppose there are two goods $j, h$ and one agent $i$ such that $\alpha_{ij} \geq \alpha_{ii}$ and $\alpha_{ih} \geq \alpha_{ii}$. Then, we have $|\alpha_{ij} - \alpha_{ih}| = |\alpha_{ij} - \alpha_{ii} + \alpha_{ii} - \alpha_{ih}| \leq |\alpha_{ij} - \alpha_{ii}| + |\alpha_{ii} - \alpha_{ih}| \leq 2m_i$. Thus, the IM condition implies that the difference between

any two valuations should be lower than the double of the initial amount of money that the agent has. The main result of this section is the following.

*Theorem 1. Consider an economy $E = \{(e_i)_{i=1,\dots,n}, V\}$.*

1. *If $E$ satisfies the IM condition, there is a competitive equilibrium which can be reached by the DGS mechanism.*

2. *If $E$ satisfies the WMS condition, there is a competitive equilibrium which can be reached by the TTCPS mechanism.*

*Proof.* A proof of the first part can be found in Quinzii (17) and Demange, Gale and Sotomayor (6).

For the second part we have to prove that $p^{TTC}$ and $((\omega_i, \sigma^{TTC}(i))_{i \in A}$ is a competitive equilibrium. Fix an agent $i$ and denote by $\sigma(i)$ the good assigned by the TTC algorithm to this agent. We shall prove that:

(1) $\omega_i + p_i - p_{\sigma(i)} \geq 0$ and,

(2) $\alpha_{i\sigma(i)} + p_i - p_{\sigma(i)} \geq \alpha_{ih} + p_i - p_h$ for all $h$ such that $\omega_i + p_i - p_h \geq 0$.

Assertion (1) is straightforward since good $\sigma(i)$ leaves the market with good $i$ and then we have $p_i = p_{\sigma(i)}$.

For (2), suppose that agent $i$ leaves the market in the step $k - r$ of the procedure ($0 \leq r < k$). Then, for all good $l$ such that $a_l$ leaves the market in the step $j < k - r$, we have $p_l \geq p_s = \omega^k + \omega^{k-1} + \dots + \omega^{k-r} + r + 1$ for all $s$ such that $a_s \in C^{k-r-1}$ with $\omega^{k-r} \geq \omega_i$. Also, $p_i = \omega^k + \omega^{k-1} + \dots + \omega^{k-r+1} + r$. Thus, $p_i - p_s = -\omega^{k-r} - 1 = -(\omega^{k-r} + 1)$ with $\omega^{k-r} \geq \omega_i$. This means that $i$ cannot achieve goods of agents that leave the market in any step $j < k - r$.

Finally, we have to verify that for any good $j$ such that $a_j \in C^t$ with $t = k - r, \dots, k$ the following holds: $\alpha_{i\sigma(i)} \geq \alpha_{ij} + p_i - p_j$. By way of contradiction, suppose there is a good $j \in C^t$ with $t = k - r, \dots, k$ such that $\alpha_{i\sigma(i)} < \alpha_{ij} + p_i - p_j$. Then, $0 \leq \alpha_{i\sigma(i)} - \alpha_{ij} < p_i - p_j \leq$

$p_i = \omega^k + \omega^{k-1} + ... + \omega^{k-r+1} + r$. Since $r \leq k-1$, we have that $\alpha_{i\sigma(i)} - \alpha_{ij} < \omega^k + \omega^{k-1} + ... + \omega^2 + k - 1 \leq \sum_{j \in A \setminus \{i\}} \omega_j + k - 1$. Thus, we conclude that $\alpha_{i\sigma(i)} - \alpha_{ij} < \sum_{r \in A \setminus \{i\}} \omega_r + k - 1$, which is a contradiction. ■

Theorem 1 shows that there is a discontinuity in the existence of competitive equilibrium in relation to the total amount of money that initially exists in the economy. If there is abundant money in the economy (IM condition) or if money is very scarce (WMS condition), there is a competitive equilibrium. For other cases between those situations (for instance, Example 1), we cannot guarantee the existence of a competitive equilibrium. Also, as we note in the following example, the WMS condition cannot be relaxed.

*Example* 2. Consider the following economy.

|       | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $\omega$ |
|-------|-------|-------|-------|-------|----------|
| $a_1$ | 1     | 25    | 13    | 37    | 4        |
| $a_2$ | 1     | 25    | 13    | 37    | 2        |
| $a_3$ | 1     | 25    | 13    | 37    | 3        |
| $a_4$ | 25    | 1     | 13    | 36    | 1        |

In the first place, note that the WMS condition holds for agents $a_1, a_2$ and $a_3$. For agent $a_4$, the condition does not hold because $\alpha_{44} - \alpha_{41} = 11 < \sum_{i=1}^{3} \omega_i + 3 = 12$. It is straightforward to verify that the unique allocation in the core of the economy is $((4, g_1), (2, g_2), (3, g_3), (1, g_4))$. In Appendix A.1 we prove that there is no integer price vector $p$ such that the allocation defined previously and prices $p$ are a competitive equilibrium.[25] Thus, we conclude that there is no competitive equilibrium in this economy. ■

Note that price vector $p^{TTC}$ satisfies that $p_i = p_{\sigma(i)}$ for all agent $i$. Thus, at the competitive equilibrium found by the TTCPS mechanism, there are no money transfers between agents and the final assignment of money is equal to its initial distribution. In this sense, we say that

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[25]Note that if we allow real price vectors, the allocation is a competitive equilibrium associated with, for example, $p = (0.9, 9, 5, 11.9)$

the TTCPS minimizes money transactions. Moreover, the assignment of goods defined by the TTCPS is the unique assignment with that property. Thus, we have a characterization of the TTCPS, which is introduced after presenting the mechanism definition.

*Definition* 4. A **mechanism** is a function that assigns a competitive equilibrium for each economy.

*Proposition* 1. A mechanism minimizes money transactions if and only if it is the TTCPS mechanism.

(See Appendix A.2 for a proof)

As we just noted, at the competitive equilibrium found by the TTCPS mechanism, there are no money transfers between agents and the final assignment of money is equal to its initial distribution. In order to adjust prices, we introduce a mechanism which includes a final phase that finds the minimum equilibrium price vector associated with the assignment found by the TTC mechanism.

Consider an economy $E = \{(e_i)_{i=1,\ldots,n}, V\}$. Let $p^{TTC}$ and $((\omega_i, \sigma^{TTC}(i)))_{i \in A}$ be the competitive equilibrium defined by the TTCPS. Starting with the price vector $p^{TTC}$ we apply the following algorithm:

**Step 1:** Decrease the price of good 1 by one unit until either an overdemanded set appears or it is zero. In the last case, go to the following step. When an overdemanded set is found, fix the price of good 1 in its previous value (at which there is no overdemanded set) and continue with the next step.

**Step $t \leq n$:** Decrease the price of good $t$ by one unit until either an overdemanded set appears or it is zero. When an overdemanded set is found, fix the price of good $t$ in its previous level (at which there is no overdemanded set). If the final price of good $t$ is lower than its value at the beginning of the step, go to Step 1. Otherwise, go to the following step.

The algorithm ends when no price can be decreased.

Let $\hat{p}$ denote the final price vector obtained by this procedure. The new proposed mechanism is the following.

*Definition* 5. The **extended top trading cycles (ETTC)** mechanism is the function that assigns for each economy $E = \{(e_i)_{i=1,\ldots,n}, V\}$ the competitive equilibrium formed by the price $\hat{p}$ and the allocation $(\omega_i + \hat{p}_i - \hat{p}_{\sigma^{TTC}(i)}, \sigma^{TTC}(i))_{i \in A}$.

It is straightforward to verify that $\hat{p}$ is the minimum equilibrium price vector associated with the allocation of goods $(\sigma^{TTC}(i))_{i \in A}$. That is, any price vector that constitutes a competitive equilibrium with the previous allocation, verifies that its components are greater than or equal to the components of $\hat{p}$. However, as we show in the next example, $\hat{p}$ may not be the minimum equilibrium price vector.

*Example* 3. Consider the following economy.

|       | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $\omega$ |
|-------|-------|-------|-------|-------|----------|
| $a_1$ | 21    | 31    | 11    | 1     | 1        |
| $a_2$ | 28    | 1     | 10    | 19    | 2        |
| $a_3$ | 25    | 1     | 9     | 17    | 3        |
| $a_4$ | 1     | 15    | 22    | 8     | 4        |

Note that the economy in the example satisfies the WMS condition. In the first place, we apply the TTC in order to find the TTC partition; we get $A = C^1 \cup C^2$ with $C^1 = \{a_1, a_2\}$ and $C^2 = \{a_3, a_4\}$. Since $\omega^2 = \omega_4 = 4$ we have the following TTC induced price vector $p = (p_1, p_2, p_3, p_4) = (5, 5, 0, 0)$. In the final phase we obtain the price vector: $\hat{p} = (4, 2, 0, 0)$. Thus, the competitive equilibrium found by the ETTC is $\hat{p} = (4, 2, 0, 0)$ and $((3, g_2), (0, g_1), (3, g_4), (4, g_3))$. However, the minimum equilibrium price vector is $\hat{p} = (3, 0, 0, 0)$ which is associated with the allocation $((4, g_2), (2, g_4), (0, g_1), (4, g_3))$. ■

One important result of Shapley and Shubik (20) is the existence of a unique minimum equilibrium price vector, smallest in the strong sense that this price vector is at least as

small in every component as any other equilibrium price vector. We finish the section by proving that this result does not hold in our model.

*Proposition* 2. Consider an economy $E = \{(e_i)_{i=1,\ldots,n}, V\}$. Then, there may not exist a minimum equilibrium price vector.

(See Appendix A.3 for a proof)

## 3.5  Strategic questions

In this section we study some strategic questions about the mechanisms introduced previously. After some comments on the TTCPS mechanism, it is shown that the ETTC is strategy-proof if a slightly stronger condition than WMS holds, and that the DGS mechanism does not satisfy this property, even under the same condition. Then, we analyze the existence of Nash equilibria in the revelation game induced by the DGS mechanism. A link between these two mechanisms is specified by showing that the assignment of goods in a Nash equilibrium coincides with the assignment found by the ETTC mechanism; thus, it is Pareto efficient. Finally, we identify some conditions under which the DGS mechanism stops and finds a competitive equilibrium.

### 3.5.1  Strategy-proofness

Before presenting the results, we need some definitions.

Denote by $\phi_i(e, V)$ the assignment of agent $i$ by the mechanism $\phi$ when endowments are $e = (e_i)_{i=1,\ldots,n}$ and agents' valuations are $V$.

A mechanism induces a game in which agents have to reveal some information (for example, valuations) and the mechanism determines the assignment of each agent. In this section we study two games which have the following common elements:

1. the set $A$ of all players,

2. the set of strategies of each player $i$: $S_i$

3. the set of strategy profiles defined by $S = \prod_{i=1}^{n} S_i$, and

4. the payoff function of each player, defined by $u_i : S \to R$ such that if $i$ gets $g_j$ at price $p_j$ and her good's price is $p_i$, her payoff is $\omega_i + p_i + \alpha_{ij} - p_j$ and $\alpha_{ii} + \omega_i$, otherwise.

In the first place, we consider the game in which each agent has to declare her endowment in money and valuations. Moreover, we suppose that true and reported valuations belong to a certain domain of valuations $D \subset \mathbb{Z}_+^n$. Thus, the set of strategies of each agent is $S_i = R_+ \times D$. If truthful revelation is a weakly dominant strategy for each agent in the preference revelation game induced by the mechanism, then we say that the mechanism is **strategy-proof on the domain** $D$. In the second place, suppose that the endowment in money of each agent is common knowledge and then, the set of strategies of each agent is $S_i = D$. If truthful revelation is a weakly dominant strategy for each agent in the preference revelation game induced by the mechanism, then we say that the mechanism is **strategy-proof on valuations on the domain** $D$. Formal definitions are the following.

*Definition* 6. Suppose an economy in which each agent's valuations belong to a certain domain of valuations $D \in \mathbb{Z}_+^n$. A mechanism $\phi$ **is strategy-proof on the domain** $D$ if for each $V \in D^n$, $e$ and $i \in A$, $u_i(\phi_i(e_{-i}, e_i, V_{-i}, v_i)) \geq u_i(\phi_i(e_{-i}, \hat{e}_i, V_{-i}, \hat{v}_i))$ for all $\hat{v}_i \in D$ and $\hat{e}_i \in \{g_i\} \times \mathbb{R}_+$.[26]

*Definition* 7. Suppose an economy in which each agent's valuations belong to a certain domain of valuations $D \subset \mathbb{Z}_+^n$ and that the endowment in money of each agent is common knowledge. A mechanism $\phi$ is **strategy-proof on valuations on the domain** $D$ if for each $V \in D^n$ and $i \in A$, $u_i(\phi_i(e, V_{-i}, v_i)) \geq u_i(\phi_i(e, V_{-i}, \hat{v}_i))$ for all $\hat{v}_i \in D$.

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[26]We adopt the notation $e_{-i}$ for the endowment of all agents different from $i$.

### 3.5.1.1 The top trading cycles with price setting mechanism

As we state in the next theorem, the TTCPS mechanism is strategy-proof on the domain defined by the WMS condition.

Let $D^{\text{WMS}}$ denote the domain of valuations defined by the WMS condition.

*Theorem* 2. *Suppose an economy that satisfies the WMS condition, then the TTCPS mechanism is strategy-proof on* $D^{\text{WMS}}$.

*Proof.* Note that the assignment of goods is computed by the TTC algorithm, without considering the initial amount of money of each agent. Thus, the proof presented in Abdulkadiroğlu and Sönmez (2) can be adapted to our framework to show that no agent can benefit by announcing valuations different from her true valuations.[27] In the phase 2 when prices are set, by announcing a different endowment of money, an agent does not change the price of her good neither the price of the good she is assigned to (which are always equal). Thus, truthful revelation is a weakly dominant strategy for each agent. ∎

### 3.5.1.2 The extended top trading cycles mechanism

In Appendix A.4 we present an economy that satisfies the WMS condition in which truthful revelation is not a weakly dominant strategy for some agents. Thus, the mechanism is not strategy-proof on valuations on the domain defined by the WMS condition (which implies that it is not strategy-proof). The economy should satisfy the following condition, slightly stronger than the WMS, to guarantee that the mechanism is strategy-proof on valuations.

*Definition* 8. Consider an economy $E = \{(e_i)_{i=1,...,n}, V\}$. Suppose that the TTC algorithm consists of $k$ steps. The economy satisfies the **money scarcity** (MS) condition if for any agent $i$ and different goods $h, l$ we have $|\alpha_{ih} - \alpha_{il}| \geq \sum_{j=1}^{n} \omega_j + k - 1$.

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[27]See Appendix A.5.

Let $D^{\mathrm{MS}}$ denote the domain of valuations defined by the MS condition.

*Theorem* 3. *Suppose an economy that satisfies the MS condition, then the ETTC mechanism is strategy-proof on valuations on $D^{MS}$.*

(See Appendix A.5 for a proof)

We must note that even if the MS condition holds, the ETTC mechanism is not strategy-proof on $D^{MS}$. Consider the example of Section 2. The economy satisfies the MS condition and the ETTC mechanism finds the following competitive equilibrium: $p = (0, 8, 5, 7)$ and $((4, g_1), (4, g_4), (2, g_3)(0, g_2))$. If $a_4$ declares the following endowment of money: $\omega_4 = 0$, final prices are $\hat{p} = (0, 8, 5, 8)$ and she benefits since she exchanges her good with agent $a_2$ but she does not pay for it.

### 3.5.1.3 The Demange, Gale and Sotomayor mechanism

The next example shows that the DGS mechanism is not strategy-proof on valuations on the domain defined by the MS condition (and thus, it is not strategy-proof).

*Example* 4. Consider the following economy.

|       | $g_1$ | $g_2$ | $g_3$ | $\omega$ |
|-------|-------|-------|-------|----------|
| $a_1$ | 8     | 15    | 1     | 1        |
| $a_2$ | 16    | 2     | 9     | 2        |
| $a_3$ | 18    | 11    | 4     | 3        |

The DGS mechanism finds the following competitive equilibrium: $p = (3, 0, 0)$ and $\sigma = ((4, g_2), (2, g_3), (0, g_1))$. Note that if agent $a_2$ announces its own good as her second option, she receives $g_1$. Indeed, if $a_2$ reveals valuations $\hat{v}_2 = (16, 9, 2)$ the competitive equilibrium found by the mechanism is $\hat{p} = (4, 4, 0)$ and $\hat{\sigma} = ((1, g_2), (2, g_1), (3, g_3))$. Therefore, $a_2$ obtains a higher utility if she reveals a vector of valuations different from her true valuations. Finally, note that the economy satisfies the MS condition.∎

We must note that, in the last example, for agent $a_2$ to manipulate the mechanism, she does not need any information about other agents' valuations. She only needs to observe which objects were demanded by each of the other agents. The same observation was noted by Sotomayor (22) in the mechanism that she proposes. As she remarks, this situation contrasts with the existent results in the literature, where an agent needs to have a large amount of information to identify profitable strategic manipulations.

We know that truthful revelation is not a weakly dominant strategy in the revelation game induced by the DGS mechanism. Next, we study the existence of equilibrium in this game. We focus on the game in which each agent has to reveal her valuations and endowments in money are common knowledge, and we search for a Nash equilibrium.

A strategy profile $(s_1^*, ..., s_n^*)$ is a **Nash equilibrium** of the game induced by the DGS mechanism, if there is no individual $i \in A$ and no strategy $s_i \in S_i$ such that $u_i(s_i, s_{-i}^*) > u_i(s_1^*, ..., s_n^*)$.

In order to find a Nash equilibrium of the revelation game, we introduce a set of strategies, called dropping strategies. A **dropping strategy** consists of a vector $s_i \in \mathbb{Z}_+^n$ such that if $\sigma^{TTC}(i)$ is the good assigned to agent $i$ by the TTC procedure, the highest valuation in $s_i$ corresponds to $\sigma^{TTC}(i)$ and the second highest valuation corresponds to $g_i$ (her own good). A **dropping profile** is a strategy profile $(s_1, ..., s_n)$ such that each $s_i$ is a dropping strategy. [28]

As we show in the next example, a dropping profile may not be a Nash equilibrium of the revelation game induced by the DGS mechanism, even when the economy satisfies the MS condition. In order for a dropping profile to be a Nash equilibrium, the economy should satisfy a stronger condition, which is defined after the example.

*Example* 5. Suppose the following economy.

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[28]This type of strategies was originally introduced by Kojima and Pathak (12).

| | $g_1$ | $g_2$ | $\omega$ |
|---|---|---|---|
| $a_1$ | 1 | 3 | 1 |
| $a_2$ | 2 | 1 | 0 |

Note that each agent plays a dropping strategy and that the DGS mechanism stops at the first round. Also, we have that $k = 1$. Since $\alpha_{21} - \alpha_{22} = 1 = \omega_1 + \omega_2$, the economy satisfies the MS condition. The competitive equilibrium found by the DGS mechanism is: $p = (0,0)$ and $((1, g_2), (0, g_1))$, and agent $a_2$ gets an utility of 2. If $a_2$ announces the following valuations, $v_2' = (1, 2)$, the new competitive equilibrium found by the mechanism is $p' = (0, 1)$ and $((0, g_2), (1, g_1))$, and agent $a_2$ gets an utility of 3. Thus, in this economy that satisfies the MS condition, a dropping profile is not a Nash equilibrium. ∎

*Definition* 9. Consider an economy $E = \{(e_i)_{i=1,\ldots,n}, V\}$. The economy satisfies the **strong money scarcity condition** if for any agent $i$ and different goods $h, l$, we have $|\alpha_{ih} - \alpha_{il}| \geq \sum_{j \in A \setminus \{i\}} \omega_j + n - 1$.

Example 5 presents an economy which satisfies the MS condition but does not satisfy the strong version of the condition. Indeed, since $n = 2$ we have that $\alpha_{21} - \alpha_{22} = 1 < m_1 + 1 = 2$.

A dropping profile satisfies the strong MS condition if each agent's strategy satisfies the strong MS condition.

*Theorem 4. Suppose an economy that satisfies the strong MS condition, then a dropping profile that satisfies the strong MS condition is a Nash equilibrium of the revelation game induced by the DGS mechanism.*

(See Appendix A.6 for a proof)

The last theorem shows that the assignment of goods of the ETTC mechanism is the outcome of a Nash equilibrium in the revelation game induced by the DGS mechanism. Note that final prices of each mechanism may not be equal.

### 3.5.2 Convergence of the DGS mechanism

In Section 4 we proved that the WMS condition guarantees the existence of a competitive equilibrium which can be found by the ETTC mechanism. In this section we investigate if the DGS mechanism also finds a competitive equilibrium whenever it exists.

The economy introduced in Section 2 satisfies the strong MS condition and, however, the DGS mechanism fails to find a competitive equilibrium. In that economy the mechanism does not stop and prices increase indefinitely. On the other hand, the outcome of the ETTC mechanism is $p = (0, 8, 5, 7)$ and $((4, g_1), (4, g_4), (2, g_3)(0, g_2, ))$.

The example of Section 2 raises the following question: under which conditions does the DGS mechanism stop and find a competitive equilibrium? We finish this section with the definition of a strong condition to guarantee that this mechanism stops. It is worth noting that the DGS mechanism always stops and finds a competitive equilibrium when it is applied to a three agents economy whenever the economy satisfies the WMS condition and that $\omega_i \neq \omega_j$ for all $i, j \in A$ such that $i \neq j$.

The conditions that ensure the convergence of the mechanism to a competitive equilibrium are very restrictive, however, we include the result in order to answer the question posited at the beginning of the paper and, to stress the inadequacy of this mechanism in our framework.

For each $i \in A$, her valuation vector $v_i$ defines a ranking of goods: $j$ is ranked above $h$ if $\alpha_{ij} > \alpha_{ih}$. Let $O^i(j)$ be the place of good $j$ in the ranking of agent $i$ (for instance, if $j$ is the good with highest valuation at $v_i$, $O^i(j) = 1$).

*Theorem 5. Suppose an economy $E = \{(e_i)_{i=1,\dots,n}, V\}$ such that:*

1. *it satisfies the WMS condition,*

2. *for each agent $i$, $O^i(i) \geq 2$; that is, for each agent there is at most one good preferred over her own good.*

*Then, the DGS mechanism stops and finds a competitive equilibrium.*

(See Appendix A.7 for a proof)

In the last proposition, the WMS condition cannot be discarded. We finish this section with an example illustrating this remark.

*Example* 6. Suppose the following economy:

|       | $g_1$ | $g_2$ | $g_3$ | $\omega$ |
|-------|-------|-------|-------|----------|
| $a_1$ | 3     | 6     | 1     | 2        |
| $a_2$ | 3     | 2     | 1     | 3        |
| $a_3$ | 1     | 7     | 2     | 4        |

Note that the economy satisfies the second condition of the last theorem but does not satisfy the WMS condition. One can easily show that the DGS mechanism does not stop when it is applied to this economy. Thus, the WMS condition is a necessary assumption for the last result. ∎

## 3.6   Concluding remarks

In this paper, we study an economy with indivisible goods, money and budget-constrained agents with unit-demand, who act as both sellers and buyers. In this model, a widely used assumption is the indispensability of money, originally stated by Quinzii (17). As many authors have remarked, the assumption can be understood as abundance of money in the economy. We show that if this assumption holds, budget constraints never bind. On the other hand, the assumption is crucial to guarantee the existence of competitive equilibrium.

We contribute to the literature by extending the original result of Quinzii on the existence of competitive equilibrium to cases where money is scarce and utility functions are quasi-linear. Moreover, an extended version of the top trading cycles algorithm, which is strategy-proof

under the money scarcity condition, is defined in order to find a competitive equilibrium. We show also that the mechanism of Demange, Gale and Sotomayor is not strategy-proof and we find a set of Nash equilibria of the revelation game induced by this mechanism. Finally, we identify a stronger condition than money scarcity to guarantee the convergence of this mechanism to a competitive equilibrium in the general model.

The results proven in the paper show that the ETTC is a suitable mechanism to find a competitive equilibrium in the economies that have been analyzed. In particular, we have demonstrated that the mechanism is strategy-proof on valuations under the money scarcity condition. The next step involves a characterization of those mechanisms that are strategy-proof in this framework. Recent studies have made some advances in this sense. For example, Morimoto and Serizawa (14) have characterized the rule that assigns a minimum price Walrasian equilibrium allocation in the Assignment Game. A similar characterization in our model where budget-constrained agents act as both sellers and buyers, requires a deep analysis and goes beyond the scope of the present work.

It could be interesting to study the extension of these results to the case in which there are also agents that only have money and do not own any good. In that case, the extended version of the top trading cycles cannot be used because agents without a good will fail to participate in a cycle. Thus, a new mechanism should be designed to find a competitive equilibrium in that setting.

## 3.7 Appendix

### A.1 Proof of Example 2

It is straightforward to verify that the unique allocation in the core of the economy is $((4, g_1), (2, g_2), (3, g_3), (1, g_4))$. We have to prove that there is no integer price vector $p$ such that the allocation defined previously and prices $p$ are a competitive equilibrium. By

way of contradiction, suppose there is a integer price vector $p = (p_1, p_2, p_3, p_4)$ such that $((4, g_1), (2, g_2), (3, g_3), (1, g_4))$ and $p$ are a competitive equilibrium.

In the first place, we claim that $p_4 > p_2$. Indeed, suppose that $p_4 \leq p_2$; then, $\omega_2 + p_2 - p_4 \geq 0$ and $\alpha_{24} + p_2 - p_4 > \alpha_{22}$, but this means that $a_2$ does not demand $g_2$, a contradiction. In the same way, it can be shown that $p_2 > p_3$ and $p_3 > p_1$.

Therefore, we have the following inequalities: $p_4 > p_2 > p_3 > p_1$.

In the second place, we know that each agent demands her own good; that is, $g_i \in D_i(p)$. In order to this result holds, we should have:

1. For agent $a_1$:

   (1.1) $4 + p_1 - p_2 < 0$ or, if $4 + p_1 - p_2 \geq 0$ the following should hold: $1 \geq 25 + p_1 - p_2$. But the second case is equivalent to $p_2 - p_1 \geq 24$ if $p_2 - p_1 \leq 4$, which is not possible. Then, the first relation should hold.

   (1.2) $4 + p_1 - p_3 < 0$ or, if $4 + p_1 - p_3 \geq 0$ the following should hold: $1 \geq 13 + p_1 - p_3$. As in the previous case, the second relation is not possible.

   (1.3) $4 + p_1 - p_4 < 0$ or, if $4 + p_1 - p_4 \geq 0$ the following should hold: $1 \geq 37 + p_1 - p_4$. As in the first case, the second relation is not possible.

2. For agent $a_2$:

   (2.1) Since $p_2 > p_1$, we should have $25 \geq 1 + p_2 - p_1$, and thus, $24 \geq p_2 - p_1$.

   (2.2) Since $p_2 > p_3$ we should have $25 \geq 13 + p_2 - p_3$, and thus $12 \geq p_2 - p_3$.

   (2.3) $2 + p_2 - p_4 < 0$ or, if $2 + p_2 - p_4 \geq 0$ then, $25 \geq 37 + p_2 - p_4$. But, as with agent $a_1$, the last case is not possible.

3. For agent $a_3$:

   (3.1) Since $p_3 > p_1$, we should have $13 \geq 1 + p_3 - p_1$, and thus $12 \geq p_2 - p_1$.

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(3.2) $3 + p_3 - p_2 < 0$ or, if $3 + p_3 - p_2 \geq 0$ then, $13 \geq 25 + p_3 - p_2$. Note that the last case is not possible.

(3.3) $3 + p_3 - p_4 < 0$ or, if $3 + p_3 - p_4 \geq 0$ then, $13 \geq 37 + p_3 - p_4$. But note that the last case is not possible.

4. For agent $a_4$:

(4.1) Since $p_4 > p_1$, then we should have $36 \geq 25 + p_4 - p_1$, and thus $11 \geq p_4 - p_1$.

(4.2) Since $p_4 > p_2$, then we should have $36 \geq 1 + p_4 - p_2$, and thus $35 \geq p_4 - p_2$.

(4.3) Since $p_4 > p_3$, then we should have $36 \geq 13 + p_4 - p_3$, and thus $23 \geq p_4 - p_3$.

Thus, the following inequalities should hold:

(1) $4 < p_2 - p_1 \leq 24$

(2) $4 < p_3 - p_1 \leq 12$

(3) $4 < p_4 - p_1 \leq 11$

(4) $3 < p_2 - p_3 \leq 12$

(5) $2 < p_4 - p_2 \leq 35$

(6) $3 < p_4 - p_3 \leq 23$

Note that $11 \geq p_4 - p_1 = p_4 - p_2 + p_2 - p_1$ and, since $p_4 - p_2 > 0$, we have $p_2 - p_1 \leq 10$. Following in the same way:

$11 \geq p_4 - p_1 = p_4 - p_3 + p_3 - p_1$ and, since $p_3 - p_1 > 0$, we have $p_4 - p_3 \leq 10$.
$11 \geq p_4 - p_1 = p_4 - p_2 + p_2 - p_3 + p_3 - p_1$ and then, $p_4 - p_2 \leq 9$, $p_2 - p_3 \leq 9$, $p_3 - p_1 \leq 9$.
$p_4 - p_3 = p_4 - p_2 + p_2 - p_3$ and since inequalities (4) and (5), we have $p_4 - p_3 > 5$.
$p_4 - p_1 = p_4 - p_2 + p_2 - p_3 + p_3 - p_1$ and since inequalities (2), (4) and (5), we have $p_4 - p_1 > 9$.

Thus, the last set of inequalities becomes:

(1) $4 < p_2 - p_1 \leq 10$

(2) $4 < p_3 - p_1 \leq 9$

(3) $9 < p_4 - p_1 \leq 11$

(4) $3 < p_2 - p_3 \leq 9$

(5) $2 < p_4 - p_2 \leq 9$

(6) $6 \leq p_4 - p_3 \leq 10$

We can suppose, without loss of generality, that $p_1 = 0$; therefore, we have only two possible values for $p_4$: $p_4 = 10$ or $p_4 = 11$.

1. If $p_4 = 10$, by inequality (6) we know that $p_3 \leq 4$, which is not possible since inequality (2).

2. If $p_4 = 11$, by inequalities (5) and (6), we know that $2 \leq p_2 < 9$ and $1 \leq p_3 \leq 5$. But, since this last relation and inequality (2) we have that $p_3 = 5$ and then, by (4), $p_2 > 8$ which contradicts that $2 \leq p_2 < 9$.

Therefore, there is no integer price vector such that the unique allocation of the core and prices $p$ are a competitive equilibrium.

## A.2 Proof of Proposition 1

*Proposition 1: A mechanism minimizes money transactions if and only if it is the TTCPS mechanism.*

*Proof.* At the competitive equilibrium found by the TTCPS, there are no money transfers. Denote by $p^{TTC}$, $(\omega_i, \sigma^{TTC}(i))$ the TTCPS competitive equilibrium. We know that $p_i = p_{\sigma^{TTC}(i)}$. We will prove that there is no other competitive equilibrium with an allocation of goods $\tau$ such that $p_i = p_{\tau(i)}$. Suppose, by way of contradiction, that there is a competitive

equilibrium $q$, $(\omega_i, \tau(i))$, such that $q_i = q_{\tau(i)}$. Consider the TTC partition, $A = C^1 \cup C^2 \cup ... \cup C^k$. Note that, at $\sigma^{TTC}$, all agents in $C^1$ receive their most preferred good. Assume there is an agent in $C^1$ such that she receives at $\tau$ a different good from the one she receives at $\sigma$. Then, the allocation of goods defined by $C^1$ is feasible for $C^1$ and no member is worse off and at least one of them is strictly better off at $C^1$ respect to the assignment $\tau$. Thus, the assignment $(\omega_i, \tau(i))$ is not in the strong core of the economy, which is a contradiction. Then, both competitive equilibria have the same allocation of goods for agents in $C^1$. Suppose now that there is an agent in $C^2$ such that she receives at $\tau$ a different good from the one she receives at $\sigma$. All agents in $C^2$ receive at $\sigma^{TTC}$ their most preferred good of those that were left. A similar argument as before shows that the allocation of goods defined by $C^1 \cup C^2$ is feasible for $C^1 \cup C^2$ and no member is worse off and at least one of them is strictly better off at $\sigma^{TTC}$ respect to the assignment $\tau$. Then, $(\omega_i, \tau(i))$ is not in the strong core of the economy. Proceeding in this manner for all the sets in the TTC partition, we see that $\sigma^{TTC}(i) = \tau(i)$.
∎

## A.3 Proof of Proposition 2

Proposition 2: Consider an economy $E = \{(e_i)_{i=1,...,n}, V\}$. Then, there may not exist a minimum equilibrium price vector.

*Proof.* Consider the following economy.

|       | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $\omega$ |
|-------|-------|-------|-------|-------|----------|
| $a_1$ | 8     | 1     | 22    | 15    | 1        |
| $a_2$ | 1     | 8     | 22    | 15    | 1        |
| $a_3$ | 20    | 13    | 7     | 1     | 2        |
| $a_4$ | 12    | 18    | 6     | 1     | 3        |

First note that the economy of the example satisfies the WMS condition. The outcome of the TTCPS mechanism is $p = (p_1, p_2, p_3, p_4) = (4, 0, 4, 0)$ and $((1, g_3), (1, g_4), (2, g_1), (3, g_2))$.

In the final phase we obtain the price vector: $\hat{p} = (1, 0, 2, 0)$. Thus, the competitive equilibrium found by the ETTC is $\hat{p} = (1, 0, 2, 0)$ and $((0, g_3), (1, g_4), (3, g_1), (3, g_2))$. But the vector $\tilde{p} = (0, 1, 2, 0)$ is also an equilibrium price vector associated with the allocation $((1, g_4), (0, g_3), (4, g_1)(2, g_2))$ and it is straightforward to verify that there is no equilibrium price vector $q$ such that $q \leq p$ and $q \leq \tilde{p}$. ∎

## A.4 Example: an economy that satisfies the WMS condition and truthful revelation is not a dominant strategy

Consider the following economy.

|       | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $\omega$ |
|-------|-------|-------|-------|-------|-------|----------|
| $a_1$ | 4     | 13    | 7     | 1     | 10    | 3        |
| $a_2$ | 20    | 13    | 27    | 1     | 7     | 0        |
| $a_3$ | 27    | 7     | 13    | 20    | 1     | 0        |
| $a_4$ | 7     | 27    | 1     | 20    | 13    | 0        |
| $a_5$ | 6     | 1     | 21    | 16    | 11    | 1        |

First, note that the economy satisfies the WMS condition. The outcome of the ETTC mechanism is $p = (0, 3, 2, 2, 0)$ and $((0, g_2), (1, g_3), (2, g_1), (0, g_4), (1, g_5))$. Thus, the utility of agent $a_1$ is 13. Suppose that this agent announces the following vector valuation: $v_1' = (4, 10, 7, 1, 13)$. The new outcome of the ETTC mechanism is $p = (2, 1, 2, 0, 1)$ and $((4, g_5), (0, g_2), (0, g_1), (0, g_4), (0, g_3))$; thus, the utility obtained by $a_1$ is 14. Therefore, an agent can benefit by announcing other valuations than her true ones; thus, the mechanism is not strategy-proof on valuations.

## A.5 Proof of Theorem 3

The ETTC consists in two steps. In the first, the TTC mechanism is applied and prices are computed. In the second step, in those cases where it is possible, prices are decreased in the

way defined previously. To show that in the first step the mechanism is strategy-proof on valuations, the proof presented in Abdulkadiroğlu and Sönmez (2) can be adapted to our framework. For the sake of completeness, we include the adapted proof. Then, we present the proof for the second step of the mechanism. We begin with the following lemma.

*Lemma* 1. Fix the valuations of all agents except $i$ at $V_{-i}$. Suppose that in the algorithm agent $i$ is removed from the market at Step $t$ under $v_i$ and at Step $t'$ under $v'_i$. Suppose $t < t'$. Then, the remaining agents at the beginning of Step $t$ are the same whether agent $i$ announces $v_i$ or $v'_i$.

*Proof.* Since agent $i$ fails to participate in a cycle prior to Step $t$ in either case, the same cycles form. ∎

*Theorem 3: Suppose an economy that satisfies the MS condition, then the ETTC mechanism is strategy-proof on $D^{MS}$.*

*Proof.* **First step of the mechanism:**

Consider an agent $i$ with true valuations $v_i$. Fix an announced valuations profile $V_{-i}$ for every agent except $i$. We want to show that revealing her true valuations $v_i$ is at least as good as announcing any other valuations $v'_i$. Let $t'$ be the step at which agent $i$ leaves under $v'_i$, $(i, h, i_1, i_2, ..., i_k)$ be the cycle she joins, and thus, she is assigned to good $h$. Let $t$ be the step at which she leaves under her true valuation $v_i$. Two cases are considered.
**Case 1:** $t \geq t'$. Suppose agent $i$ announces her true valuation $v_i$. By Lemma 1, the same agents remain in the market at the beginning of Step $t'$, whether agent $i$ announces $v_i$ or $v'_i$. Therefore at this Step, agent $h$ points to agent $i_1$, agent $i_1$ points to agent $i_2$,..., agent $i_k$ points to agent $i$, and all these agents keep doing so as long as agent $i$ remains in the market. Since agent $i$ truthfully points to her best remaining choice at each step, she either receives a good that is at least as good as good $h$ or eventually joins to the cycle $(i, h, i_1, i_2, ..., i_k)$.
**Case 2:** $t < t'$. By the Lemma the same agents remain in the algorithm at the beginning of

Step $t$ whether agent $i$ announces $v_i$ or $v_i'$. Moreover, agent $i$ is assigned a good at her best choice remaining at step $t$ under $v_i$. Therefore, in this case too, her assignment under the true valuation is at least as good as $h$.

**Second step of the mechanism:**

Let $v_i = (\alpha_{ij})_{j=1,\dots n}$ be agent $i$ true valuations. Since the economy satisfies the MS condition, we know that $|\alpha_{ih} - \alpha_{il}| \geq \sum_{j=1}^{n} \omega_j + k - 1$ for all goods $l, p, l \neq p$.

Let $j$ ($h$) be the good to which $i$ is assigned when she declares $v_i$ ($v_i'$). Denote by $p$ ($p'$) the price vector found by the TTCPS when $i$ announces $v_i$ ($v_i'$). By the last result we know that $\alpha_{ij} \geq \alpha_{ih}$. Suppose that when prices are set, we have that $\alpha_{ij} + \omega_i + p_i - p_j < \alpha_{ih} + \omega_i + p_i' - p_h'$, and so agent $i$ benefits by announcing $v_i'$. Since prices of goods that belong to the same cycle are equal, we have $p_i = p_j$ and $p_i' = p_h'$, and then, the last inequality is not possible.

Finally, suppose, by way of contradiction, that in the the final phase, when prices are decreased, agent $i$ has a higher utility by announcing $v_i'$; that is, $\alpha_{ij} + \omega_i + \hat{p}_i - \hat{p}_j < \alpha_{ih} + \omega_i + \hat{p}_i' - \hat{p}_h'$, where $\hat{p}$ and $\hat{p}'$ are the final price vectors found by the ETTC mechanism. To make notation lighter, denote $\hat{p}'$ as $q$.

Therefore, suppose that $0 \leq \alpha_{ij} + \hat{p}_i - \hat{p}_j < \alpha_{ih} + q_i - q_h$.

First, note that if $h = i$ we have a contradiction. Indeed, in this case $0 \leq \alpha_{ij} - \alpha_{ih} < \hat{p}_j - \hat{p}_i \leq \sum_{j=1}^{n} \omega_j + k - 1$.

Second, denote as before by $C' = \{i, h, i_1, ..., i_k\}$ the top trading cycle to which agent $i$ joins when she declares $v_i'$. We claim that $q_i - q_h \leq \omega_k + ... + \omega_1 + \omega_h$. Indeed, we have the following inequalities:

$\omega_k + q_k - q_i \geq 0$

$\omega_{k-1} + q_{k-1} - q_k \geq 0$

...

$\omega_1 + q_1 - q_2 \geq 0$

$\omega_h + q_h - q_1 \geq 0$

Therefore:

$\omega_k \geq q_i - q_k$

$\omega_{k-1} \geq q_k - q_{k-1}$

...

$\omega_1 \geq q_2 - q_1$

$\omega_h \geq q_1 - q_h$

Thus, $q_i - q_h = q_i - q_k + q_k - q_{k-1} + ... + q_2 - q_1 + q_1 - q_h \leq \omega_k + ... + \omega_1 + \omega_h$.

Since $\omega_i + \hat{p}_i - \hat{p}_j \geq 0$, we know that $\hat{p}_j - \hat{p}_i \leq \omega_i$.

Finally, we have $0 \leq \alpha_{ij} - \alpha_{ih} < \hat{p}_j - \hat{p}_i + q_i - q_h \leq \omega_i + \omega_k + ... + \omega_1 + \omega_h \leq \sum_{j=1}^{n} \omega_j + k - 1$, which is a contradiction.

We must note that, it may seem to be enough to assume $\alpha_{ij} - \alpha_{ih} \geq \sum_{j=1}^{n} \omega_j$ in order to obtain the result. However, we need to guarantee that the ETTC mechanism finds an equilibrium in the economy $E$. Otherwise, it is straightforward that the mechanism may not be strategy-proof on valuations. Thus, in order to obtain the result the MS condition should hold. ∎

### A.6 Proof of Theorem 4

*Theorem 4: Suppose an economy that satisfies the strong MS condition, then a dropping profile that satisfies the strong MS condition is a Nash equilibrium of the revelation game induced by the DGS mechanism.*

*Proof.* Let $(\alpha_{ij})_{j \in G}$ be agent $i$ true valuations and $s^*$ a dropping profile.
When all agents play a dropping strategy, the DGS mechanism stops at the first step and final prices are zero. Suppose that agent $i$ gets good $\sigma(i)$ under $s^*$ and thus, she gets a payoff

$\omega_i + \alpha_{i\sigma(i)}$. We have to prove that no agent benefits from deviating.

Let us analyze the benefits from deviating that an agent $j_1$ may have. If she announces other valuations but the good with the highest valuation is $\sigma(j_1)$, she gets the same payoff. Then, suppose that she reveals a valuation vector in which the good with highest valuation is different from $\sigma(j_1)$. We have two cases.

**Case 1:** In the first step of the algorithm $j_1$ demands a good that belongs to an agent who is not in her original cycle. This case is described in Figure 1.
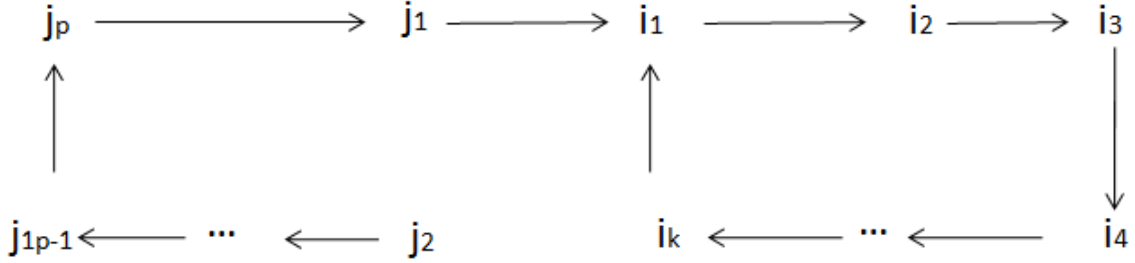


Figure 1

As in the figure, suppose agent $j_1$ demands the good of agent $i_1$ who belongs to the cycle $(i_1, i_2, ..., i_k)$. So the price of $g_{i_1}$ increases until agent $j_1$ or $i_k$ cannot demand it.

**Subcase 1.1:** If $\omega_{i_k} < \omega_{j_1}$ at some step $i_k$ stops demanding $g_{i_1}$ and begins to demand her own good. As $g_{i_k}$ is also demanded by $i_{k-1}$, the price of $g_{i_k}$ increases and $i_k$ demands $g_{i_1}$ again. This process continues until agent $j_1$ or $i_{k-1}$ changes her demand. If $\omega_{j_1} < \omega_k + \omega_{k-1} + 2$ at some step of the mechanism agent $j_1$ cannot demand $g_{i_1}$ and demands another good, and we are in Subcase 1.2. If $\omega_{j_1} > \omega_k + \omega_{k-1} + 2$ we have that $p_{i_1} = \omega_k + \omega_{k-1} + 2$, $p_{i_k} = \omega_{k-1} + 1$, agent $j_1$ demands $i_1$'s good, agents $i_k$ and $i_{k-1}$ demand their own goods and $g_{i_{k-1}}$ is overdemanded. The process continues until $j_1$ cannot demand $g_{i_1}$ or the following step is reached: $p_{i_2} = 0, p_{i_3} = \omega_{i_2} + 1, p_{i_4} = \omega_{i_2} + \omega_{i_3} + 2, ..., p_{i_k} = \sum_{s=2}^{k-1} \omega_{i_s} + k - 2, p_{i_1} = \sum_{s=2}^{k} \omega_{i_s} + k - 1$. Since at all steps of the process $p_{i_2} < p_{i_1}$, good $g_{i_2}$ is always affordable by agent $i_1$. Thus, this subcase finishes when $p_{i_1} = \omega_{j_1} + 1$ and $j_1$ no longer demands good $g_{i_1}$ and Subase 1.2 is reached.

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**Subcase 1.2:** If $\omega_{i_k} > \omega_{j_1}$, at some step of the mechanism agent $j_1$ cannot demand $g_{i_1}$ and demands the second good in her vector valuation. We have the following situations:

1. If $j_1$ demands some good of the cycle $(i_1, i_2, ..., i_k)$, for example $g_{i_2}$, the previous process applies until $p_2 = \omega_{j_1} + 1$. We have the same situation if $j_1$ demands a good of a cycle different from $(i_1, i_2, ..., i_k)$ and from the cycle to which she originally joins.

2. If $j_1$ demands a good of her cycle but different from her own good, we are in Case 2.

3. The last case is when she demands her own good. In that case, her good $g_{j_1}$ is overdemanded and its price increases. Once again, as good $g_{i_2}$ is always attainable by agent $i_1$, the process finishes when $p_{i_1} = \sum_{s=1}^{p} \omega_{j_s} + p$ and all agents in the original cycle $(j_1, ..., j_p)$ demand their own goods. It is in this case that $p_{i_1}$ reaches its highest value of all the analyzed cases, which is less or equal than $\sum_{\substack{i=1 \\ i \neq i_1}}^{n} \omega_i + n - 1$ (because $p \leq n-1$); this implies that at the end of the process $i_1$ still demands $g_{i_2}$ and $i_k$ still demands $g_{i_1}$. Indeed, since the economy satisfies the strong MS condition and $p_{i_1} \leq \sum_{\substack{i=1 \\ i \neq i_1}}^{n} \omega_i + n - 1$, we have that $\alpha_{i_1 i_2} - \alpha_{i_1 h} \geq p_{i_1}$ for all good $h$. With the same argument it can be shown that $\alpha_{i_k i_1} - \alpha_{i_k h} \geq p_{i_1}$ for all good $h$.

**Case 2:** Finally, we have to study the case in which agent $j_1$ demands a good from an agent of her cycle as in Figure 4 (the dotted arrow represents agent's demand under $s^*$).
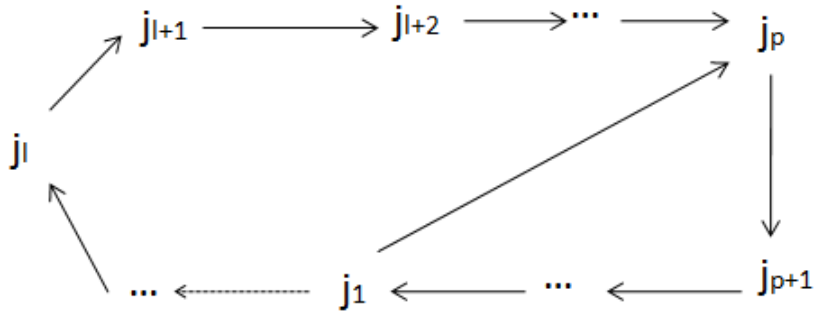


Figure 2

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Suppose that when $j_1$ changes her valuations, she demands the good of agent $j_p$. So the price of $g_{j_p}$ increases until one of the agents cannot demand it. If $\omega_{j_{p-1}} > \omega_{j_1}$, at some step of the mechanism agent $j_1$ cannot demand $g_{j_p}$ and has to demand another good. If $\omega_{j_{p-1}} < \omega_{j_1}$, the price of good $g_{j_p}$ increases until $\omega_{j_{p-1}} + 1$ and agent $j_{p-1}$ demands her own good, which is overdemanded. The process continues until $j_1$ no longer demands good $g_{j_p}$ or the cycle $(j_1, j_p, ..., j_q)$ is implemented and all agents of the original cycle that no belong to $(j_1, j_p, ..., j_q)$ are assigned to their own goods. In the last situation, since $\alpha_{ip} < \alpha_{i\sigma(i)}$ the payoff of agent $i$ is lower than the payoff when she declares a dropping strategy. ■

## A.7 Proof of Theorem 5

*Theorem 5: Suppose an economy $E = \{(e_i)_{i=1,...,n}, V\}$ such that:*

1. *it satisfies the WMS condition,*

2. *for each agent $i$, $O^i(i) \geq 2$; that is, for each agent there is at most one good preferred over her own good.*

*Then, the DGS mechanism stops and finds a competitive equilibrium.*

*Proof.* In the first round of the algorithm there is at least one cycle. The final assignment of goods is the one defined by this cycle (or cycles) and other agents are assigned to their own good. The formal proof is similar to the one of Theorem 3, Case 1. Specifically, we have to show that the cycles that appear in the first step of the mechanism remain formed until the mechanism finishes. The maximum level that a price $p_i$ can achieve is when at the initial step of the mechanism agents can be indexed in an order $\{i, 1, 2, ..., n - 1\}$ such that $i$ demands her good, 1 demands the good of agent $i$ and $j$ demands the good of agent $j - 1$ for $j = 2, ..., n - 1$. In this case, the final value of $p_i$ is equal to $\sum_{j=1}^{n-1} \omega_j + n - 1$. Since in this situation $n = k$, we finish the proof. ■

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# Chapter 4

# Strategic Uncertainty and Unraveling in Matching Markets

## 4.1 Introduction

We study unraveling in labor markets, and in matching markets in general. Unraveling is a phenomenon by which matches are made too early. They are made at a point in time when there is too little information about the quality of a match. The literature has documented many episodes of unraveling: the market for medical interns is a famous example, in which labor contracts for interns were signed two years before the future interns would graduate (see Roth (19) or Roth and Sotomayor (18)). Other examples of unraveling include the market for federal court clerks (Avery, Jolls, Posner, and Roth (1); Roth (23)), for gastroenterology fellows (Niederle and Roth (14, 15)), for college football games (Fréchette, Roth, and Ünver (7); Roth (22)), and for placement in sororities (Mongell and Roth (13)).

We explain unraveling of the timing of offers as the result of *strategic unraveling.* If some agents go early, it becomes more attractive for other agents to go early, which makes it more attractive for even more agents to go early. Our explanation is reminiscent of models of bank

runs, where strategic complementarity makes agents undertake an inefficient action because they are concerned that others may take this inefficient action (Diamond and Dybvig (4)). As we shall see, the matching environment is quite different from models of bank runs, but the basic logic of strategic unraveling is similar.

Strategic unraveling in our model proceeds as follows. There is a loss in efficiency when some agents go early: Information about the quality of the matches arrives late, so it is better for efficiency to wait until the information has arrived to make a match. If some agents go early anyway, this forces later matches to be less efficient. The result is a negative externality that makes it more tempting for all agents to go early. So the externality may push some additional agents over the threshold by which they decide to go early. In turn, these additional agents going early makes it even more tempting to go early—and so on and so forth.

We show that, as a result of such strategic unraveling, *any given agent is more likely to go early than go late.* Our model assumes that there are two periods, and that there is incomplete information over the agents' discount factor. We view the incomplete information simply as modeling device: as a way of generating the strategic uncertainty that allows the logic of unraveling to apply. If an agent goes early, she has no information about the quality of a match. If an agent goes late, then all information has been released, and matching is assortative on the quality of an agent as a partner (highest quality agents match with each other, the second highest match with each other, and so on).

The qualitative effects of the negative externality are intuitive, but how far do they go? In our model, we can precisely *calculate* (and the heart of our arguments is indeed very combinatorial) the extent to which strategic unraveling pushes agents to go early. It turns out that unraveling goes all the way to making each individual more likely to go early than to go late. There are ways in which our model is rigged against unraveling. It makes late matching particularly attractive, and rules out unraveling purely as the result of coordination

failure (see Section 4.4). Yet the model produces early contracting as the modal outcome.[1]

A more precise statement of our results follows. We first assume that only firms are strategic. Workers always accept the offers they receive. In this environment, we show that there is always a full unraveling Bayesian Nash equilibrium, in which all firms make early offers. Further, in *any* symmetric Bayesian Nash equilibrium, a firm makes an early offer with probability at least 3/4.

If we assume that the prior over discount factors is uniform, we can say more. There are exactly two symmetric equilibria when the size of the market is at least 11. One is the full unraveling equilibrium, but it is unstable. In the second equilibrium, which is stable, agents go early with probability larger than 3/4. As the size of the market grows, the probability of going early in the second equilibrium converges to 3/4. If the number of agents is lower than or equal to 10, the unique symmetric Bayesian Nash equilibrium is the full unraveling equilibrium.

In second place, we consider a model where both sides of the market are strategic. Our results continue to apply (there is actually not a substantial conceptual difference between the two models). Among other things, we prove that in any symmetric Bayesian Nash equilibrium, the expected proportion of agents that match early is at least 1/2.

Our results reveal that there may exist an equilibrium pattern of adherence and non-adherence to the hiring dates. The market may become divided in equilibrium, with one segment hiring early, and the other waiting to match in the final period with full information about agents' qualities. We demonstrate that a mixed level of adherence can be sustainable

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[1]Continuing with the similarity with bank runs, the result is reminiscent of the literature on global games, where basic assumptions on the structure of signals give a precise calculation of how far iterated elimination of dominated strategies will go (Frankel, Morris, and Pauzner (6)). There is, however, a clear difference with the literature on bank runs. A run can be explained purely by coordination failure. Agents' payoffs in our model are biased against unraveling, and coordination failure alone would not suffice to make agents unravel (see Section 4.4 for a discussion of this issue).

in an equilibrium, which is consistent with the empirical evidence Avery, Jolls, Posner, and Roth (1).

## 4.2 Related Literature

Ours is the first theoretical study that identifies strategic uncertainty as the main force behind the unraveling of matching markets. One empirical investigation of the market for medical interns also attributes unraveling to strategic uncertainty: Wetz, Seelig, Khoueiry, and Weiserbs (27) write that early contracting is motivated by concerns over losing interns to other programs who operate outside of the centralized algorithm. Their explanation, based on agents' observed behavior in the market, is essentially what we have tried to capture formally in the present paper.

The best-known episode of unraveling is the case of the market for hospital interns before 1945 (Roth (19); Roth and Sotomayor (18); Roth (21)). There is evidence that unraveling still exists in this market: Wetz, Seelig, Khoueiry, and Weiserbs (27) study out-of-match residency offers during the year 2007. In the market for interns, some interns are allowed to take outside-the-match offers (for instance, osteopathic medical students and international medical graduates). Wetz, Seelig, Khoueiry, and Weiserbs (27) find that 15.7% of the total number of postgraduate year-1 positions available in the three primary care and four procedural and/or lifestyle-oriented specialities studied, were offered outside the match. The authors conclude that about one in five positions in nonprocedural, primary care specialties were offered outside the match and, thus, the situation is similar to that which existed before 1952.

One classic explanation of unraveling is the "stability hypothesis," as formulated by Roth (20) and Kagel and Roth (9). This hypothesis affirms that unraveling will be prevented if once the relevant information is revealed, a stable matching is implemented through a clearinghouse. The idea is that, in some sense, the market is trying to establish a stable matching. It simply may be doing so in an inefficient manner. Our paper provides some

justification for central clearing houses. There is a clear efficiency gain from late contracting in our model, and late contracting equals a stable matching. The agents' strategic behavior prevents the market from reaching this stable matching, and makes the market unravel.

A handful other papers provide theoretical explanations for unraveling. They focus on different mechanisms than the one we have studied here.

Li and Rosen (10) and Li and Suen (11) study a model with transfers (a model based on Shapley and Shubik (25) assignment game) in which early contracting provides insurance. They show that unraveling may occur among workers who appear to be most promising a priori, before full information is revealed. In a similar framework, Li and Suen (12) allow for unproductive firms and find multiple equilibria with unraveling. They show that more firms and workers will contract early if the uncertainty about the number of productive workers is higher and the more risk-adverse agents are. As we explain in Section 4.4, our model does not have an insurance motive for early contracting, and focuses on a different explanation for unraveling.

Damiano, Li, and Suen (3) present an explanation of unraveling that is based on search and matching. Agents know their qualities, so there is no informational gain from matching late, but an agent may not meet a partner of sufficiently high quality in a given period. If there are costs to searching, then there is unraveling in how willing agents are to accept a partner. In Damiano, Li, and Suen (3), unraveling is triggered by search costs. In our model, it is triggered by incomplete information.

Du and Livne (5) consider the role of transfers in unraveling. They show that, in the absence of transfers, and in the limit as the market size grows, a substantial number of agents will contract early. Unraveling in their paper happens because new agents arrive over time, and agents who are in relatively high positions may want to contract early because the new arrivals may be of higher match qualities. In contrast, in a flexible-transfer regime, agents will not unravel.

Niederle, Roth, and Ünver (16) explain unraveling as the result of an imbalance between demand and supply. Unraveling arises when there is a surplus of applicants, but a shortage of high quality applicants. When a worker does not know if she will be in the long or short side of the market, she may find early offers made by low quality firms attractive. For such firms, early offers is the only way to employ high quality workers.

Hałaburda (8) proposes that the key to explaining unraveling is the similarity of firms' preferences. Workers' preferences for firms are identical, and known from the start, but firms learn their preferences for workers in the second period. If firms' preferences are similar, then firms tend to prefer the same workers. Thus, worse firms may have better chances to hire their most preferred candidates if they make early offers. So, if firms' preferences are sufficiently similar, it is likely that some firms will go early. In our model, although preferences are identical, this feature does not explain unraveling. An agent may be concerned about being one of the worst agents in the market, but she would still prefer to wait and contract in the second period. Early contracting in our model is inefficient for every agent. As we show below, the strategic uncertainty over how many other agents go early is the main mechanism behind incentives for some agents to match early.

## 4.3   The model and results

We present a model of one-to-one matching between workers and firms. In our model, we adopt the language of the medical interns market. The workers are doctors, and firms are hospitals.

Let $H$ and $D$ be two finite and disjoint sets: $H$ is the set of hospitals, and $D$ the set of doctors. Suppose that $|H| = |D| = n$, so we can identify $H$ and $D$ with (copies of) $\{1, \ldots, n\}$.

A *matching* is a function $\mu : H \cup D \to H \cup D$ such that, for all $h \in H$ and $d \in D$,

1. $\mu(h) \in D \cup \{h\}$ and $\mu(d) \in H \cup \{d\}$

2. $d = \mu(h)$ if and only if $h = \mu(d)$.

The meaning of $\mu(h) = h$ is that the position of hospital $h$ remains unfilled, and $\mu(d) = d$ means that doctor $d$ does not find a job.

Each doctor $d$ and hospital $h$ is assigned a *quality*

$$\pi^D(d) \in \{1, \ldots, n\} \text{ and } \pi^H(h) \in \{1, \ldots, n\}.$$

Suppose that $\pi^H$ and $\pi^D$ are permutations of $\{1, \ldots, n\}$, so we can think of quality as the rank of a hospital or doctor in the market. The highest-ranked hospital is $h$ such that $\pi^H(h) = n$, for example. If doctor $d$ is hired by hospital $h$, then they obtain utilities that depend on their qualities, $u_d(\pi^D(d), \pi^H(h))$ is the utility to $d$ and $u_h(\pi^D(d), \pi^H(h))$ is the utility to $h$. If an agent remains unmatched, then she obtains a utility of zero.

A matching $\mu$ is *stable* if there is no pair $(h, d)$ such that

$$u_d(\pi^D(d), \pi^H(h)) > u_d(\pi^D(d), \pi^H(\mu(d))) \text{ and } u_h(\pi^D(d), \pi^H(h)) > u_h(\pi^D(\mu(h)), \pi^H(h)).$$

We assume that $u_d$ and $u_h$ are *multiplicative*; that is: $u_d(i, j) = u_h(i, j) = ij$.

*Remark 1.* There is a unique stable matching, the matching $\mu(i) = i$ (the identity matching).

### 4.3.1 Matching over time: early or late offers

The model is a stylized environment with two periods. In the first period, match qualities $\pi^H$ and $\pi^D$ are not known. In the second period, a pair $(\pi^H, \pi^D)$ is drawn at random, uniformly and independently. A match is formed among the agents who wish to match in period $t = 0$: all agents are identical at that point, so the matching is purely random. In the second period, when match qualities are known, a stable matching is formed among the agents that did not match in the first period.

Our purpose is to focus on the strategic motivations for going early: we study the simultaneous-move game in which hospitals decide whether to go early and match at time $t = 0$, or to

wait and match at time $t = 1$. In particular, we assume that only hospitals are strategic and that matchings are automatic. In period $t = 1$ the matching is assortative among the agents who have not matched in period $t = 0$; the assortative matching is the unique stable matching under our assumptions. In period $t = 0$, matching is random because no agent has any information on match qualities.

In Section 4.3.4 we present results where both doctors and hospitals are strategic. Our results essentially continue to hold when both sides are strategic, but we choose to present first the model in which only hospitals are strategic. The reason is twofold. First, there is no deep conceptual difference between the two cases. Indeed, we use the results in this section to prove the results of Section 4.3.4. In second place, the discussion of unraveling in Roth (19) suggests that, in the hospital-interns market, only hospitals are strategic.

Each agent $i \in H \cup D$ has a discount factor $\delta_i$. The utility at $t = 0$ when $h$ and $d$ match in period $t$ is given by

$$\delta_h^t u_h(\pi^D(d), \pi^H(h)) = \delta_h^t \pi^D(d)\pi^H(h), \text{ and}$$

$$\delta_d^t u_d(\pi^D(d), \pi^H(h)) = \delta_d^t \pi^D(d)\pi^H(h),$$

to $h$ and $d$, respectively.

The following timeline describes how events unfold.
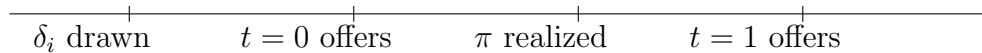


We proceed to describe the payoffs from making an early vs. a late offer to match. At time 0, qualities are purely random. So if a hospitals $h$ matches in period 0 its expected utility is $\mathcal{U}_e = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} ij$, the expected value of the product $ij$ when $i$ and $j$ are random.

In period 1, agents have learned the values of $\pi^D$ and $\pi^H$. The matching will be assortative among the agents who have not matched early. Assortative means that the doctor with

the highest value of $\pi^D(d)$ will match with the hospital with the highest value of $\pi^H(h)$, the doctor with the next-highest value of $\pi^D(d)$ will match with the hospital with the next-highest value of $\pi^H(h)$, and so on.

Now, it is complicated to calculate the expected utility of going late because the calculation depends on how many agents go early. If $m$ agents have left the market in time $t = 0$, then the assortative matching matches the highest *available* hospital and doctor, but the actual highest-quality matches may have left early. The problem is compounded as we consider the second-highest qualities, the third-highest, and so on.

One special case is simple to calculate. Consider a given hospital $h$. If *all other* hospitals wait to make offers in period $t = 1$, then the expected utility to hospital $h$, in period 0, of waiting for period 1 is $\delta_h \frac{1}{n} \sum_{i=1}^{n} i^2$.

In general, if $m$ hospitals have left the market, we write $\mathcal{U}_m$ for the expected value of $\pi^H(h)\pi^D(\tilde{\mu}(h))$, where $\pi^H$ and $\pi^D$ are random, and $\tilde{\mu}$ is the (random) assortative matching in period 1. The matching $\tilde{\mu}$ is determined by the realization of match qualities $\pi^H$ and $\pi^D$, including the qualities of the $m$ hospitals, with corresponding doctors, who have left the market. That is, when $m$ hospitals exit the market at $t = 0$, $\mathcal{U}_m$ is the expected utility to a hospital of waiting for $t = 1$.

The following is an important technical result in our paper.

*Lemma* 1.
$$\mathcal{U}_m = \frac{(n+1)^2(2(n-m)+1)}{6(n-m+1)}.$$

An important consequence of Lemma 1 is that $\mathcal{U}_m > \mathcal{U}_{m+1}$. The difference $\mathcal{U}_{m+1} - \mathcal{U}_m$ is the *negative externality* imposed by a hospital-doctor pair who match early on the agents who decide to match late. It is important to note that the negative externality increases with $m$, so that additional agents going early increase the incentives of any given agent to go early. This effect vanishes as the market grows large, which helps to stabilize the number of agents who go early in a large market (see the discussion after Corollary 3 on page 116).

Section 4.5 gives a precise definition of the quantity $\mathcal{U}_m$ and presents a proof of Lemma 1.

### 4.3.2  Incomplete information

We now introduce a Bayesian game in which hospitals may make early offers due to the strategic uncertainty over how many other hospitals go early.

We assume that $\delta_h \in [0, 1]$ is the private information of hospital $h$. The *type* of an agent $h$ is therefore $\delta_h$. All agents share the prior that the different $\delta_h$ are drawn independently from a distribution over $[0, 1]$ with cumulative distribution function (cdf) $F$. We assume that $x \leq F(x)$ for all $x \in [0, 1]$: the assumption is satisfied for any distribution with a concave cdf. For example the uniform, or truncated normal, distributions on $[0, 1]$ satisfy our assumption.

A strategy for a hospital $h$ is a function

$$s_h : [0, 1] \rightarrow \{0, 1\},$$

where $s_h(\delta_h)$ is the period in which hospital $h$ makes its offer. In our model, there is no decision to be made other than *when* to match.

Given a profile of strategies $s = (s_1, \ldots, s_n)$, we write $s_{-h}$ for the profile of strategies of hospitals other than $h$. Given a profile $s_{-h}$, for each realization of $\delta_{-h}$, $s_{-h}$ determines $m$, the number of hospitals, other that $h$, that go early. Thus, $s_{-h}$ defines a probability distribution for $m$. Given a profile $s_{-h}$, $m$ is a random variable, and so it is $\mathcal{U}_m$ with a distribution defined by $F$. Then, we can compute the expected value of $\mathcal{U}_m$ given $s_{-h}$ (see Lemma 1), which is denoted by $\mathbf{E}_{s_{-h}}\mathcal{U}_m$. We write $\delta_h \mathbf{E}_{s_{-h}}\mathcal{U}_m$ for the expected utility at time 0, to hospital $h$, of waiting for $t = 1$ to make an offer, if all hospitals other than $h$ have the profile of strategies $s_{-h}$: $\mathbf{E}_{s_{-h}}\mathcal{U}_m = \sum_{i=0}^{n-1} Pr(m = i) \, \mathcal{U}_i$, where as we just noted, $Pr(m = i)$ is calculated from $s_{-h}$ and $F$.

Given a profile $s_{-h}$, a hospital $h$ will decide to go early if and only if

$$\mathcal{U}_e \geq \delta_h \mathbf{E}_{s_{-h}} \mathcal{U}_m \tag{1}$$

(recall that $\mathcal{U}_e$ is the expected utility of making an early offer). So a strategy $s_h$ is a best response to $s_{-h}$ if for every $\delta_h$, $s_h(\delta_h) = 0$ iff (1) is satisfied.

A profile of strategies $s = (s_1, \ldots, s_n)$ is a *Bayesian Nash equilibrium (BNE)* if (1) is satisfied for each $h \in H$. A BNE is *symmetric* if $s_h = s_{h'}$ for all $h, h' \in H$. A BNE is *full unraveling* if $s_h = 0$ for all $h \in H$. Thus, in a full unraveling BNE, all agents go early no matter their type.

*Theorem* 1. If $n \leq 10$ then the unique symmetric BNE is the full unraveling BNE. If $n > 10$ then there is at least one symmetric BNE, namely the full unraveling BNE; moreover, in any symmetric BNE $s = (s_1, \ldots, s_n)$ we have that

$$Pr(s_h = 0) \geq F(3/4) \geq 3/4,$$

for all $h \in H$.

Theorem 1 says that any hospital, in any symmetric BNE, is more likely to go early than late. The equilibrium probability of going early is at least $3/4$. It is therefore immediate that:

*Corollary* 1. In any symmetric BNE, the expected number of hospitals that go early is at least $nF(3/4) \geq n3/4$.

### 4.3.3 Stability of BNE – Uniform $F$

In this section we entertain an additional assumption. We suppose that the prior distribution $F$ is the uniform cdf. In this case, we can make more precise statements about the set of BNE in our game. We can also talk about the stability of equilibria.

As we shall see, for large $n$, in the unique stable equilibrium, the market is divided. Most of the market ($3/4$ of all hospitals) go early, while the rest wait and contract late. Thus our

results with a uniform $F$ can explain some of the empirical findings where only part of the market unravels.

*Theorem 2.* Let $F$ be the uniform cdf. If $n \leq 10$ then the unique symmetric BNE is the full unraveling BNE. If $n > 10$ then there are exactly two symmetric BNE. One is the full unraveling BNE. The second is a BNE $s^n = (s_1^n, \ldots, s_n^n)$ in which for every $h \in H$

$$Pr(s_h^n = 0) \geq 3/4 = \lim_{n \to \infty} Pr(s_h^n = 0).$$

*Remark 2.* The proof of Theorem 1 actually follows from Theorem 2. We lay out the details in Section 4.7.

We discuss a notion of stability of BNE. Stability allows us to select a symmetric BNE in the cases in which there is more than one. It turns out that the full unraveling BNE is stable when $n \leq 10$ and the equilibrium denoted by $s^n$ in Theorem 2 is the unique stable symmetric BNE when $n > 10$.

A strategy $s_h$ satisfying Equation (1) is characterized by a *threshold* $\bar{\delta}_h \in [0,1]$ such that $s_h(\delta_h) = 0$ if $\delta_h \leq \bar{\delta}_h$ and $s_h(\delta_h) = 1$ if $\delta_h > \bar{\delta}_h$. Given identical thresholds $\bar{\delta}_{-h} = \bar{\delta}$ for all hospitals other than $h$, we can let $\beta^n(\bar{\delta})$ be the threshold for hospital $h$ defined by Equation (1).

A symmetric BNE is then described by a single $\bar{\delta} \in [0,1]$ with the property that

$$\bar{\delta} = \beta^n(\bar{\delta}).$$

The function $\beta^n$ is the best-response function of our game. The symmetric BNE are the fixed points of $\beta^n$. The following figure shows the graph of $\beta^n$ for $n = 3, 7, 11, 15, 17$.

A symmetric BNE $\bar{\delta}$ is *stable* if there is an open interval $I$ of $\bar{\delta}$ in $[0,1]$ such that for all $\delta \in I$

1. $\delta < \beta^n(\delta)$ when $\delta < \bar{\delta}$, and
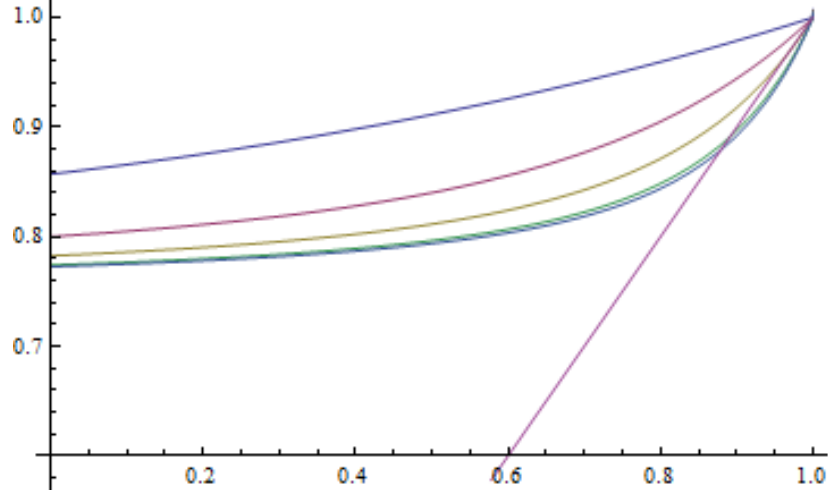
2. $\delta > \beta^n(\delta)$ when $\delta > \bar{\delta}$.

Figure 3: The graph of $\beta^n$ for $n = 3, 7, 11, 15, 17$.

A symmetric BNE that is not stable is *unstable*.

In the examples in Figure 3, it is evident that the full unraveling BNE is stable when it is unique. For larger $n$, we have two BNE. The smaller BNE is stable, while the full unraveling BNE is unstable. The picture that emerges from Figure 3 holds more generally:

*Proposition* 1. Let $F$ be the uniform cdf. If $n \leq 10$ then the full unraveling BNE is stable. If $n > 10$ then the symmetric BNE denoted by $s^n$ in Theorem 2 is stable while the full unraveling BNE is unstable.

### 4.3.4 Strategic doctors

We now assume that doctors are strategic as well. We consider the simultaneous-move game in which the players are $H \cup D$. Each agent has to decide whether to match in period $t = 0$ or $t = 1$. So the set of available actions is $\{0, 1\}$ to each player. Agents' strategies are functions $s_i : [0, 1] \rightarrow \{0, 1\}$, with $i \in H \cup D$.

When doctors are strategic, the probability that $m$ agents go early is the probability that the *minimum* between the hospitals and the doctors that make offers at period $t = 0$, equals $m$. For any profile of strategies $s$, and any realization of types $(\delta_i)$, the number of agents who

exit the market is the minimum of two quantities, the number of hospitals $h$ with $s_h(\delta_h) = 0$, and the number of doctors $d$ with $s_d(\delta_d) = 0$.

Thus, given a profile of strategies of *all* agents other than $h$, the expected value of $\mathcal{U}_m$, $\mathbf{E}_{s_{-h}}\mathcal{U}_m$, involves the probability distribution of the minimum of two independent binomial random variables, instead of a single binomial random variable as in the previous case. The number $m$ is drawn according to the minimum of two binomial distributions.

The calculations performed in the proof of Theorem 1 are still sufficient to give us the following result.

*Theorem* 3. There is at least one symmetric BNE, namely the full unraveling BNE. In any symmetric BNE $s = (s_i)_{i \in H \cup D}$, for every $i \in H \cup D$ we have that

$$Pr(s_i = 0) \geq F(1/2) \geq 1/2.$$

*Corollary* 2. In any symmetric BNE, the expected number of agents that go early is at least $nF(1/2) \geq n/2$.

The results in Section 4.3.3 extend to the case when doctors are strategic. We obtain the following result.

*Theorem* 4. Let $F$ be the uniform cdf. If $n > 10$, then there are exactly two symmetric BNE. One is the full unraveling BNE, which is unstable. The second is a stable BNE $s = (s_i)_{i \in H \cup D}$ such that $Pr(s_i = 0) \geq 1/2$ for every $i \in H \cup D$.

## 4.4   A discussion of our model

Our model has two specific assumptions that merit some additional discussion.

**Payoffs.**   We assume that payoffs are multiplicative, a common assumption in applied matching theory (see e.g. Bulow and Levin (2), Damiano, Li, and Suen (3), and many other papers). In our particular case, there are two reasons for working with multiplicative payoffs.

First, a parametric assumption about payoffs is unavoidable when we are trying to precisely calculate the probability that an agent will go early. As such, the multiplicative form is natural.

The multiplicative assumption also makes sense as a way of abstracting from other possible explanations of unraveling. We did not want an explanation of unraveling that was based on the insurance value of going early (an avenue explored by Li and Rosen (10)). We assumed payoffs for which there is a clear advantage to going late, not early. In our model, agents are risk neutral, and even though an agent may end up with a low quality, there is not enough insurance in going early to compensate from the gain in efficiency from a late assortative matching. The multiplicative model implies that, even though an agent may be concerned about a bad draw of their quality, the gains from matching assortatively outweigh the temptation to match to an average partner in $t = 0$.

Roth (19) suggests that unraveling is the result of a prisoners' dilemma game among the hospitals. The implication is that it is a dominant strategy for the hospitals to go early. Our focus is on the strategic channel, whereby agents go early because of their concerns that others go early (and the consequence negative externality). By our assumptions on preferences, we rule out that it is dominant for agents to go early.

It is still possible to generate unraveling by way of a coordination failure, as in the literature on bank runs (Diamond and Dybvig (4)). In our model, however, and in contrast to the model of bank runs, such unraveling is unstable. Only when all agents are *certain* that all other agents want to go early, are they willing to go early. This would be an unstable situation: It is easy to rule out such an outcome if agents' beliefs may depart from certainty that everyone goes early. In contrast, we show that there is in our model a stable equilibrium in which agents are more likely to go early than to go late. Coordination failure is still present in that equilibrium, but unraveling arises through the channel of strategic unraveling.

Finally, the multiplicative model also captures the negative externality imposed by agents who go early on the rest of the market. There is an efficiency loss when some agents go

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early; they hurt the rest of the agents (even in a model without transfers like ours).

**Information.** The second assumption that deserves mention is our informational assumption. We assume that agents are completely ignorant about match qualities at date $t = 0$. The assumption is extreme, and it is meant to focus the model on the trade-off between the value of the information revealed at $t = 1$, and the incentives to go early. By assuming that there is no information at time $t = 0$, and full information at $t = 1$, we have biased the model against the unraveling outcome.

That said, it may not be an unrealistic assumption. From Roth and Xing (24): "offers are being made so early that there are serious difficulties in distinguishing among the candidates." So our assumption of complete ignorance over match qualities may reflect the actual situation in the markets where we observe unraveling.

Finally, we use an assumption of the cdf $F$ that allows us to exploit the results obtained in the case when $F$ is uniform. The assumption that $x \leq F(x)$ means that $F$ is smaller in the sense of first-order stochastic dominance, than the uniform distribution. Again, we need the uniform distribution to make precise calculations, and then the inequality on $F$ allows us to obtain bounds. As we remarked above, the assumption on $F$ is satisfied when $F$ is concave.

## 4.5 Proof of Lemma 1

In this Section we present, in the first place, a formula for $\mathcal{U}_1$ which clarifies the meaning of this quantity. Then, a algorithm to compute $\mathcal{U}_m$ in the general case is introduced (Proposition 2). Lemmas 2 and 3 deduce a simple formula for $\mathcal{U}_m$.

Recall that $\mathcal{U}_0$ is the expected utility from waiting when all other hospitals wait. Then:

$$\mathcal{U}_0 = \frac{1}{n} \sum_{i=1}^{n} i^2 = \frac{(n+1)(2n+1)}{6}.$$

### 4.5.1 Computing $\mathcal{U}_1$

We compute the expected utility from waiting, when only one pair of hospital-doctor goes early. In period 1, after permutations $\pi^H$ and $\pi^D$ are drawn, sets $H$ and $D$ can be ordered according to agents' quality. Then, consider the sets $H$ and $D$ described as: $H = \{1, 2, \ldots, n\}$ and $D = \{1, 2, \ldots, n\}$, where the first agent is the lowest-quality agent, and the last agent is the highest-quality agent.

First, conditional on being of quality $i$, the leaving hospital is of a higher quality than $i$ with probability $(n - i)/(n - 1)$, and of a lower quality than $i$ with probability $(i - 1)/(n - 1)$. This is deduced from the fact that there are $n - 1$ possible qualities for the hospital that leaves early, $(n - i)$ of those are higher than $i$ and $(i - 1)$ lower than $i$. Figure 2 may help to make the computations.
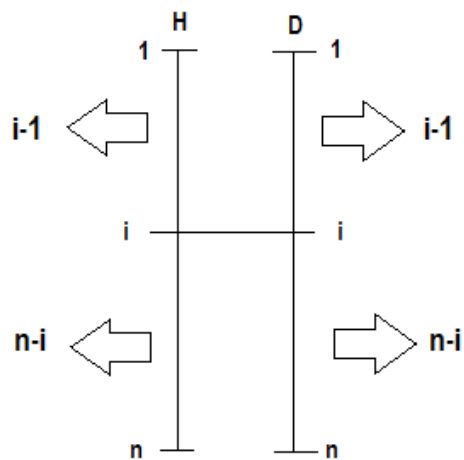


Figure 4: Computing $\mathcal{U}_1$

If the leaving hospital is of a higher quality than $i$, this means that hospital $i$ is better off, unless the doctor that leaves with hospital $i$ is also a "good" doctor: unless the doctor that leaves is one that would be matched in the second period with a hospital better than $i$. This happens with probability $(n - i)/n$. With the complementary probability, $i/n$, hospital $i$

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is better off by the better hospital leaving. Being better off means that hospital $i$ will be matched in the second period with a doctor with a quality one unit higher than $i$ (i.e., a doctor of quality $(i+1)$), which is worth $i$ to a hospital of quality $i$.

If the leaving hospital is of a lower quality than $i$, then this does not affect hospital $i$ and it gets $i^2$; unless the doctor that leaves used to be with a better hospital, or with $i$, in which case hospital $i$ goes down one step. To a hospital of quality $i$, losing one step is worth $-i$. So in the event that a hospital of lower quality than $i$ leaves (which has probability $(i-1)/(n-1)$) it gets $i^2$ for sure but it loses $-i$ with probability $(n-i+1)/n$, the probability that the partner of the hospital that goes early is of a quality greater than or equal to $i$.

So:

$$\mathcal{U}_1 = \frac{1}{n}\sum_{i=1}^{n}\left\{ \frac{n-i}{n-1}\left[ i^2 + \frac{i}{n}i \right] + \frac{i-1}{n-1}\left[ i^2 - \frac{n-i+1}{n}i \right] \right\}.$$

Since the terms that multiply $i^2$ add to 1, this gives:

$$\mathcal{U}_1 = \frac{1}{n}\sum_{i=1}^{n}\left[ i^2 + \frac{i}{n(n-1)}(n-2i+1) \right] = \frac{(2n-1)(n+1)^2}{6n}.$$

Note that $\mathcal{U}_1$ can be also expressed as:

$$\mathcal{U}_1 = \mathcal{U}_0 + \frac{1}{n}\sum_{i=1}^{n}\left[ \frac{(n-i)}{(n-1)}\frac{i}{n}i - \frac{(i-1)}{(n-1)}\frac{(n-i+1)}{n}i \right] = \mathcal{U}_0 - \frac{n+1}{6n}.$$

The intuition behind this equation is the following. Notice that with probability $(n-i)/(n-1)$ the hospital that leaves early is of a higher quality than $i$ and with probability $(i-1)/(n-1)$ is of a lower quality than $i$. Then, $((n-i)/(n-1))(i/n)$ is the probability that the hospital that leaves early is of a higher quality than $i$ and the doctor it hires is of a quality lower than or equal to $i$. In this event, hospital $i$ increases its utility by $i$. If the hospital that goes early is of quality lower than $i$ and it hires a doctor of quality higher than or equal to $i$, which happens with probability $((i-1)/(n-1))((n-i+1)/n)$, then hospital $i$ decreases its utility by $i$. Therefore, $\mathcal{U}_1$ can be expressed as $\mathcal{U}_0$ plus the expected utility derived from

the leaving of a pair of hospital-doctor. Moreover, $-\frac{n+1}{6n}$ is the negative externality imposed on the rest of the market by the first pair of hospital-doctor that decide to match early.

Clearly, this argument is very hard to generalize if we consider more than one pair of hospital-doctor that goes early. In the following Section, we develop an algorithm to compute the expected utility from waiting when $m$ pairs of hospital-doctor leave the market at $t = 0$.

### 4.5.2 An algorithm to compute $\mathcal{U}_m$

In this section, we introduce an algorithm to compute the value of $\mathcal{U}_m$ in the general case. First, we define the payoff matrix $\mathbb{U}$ as follows: the element $(i, j)$ of $\mathbb{U}$ is the utility that a doctor of quality $i$ has when she is hired by a hospital of quality $j$ (which is also the utility of the hospital). In particular, the elements of the first column of $\mathbb{U}$ are the utilities that the hospital of quality 1 has if it hires a doctor of quality $1, 2, \ldots, n$. Note that the elements of the main diagonal of $\mathbb{U}$ are: $1, 4, \ldots, i^2, \ldots, n^2$, which are the payoffs that each agent has when no pair of hospital-doctor leaves early. Thus, matrix $\mathbb{U}$ is:

$$
\begin{pmatrix}
1 & 2 & 3 & \ldots & (n-1) & n \\
2 & 4 & 6 & \ldots & 2(n-1) & 2n \\
3 & 6 & 9 & \ldots & 3(n-1) & 3n \\
. & & & & & \\
. & & & & & \\
. & & & & & \\
(n-1) & 2(n-1) & 3(n-1) & \ldots & (n-1)^2 & n(n-1) \\
n & 2n & 3n & \ldots & n(n-1) & n^2
\end{pmatrix}.
$$

When a hospital makes an offer at $t = 0$ and hires a doctor, both the hospital and the doctor may be of any quality. So, to compute the expected utility of a hospital that waits, we have to consider all possible qualities combinations. Assume that the hospital that leaves is of

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quality $j$ and the doctor that it hires is of quality $i$. If only this pair of hospital-doctor leaves the market at $t = 0$, in the second period the utilities of hospitals and doctors that do not leave the market are given by the assortative matching. Indeed, the highest quality hospital (between those that remain in the market) will hire the highest quality doctor of those that do not exit the market. The same argument holds for all agents.

Therefore, when doctor $i$ is hired at $t = 0$ by hospital $j$, the utilities of hospitals and doctors that remain in the market in the second period, are the elements of the main diagonal of the submatrix of $\mathbb{U}$ that it is obtained from deleting the row $i$ and the column $j$. To consider all possible combinations for the quality of the hospital that leaves early and the doctor that it hires, we have to go over all the elements of $\mathbb{U}$. Thus, to compute the expected utility from waiting when only one pair of hospital-doctor leaves at $t = 0$, we have to compute all the submatrices of $\mathbb{U}$ obtained by deleting one row and one column, for each one of these submatrices we find its trace, we sum all these traces and, finally, we have to divide the sum by $n^2(n-1)$, since there are $n^2$ possible pairs of qualities for the hospital and the doctor that go early, and $n - 1$ possible qualities that a hospital that waits may be assigned to in the second period..

If $m$ hospitals make an offer at $t = 0$, we generalize the previous argument as follows. Consider all submatrices of $\mathbb{U}$ that result when $m$ rows and $m$ columns are deleted. There are $\binom{n}{m}\binom{n}{m}$ submatrices that can be found. In each case, there are $(n-m)$ possible qualities for a hospital that waits. Thus, for each submatrix, compute its trace. $\mathcal{U}_m$ is the sum of all the computed traces after dividing it by $\binom{n}{m}\binom{n}{m}(n-m) = \frac{n^2(n-1)^2...(n-m+1)^2}{(m!)^2}(n-m)$.

The following proposition states this result.[2]

*Proposition* 2. Let $\mathcal{U}_m$ be expected utility to a hospital of waiting for the second period when $m$ hospitals (with their respective doctors) have left the market at $t = 0$. Denote by $T(n,m)$ the **sum** of the traces of all submatrices of $\mathbb{U}$ when $m$ rows and $m$ columns are deleted.

---

[2]The algorithm can be also applied with other functions $u_h$ and $u_d$ whenever the functions are strictly supermodular on the lattice $\{1, 2, \ldots, n\}^2$.

Then:
$$\mathcal{U}_m = \frac{T(n,m)(m!)^2}{n^2(n-1)^2\dots(n-m+1)^2}\frac{1}{(n-m)}.$$

To come up with an expression for $\mathcal{U}_m$, the next step involves the computation of $T(n,m)$. The following lemma finds a formula for $T(n,m)$. Then, we obtain a reduced expression of the formula by means of some combinatorial identities.

*Lemma* 2. Denote by $T(n,m)$ the sum of the traces of all submatrices of $\mathbb{U}$ obtained by deleting $m$ rows and $m$ columns. Then:

$$T(n,m) = \sum_{i=1}^{n}\left[i^2\sum_{k=0}^{m}\left(\binom{i-1}{k}\binom{n-i}{m-k}\right)^2\right] +$$
$$2\sum_{j=1}^{m}\left[\sum_{i=1}^{n-j}i(i+j)\left(\sum_{k=j}^{m}\binom{i+j-1}{k}\binom{n-(i+j)}{m-k}\binom{n-i}{m-k+j}\binom{i-1}{k-j}\right)\right].$$

*Proof.* First we consider the elements of the main diagonal of $\mathbb{U}$, and then, the remaining elements.

**(ii)-elements:**

Consider an element $ii$ of the matrix and suppose we delete $m$ rows and $m$ columns. Note that there are $i-1$ rows (columns) above (at the left of) the element $ii$ and $n-i$ rows (columns) below (at the right). When we delete columns and rows, the element $ii$ remains in the main diagonal if the number of rows that are deleted above $ii$ is equal to the number of columns that are deleted from the left of $ii$. That is, if we delete $k$ rows above $ii$ and $m-k$ rows below, then we have to delete $k$ columns at the left and $m-k$ columns at the right. Thus, the number of submatrices in which the element $ii$ is in the main diagonal is:

$$\sum_{k=0}^{m}\left(\binom{i-1}{k}\binom{n-i}{m-k}\right)^2.$$

Since the element $ii$ in the matrix is $i^2$, the share of $T(n,m)$ that corresponds to the elements of the main diagonal of $\mathbb{U}$ is:

$$\sum_{i=1}^{n} \left[ i^2 \sum_{k=0}^{m} \left( \binom{i-1}{k} \binom{n-i}{m-k} \right)^2 \right].$$

**(ij)-elements:**

Since $\mathbb{U}$ is a symmetric matrix, the trace of the submatrix that we obtain by deleting rows $i_1, i_2, \ldots, i_m$ and columns $j_1, j_2, \ldots, j_m$ is equal to the trace of the submatrix obtained by deleting rows $j_1, j_2, \ldots, j_m$ and columns $i_1, i_2, \ldots, i_m$. Thus, we only have to consider the elements $i(i+j)$ for $j > 0$, and take two times the final result. In particular, when only one row and one column are deleted, the elements that will be in the main diagonal of some submatrix are those of the form $i(i+j)$ for $i = 1, \ldots, n-1$ and $j = 1$. When two rows and two columns are deleted, the elements to be considered in $T(n, m)$ are the previous elements and those of the form $i(i+j)$ for $i = 1, \ldots, n-2$ and $j = 2$. In general, when $m$ rows and $m$ columns are deleted we have to consider all the elements that were contemplated when $m-1$ rows and $m-1$ columns were deleted, and those of the form $i(i+j)$ for $i = 1, \ldots, n-m$ and $j = m$.

As we just noted, when we delete $m$ rows and $m$ columns, the elements that are in the trace of some submatrix are those of the form $i(i+j)$ with $j = 1, 2, \ldots, m$. So, consider an element $i(i+j)$. This element has $i-1$ rows above and $n-i$ below. Moreover, it has $i+j-1$ columns at the left and $n-(i+j)$ columns at the right. Suppose we delete $k$ columns at the left of $i(i+j)$ and $m-(i+j)$ at the right. Now the element is in column $i+j-k$. In order to be in the main diagonal of a submatrix, it should be that: $j-k \leq 0$. Moreover, we have to delete $k-j$ rows above the element $i(i+j)$ to ensure that the element is in the main diagonal of a submatrix.

Then, the share of $T(n, m)$ that corresponds to these elements is:

$$2 \sum_{j=1}^{m} \left[ \sum_{i=1}^{n-j} i(i+j) \left( \sum_{k=j}^{m} \binom{i+j-1}{k} \binom{n-(i+j)}{m-k} \binom{i-1}{k-j} \binom{n-i}{m-(k-j)} \right) \right].$$

■

*Lemma* 3. For $n \in \mathbb{N}$ and $m \in 1, 2, \ldots, n-1$ it holds that:

$$T(n,m) = \binom{n+1}{m}^2 \left( \sum_{i=1}^{n-m} i^2 \right).$$

The following proof was provided to us by Doron Zeilberger.

*Proof.* The proof is organized in five claims.

**Claim 1:** $T(n,m)$ can be written as:

$$\sum_{i,j,k} i(i+j) \binom{i+j-1}{k} \binom{n-(i+j)}{m-k} \binom{n-i}{m-k+j} \binom{i-1}{k-j},$$

where the summation range is over all triples $(i, j, k)$, with the convention that the binomial coefficient $\binom{r}{s}$ is zero if it is not the case that $0 \le s \le r$.

**Proof Claim 1**

In the proof of the last lemma we found an expression for $T(n,m)$ using the symmetry of the matrix $\mathbb{U}$. If we do not use the symmetry we obtain the following equivalent expression:

$$T(n,m) = \sum_{i=1}^{n} \left[ i^2 \sum_{k=0}^{m} \left( \binom{i-1}{m-k}\binom{n-i}{k} \right)^2 \right] +$$

$$\sum_{j=1}^{m} \left[ \sum_{i=1}^{n-j} i(i+j) \left( \sum_{k=j}^{m} \binom{i+j-1}{k}\binom{n-(i+j)}{m-k}\binom{n-i}{m-k+j}\binom{i-1}{k-j} \right) \right] +$$

$$\sum_{i=1}^{m} \left[ \sum_{j=1}^{n-i} j(i+j) \left( \sum_{k=i}^{m} \binom{i+j-1}{k}\binom{n-(i+j)}{m-k}\binom{n-j}{m-k+i}\binom{j-1}{k-i} \right) \right].$$

Note that for each $j = 1, \ldots, m$, the range for $i$ is $1 \le i \le n-j$, and for each $i = 1, \ldots, m$, the range for $j$ is $1 \le j \le n-i$. Thus, we can write these conditions as: $1 \le i \le n$, $1 \le j \le n$ and $1 \le i+j \le n$. Now, consider the sum:

$$\sum_{i,j,k} i(i+j) \binom{i+j-1}{k}\binom{n-(i+j)}{m-k}\binom{n-i}{m-k+j}\binom{i-1}{k-j}.$$

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The implicit range for each variable is: $j \leq k \leq m$, $1 \leq i \leq n$, $1 \leq j \leq n$ and $1 \leq i + j \leq n$. This implies that both sums are equal.

**Claim 2:** The sum of Claim 1 equals:

$$\sum_{a=1}^{n} a \sum_{k=\max(0,a-(n-m))}^{\min(a-1,m)} \binom{a-1}{k}\binom{n-a}{m-k} \sum_{i=a-k}^{a-k+m} i \binom{n-i}{m-k+a-i}\binom{i-1}{k-a+i}.$$

**Proof Claim 2**

Writing $a = i + j$, (and leaving $i$ as a discrete variable, but letting $j = a - i$), the sum of the last claim is equal to:

$$\sum_{a,k,i} ia \binom{a-1}{k}\binom{n-a}{m-k}\binom{n-i}{m-k+a-i}\binom{i-1}{k-a+i} \quad .$$

Note that summation range of each variable is defined by:

1. For $a$: $1 \leq a \leq n$.

2. For $k$: $0 \leq k \leq m$, $0 \leq m-k+a-i \leq n-i$ and $0 \leq k \leq a-1$. This implies that $max(0, a-(n-m)) \leq k \leq min(a-1, m)$.

3. For $i$: $1 \leq i \leq n$, $0 \leq m-k+a-i$ and $0 \leq k-a+i$. This implies that $a-k \leq i \leq m-k+a$.

Then, the last sum equals the iterated summation:

$$\sum_{a=1}^{n} \sum_{k=\max(0,a-(n-m))}^{\min(a-1,m)} \sum_{i=a-k}^{a-k+m} ia \binom{a-1}{k}\binom{n-a}{m-k}\binom{n-i}{m-k+a-i}\binom{i-1}{k-a+i}.$$

Which is equivalent to:

$$\sum_{a=1}^{n} a \sum_{k=\max(0,a-(n-m))}^{\min(a-1,m)} \binom{a-1}{k}\binom{n-a}{m-k} \sum_{i=a-k}^{a-k+m} i \binom{n-i}{m-k+a-i}\binom{i-1}{k-a+i}.$$

**Claim 3:** The innermost sum is:

$$\sum_{i=a-k}^{a-k+m} i\binom{n-i}{m-k+a-i}\binom{i-1}{k-a+i} = (a-k)\binom{n+1}{m}.$$

**Proof Claim 3**

First note that: $i\binom{i-1}{k-a+i} = (a-k)\binom{i}{a-k}$. Then we have:

$$\sum_{i=a-k}^{a-k+m} i\binom{n-i}{m-k+a-i}\binom{i-1}{k-a+i} = (a-k)\sum_{i=a-k}^{a-k+m}\binom{n-i}{m-k+a-i}\binom{i}{a-k}.$$

Now, notice that:

$$\sum_{i=a-k}^{a-k+m}\binom{n-i}{m-k+a-i}\binom{i}{a-k} = \sum_{i=0}^{m}\binom{n-(a-k+i)}{m-i}\binom{a-k+i}{a-k}.$$

Since $\binom{a-k+i}{a-k} = \binom{a-k+i}{i}$, the last sum can be written as:

$$\sum_{i=0}^{m}\binom{n-(a-k+i)}{m-i}\binom{a-k+i}{i}.$$

Which is equal to:

$$\sum_{i=0}^{m}\binom{(n-m-a+k)+m-i}{m-i}\binom{a-k+i}{i}.$$

Finally, we use the Vandermonde-Chu identity (Sprugnoli (26), page 54):

$$\sum_{k=0}^{n}\binom{x+k}{k}\binom{y+n-k}{n-k} = \binom{x+y+n+1}{n}.$$

Defining $x = a - k$ and $y = (n - m - a + k)$, we have:

$$\sum_{i=0}^{m}\binom{(n-m-a+k)+m-i}{m-i}\binom{a-k+i}{i} = \binom{x+y+n+1}{n} = \binom{n+1}{m}.$$

**Claim 4:** For the middle sum it holds that:

$$\sum_{k=\max(0,a-(n-m))}^{\min(a-1,m)} (a-k)\binom{a-1}{k}\binom{n-a}{m-k} = a\binom{n-1}{m} - (a-1)\binom{n-2}{m-1}.$$

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## Proof Claim 4

First, we divide the sum:

$$\sum_{k=\max(0,a-(n-m))}^{\min(a-1,m)} (a-k)\binom{a-1}{k}\binom{n-a}{m-k}$$

$$= a \sum_{k=\max(0,a-(n-m))}^{\min(a-1,m)} \binom{a-1}{k}\binom{n-a}{m-k} - \sum_{k=\max(0,a-(n-m))}^{\min(a-1,m)} k\binom{a-1}{k}\binom{n-a}{m-k}.$$

We use the Vandermonde-Chu identity (Sprugnoli (26), page 53):

$$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n}.$$

And the first sum is:[3]

$$a \sum_{k=\max(0,a-(n-m))}^{\min(a-1,m)} \binom{a-1}{k}\binom{n-a}{m-k} = a\binom{n-1}{m}.$$

If we replace $k\binom{a-1}{k} = (a-1)\binom{a-2}{k-1}$ in the second sum, we have:

$$(a-1) \sum_{k=\max(0,a-(n-m))}^{\min(a-1,m)} \binom{a-2}{k-1}\binom{n-a}{m-k},$$

which is equal to:

$$(a-1)\sum_{k=0}^{m} \binom{a-2}{m-1-k}\binom{n-a}{k}.$$

By the Vandermonde-Chu identity, the sum is:

$$(a-1)\binom{n-2}{m-1}.$$

**Claim 5:** Finally, we have:

$$T(n,m) = \binom{n+1}{m}^2 \left(\sum_{i=1}^{n-m} i^2\right).$$

---

[3]Note that $\max(0, a - (n-m)) = 0$. Indeed, if $(a - (n-m)) > 0$, we have $n - a - m - k < 0$ and thus, $\binom{n-a}{m-k} = 0$. Also, we can write the sum up to $k = m$, because for $k = a, a+1, \ldots, m$, $\binom{a-1}{k} = 0$.

**Proof Claim 5**

Since the last claims we know that:

$$T(n,m) = \binom{n+1}{m}\left(\binom{n-1}{m}\left(\sum_{a=1}^{n} a^2\right) - \binom{n-2}{m-1}\left(\sum_{a=1}^{n} a(a-1)\right)\right).$$

Then, compute:

$$\binom{n+1}{m}\left(\binom{n-1}{m}\left(\sum_{a=1}^{n} a^2\right) - \binom{n-2}{m-1}\left(\sum_{a=1}^{n} a(a-1)\right)\right)$$

$$= \binom{n+1}{m}\left(\frac{(n-1)!}{m!(n-m-1)!}\frac{n(n+1)(2n+1)}{6}\right.$$
$$\left. - \frac{(n-2)!}{(m-1)!(n-m-1)!}\frac{(n-1)n(n+1)}{3}\right)$$

$$= \binom{n+1}{m}\left(\frac{(n+1)!}{m!(n-m-1)!}\frac{(2n+1)}{6} - \frac{(n+1)!}{m!(n-m-1)!}\frac{m}{3}\right)$$

$$= \binom{n+1}{m}\frac{(n+1)!}{m!(n-m-1)!}\left(\frac{2n+1}{6} - \frac{m}{3}\right)$$

$$= \binom{n+1}{m}\frac{(n+1)!}{m!(n-m-1)!}\frac{2n-2m+1}{6}$$

$$= \binom{n+1}{m}\frac{(n+1)!}{m!(n-m+1)!}\frac{(n-m)(n-m+1)(2n-2m+1)}{6}$$

$$= \binom{n+1}{m}^2\frac{(n-m)(n-m+1)(2(n-m)+1)}{6}$$

$$= \binom{n+1}{m}^2\sum_{i=1}^{n-m} i^2.$$

■ Finally, we obtain the formula for $\mathcal{U}_m$. We know that:

$$\mathcal{U}_m = \frac{T(n,m)}{\binom{n}{m}\binom{n}{m}(n-m)}.$$

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First note that:
$$\binom{n+1}{m}^2 = \left[\frac{n+1}{n-m+1}\right]^2 \binom{n}{m}^2.$$

Then, by replacing the last expression in $\mathcal{U}_m$, we obtain:

$$\mathcal{U}_m = \frac{(n+1)^2}{(n-m+1)^2(n-m)} \frac{(n-m)(n-m+1)(2(n-m)+1)}{6}.$$

By simplifying the last equation, we prove the result:

$$\mathcal{U}_m = \frac{(n+1)^2(2(n-m)+1)}{6(n-m+1)}.$$

Note that $\mathcal{U}_m$ increases with $n$, the number of agents. This means that if there are more agents in the market, the expected utility of waiting when a fixed number of agents leave the market at $t = 0$, increases.

The next result shows that $\mathcal{U}_m$ decreases with $m$, a property which will be used in the next section. Then, the expected utility of waiting and match at $t = 1$, decreases as more agents leave early.

*Corollary* 3. Let $\mathcal{U}_m$ be expected utility of a hospital that decides to wait for the second period when $m$ pairs of hospital-doctor leave the market at $t = 1$. Then for $n \in \mathbb{N}$ and $m = 0, 1, 2, \ldots, n - 1$, we have:

$$\mathcal{U}_m - \mathcal{U}_{m+1} = \frac{(n+1)^2}{6(n-m)(n-m+1)}.$$

Note that $\mathcal{U}_{m+1} - \mathcal{U}_m$ represents the negative externality imposed on the rest of the market by one pair of hospital-doctor that decides to go early, when $m$ agents have already decided to match at $t = 0$. Since $\mathcal{U}_m - \mathcal{U}_{m+1}$ increases when $m$ becomes larger, the negative externality imposed by one more pair going early increases (in absolute value) as more agents have decided to go early. Moreover, when the number of agents (that is, $n$) increases the negative

externality decreases. However, since $\lim_{n \to \infty} \mathcal{U}_{m+1} - \mathcal{U}_m = \frac{1}{6}$, it does not converge to zero as the market size goes to infinity. Thus, the negative externality becomes neutral when $n$ tends to infinity because it does not depend on the number of agents that have previously decided to match early.

## 4.6   Proof of Theorem 2

Recall that the best-response function of the game, $\beta^n$, is defined by Equation (1) in the following way. Given identical thresholds $\delta_{-h} = \delta$ for all hospitals other than $h$, $\beta^n(\delta)$ is given by the equation:

$$\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} ij = \beta^n(\delta) \mathbf{E}_{s_{-h}} \mathcal{U}_m.$$

Where $s_{-h}$ is such that $s_{\tilde{h}} = 0$ if $\delta_{\tilde{h}} \leq \delta$ and $s_{\tilde{h}} = 1$ if $\delta_{\tilde{h}} > \delta$, for all $\tilde{h} \neq h$.

Note that:

$$\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} ij = \frac{(n+1)^2}{4}.$$

When all hospitals other than $h$ have the same threshold $\delta$, the probability that $m$ hospitals make early offers is the probability $m$ hospitals have discount factors less than or equal to $\delta$, and $n - m$ hospitals have discount factors higher than $\delta$. Since discount factors are drawn independently from a uniform distribution on $[0, 1]$, the probability that $m$ hospitals leave at $t = 0$ is given by $\delta^m (1 - \delta)^{n-1-m} \binom{n-1}{m}$. Therefore:

$$\mathbf{E}_{s_{-h}} \mathcal{U}_m = \sum_{m=0}^{n-1} \delta^m (1 - \delta)^{n-1-m} \binom{n-1}{m} \mathcal{U}_m.$$

Then $\beta^n$ is defined by:

$$\beta^n(\delta) = \frac{(n+1)^2}{4 \left[ \sum_{m=0}^{n-1} \delta^m (1 - \delta)^{n-1-m} \binom{n-1}{m} \mathcal{U}_m \right]}.$$

The symmetric BNE of our game are the fixed points of the best-response function $\beta^n$. Since Lemma 1 we know that $\mathcal{U}_{n-1} = \frac{(n+1)^2}{4}$, and then $\beta^n(1) = 1$ for all $n$. Thus, full unraveling is a BNE for all $n$. In this Section we investigate the existence of other fixed points. In particular, Lemma 4 gives a simple formula for $\beta^n$. Lemma 6 shows that $\beta^n$ is an increasing function of $\delta$ and $\beta^n(0) > \frac{3}{4}$. Thus, $\beta^n$ may have, at most, one fixed point different from $\delta = 1$. Moreover, if it exists, the fixed point is higher than $\frac{3}{4}$. Lemma 6 proves that $\delta = 1$ is the unique fixed point of $\beta^n$ for all $n \leq 10$, and if $n > 10$, $\beta^n$ has exactly two fixed points. Finally, Lemma 8 studies the behavior of $\beta^n$ when $n$ tends to infinity.

It is worthwhile noting that the threshold at the BNE $s^n$ defined in Theorem 2, decreases as more agents are present in the market. This means that the probability that a hospital makes early offers, decreases as the number of agents increases. The intuition of this result is straightforward since, as we noted before, the incentives to make early offers when a fixed number of agents leave the market at $t = 0$, decreases with $n$.

*Lemma 4.*
$$\beta^n(\delta) = \frac{3}{2}\left(2 - \frac{1}{n(n+1)}\sum_{m=1}^{n}m\delta^{n-m}\right)^{-1}.$$

First we will prove the following lemma which will be useful in the proof of Lemma 4.

*Lemma 5.* For any $n \in \mathbb{N}$ and $\delta \in \mathbb{R}$ it holds:
$$\sum_{m=0}^{n}\frac{(1-\delta)^{n-m}\delta^m\binom{n}{m}}{n-m+2} = \sum_{m=0}^{n}\frac{(m+1)\delta^{n-m}}{(n+1)(n+2)}.$$

*Proof.* [4] Consider the following polynomials of degree $n$:
$$p(\delta) = \sum_{m=0}^{n}\frac{(1-\delta)^{n-m}\delta^m\binom{n}{m}}{n-m+2}, \quad \text{and}$$

$$q(\delta) = \sum_{m=0}^{n}\frac{(m+1)\delta^{n-m}}{(n+1)(n+2)}.$$

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[4] We are very grateful to Andrés Sambarino for helpful comments on this proof.

We want to prove that $p = q$ and to this end, we will show that all the derivatives of $p$ and $q$ are equal at $\delta = 0$. Denote by $p^{(k)}$ and $q^{(k)}$ the $k$th derivative of $p$ and $q$, respectively. It is straightforward to show that:

$$q^{(k)}(\delta) = \sum_{m=0}^{n-k} \frac{1}{(n+1)(n+2)}(m+1)(n-m)(n-m-1)\ldots(n-m-k+1)\delta^{n-m-k},$$

for $k = 1, 2, \ldots, n$

Then:

$$q^{(k)}(\delta) = \sum_{m=0}^{n-k} \frac{1}{(n+1)(n+2)}(m+1)\frac{(n-m)!}{(n-m-k)!}\delta^{n-m-k}.$$

When we evaluate at $\delta = 0$, we have:

$$q^{(k)}(0) = \frac{(n-k+1)k!}{(n+1)(n+2)}.$$

To compute the $k$th derivative of $p$, consider the functions:

$$g_1(\delta) = (1-\delta)^{n-m}, \text{ and}$$

$$g_2(\delta) = \delta^m.$$

Then:

$$g_1^{(i)}(\delta) = \frac{(n-m)!}{(n-m-i)!}(-1)^i(1-\delta)^{n-m-i}, \text{ and}$$

$$g_2^{(k-i)}(\delta) = \frac{m!}{(m-k+i)!}\delta^{m-(k-i)}.$$

By the general Leibniz rule we have, for $k = 1, 2, \ldots, n$:

$$(g_1 g_2)^{(k)}(\delta) = \sum_{i=0}^{k} \binom{k}{i}\frac{(n-m)!}{(n-m-i)!}\frac{m!}{(m-k+i)!}(-1)^i(1-\delta)^{n-m-i}\delta^{m-(k-i)}.$$

If $m - k \geq 0, m - (k-i) \geq 0$ for all $i$ and thus, $(g_1 g_2)^{(k)}(0) = 0$ Then, suppose $m - k \leq 0$, we have:

$$(g_1 g_2)^{(k)}(0) = \binom{k}{k-m}\frac{(n-m)!}{(n-k)!}m!(-1)^{k-m}.$$

Thus, the $k$th derivative of $p$ is:

$$p^{(k)}(0) = \sum_{m=0}^{n} \binom{n}{m} \frac{1}{(n+2-m)} \binom{k}{k-m} \frac{(n-m)!}{(n-k)!} (-1)^{k-m} m! \ .$$

As we just noted, when $m \geq k$, $p^{(k)}(0) = 0$, then, we can write the previous sum from $m = 0$ to $m = k$.

We want to prove that $p^{(k)}(0) = q^{(k)}(0)$ for all $k = 1, 2, \ldots, n$; that is:

$$\sum_{m=0}^{k} \binom{n}{m} \frac{1}{(n+2-m)} \binom{k}{k-m} \frac{(n-m)!}{(n-k)!} (-1)^{k-m} m! = \frac{(n-k+1)!k!}{(n+1)(n+2)}.$$

Note that:

$$\binom{n}{m} \frac{(n+1)(n+2)}{(n+m-2)} = \binom{n+2}{m}(n+1-m), \text{ and}$$

$$\binom{k}{k-m} \frac{(n-m)!}{(n-k)!} m! \frac{1}{(n-k+1)k!} (n+1-m) = \binom{n+1-m}{n-k+1}.$$

Thus, we have to prove that:

$$(-1)^k \sum_{m=0}^{k} \binom{n+2}{m} \binom{n+1-m}{n-k+1} (-1)^m = 1.$$

To finish the proof we use the following binomial identity (Riordan (17), page 8):

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x-k}{r} = \binom{x-n}{r-n} = \binom{x-n}{x-r}.$$

Thus:

$$(-1)^k \sum_{m=0}^{k} \binom{n+2}{m} \binom{n+1-m}{n-k+1} (-1)^m = (-1)^k \binom{-1}{k}.$$

Finally, by the Negation rule we have: $\binom{-1}{k} = (-1)^k \binom{1+k-1}{k} = (-1)^k$, and then:

$$(-1)^k \sum_{m=0}^{k} \binom{n+2}{m} \binom{n+1-m}{n-k+1} (-1)^m = (-1)^{2k} = 1.$$

∎

**Proof of Lemma 4.**

We know that:

$$\beta^n(\delta) = \frac{(n+1)^2}{4\left[\sum_{m=0}^{n-1}(1-\delta)^{n-1-m}\delta^m \binom{n-1}{m}\mathcal{U}_m\right]}, \text{ and}$$

$$\mathcal{U}_m = \frac{(n+1)^2(2(n-m)+1)}{6(n-m+1)},$$

for $m = 0, \ldots, n-1$.

We will use these two identities:

$$\frac{2(n-m)+1}{n-m+1} = 2 - \frac{1}{n-m+1},$$

$$\sum_{m=0}^{n-1}(1-\delta)^{n-1-m}\delta^m \binom{n-1}{m} = 1.$$

Then, by substituting $\mathcal{U}_m$ and since the last identities we have:

$$\beta^n(\delta) = \frac{3}{2\left(2 - \sum_{m=0}^{n-1}\frac{(1-\delta)^{n-1-m}\delta^m \binom{n-1}{m}}{n-m+1}\right)}.$$

By the previous lemma we can write $\beta^n$ as:

$$\beta^n(\delta) = \frac{3}{2\left(2 - \sum_{m=0}^{n-1}\frac{(m+1)}{n(n+1)}\delta^{n-1-m}\right)}.$$

Which is equivalent to:

$$\beta^n(\delta) = \frac{3}{2\left(2 - \frac{1}{n(n+1)}\sum_{m=1}^{n}m\delta^{n-m}\right)}.$$

$\square$

The following lemma gives more information on the nature of $\beta^n$.

*Lemma* 6.

$$\beta^n(\delta) = \begin{cases} \dfrac{3}{2\left[2 - \dfrac{1}{n(n+1)}\left(\dfrac{\delta^{n+1} - n\delta^2 + (n-1)\delta}{(1-\delta)^2}\right)\right]} & \text{if } \delta \in (0,1) \\[4ex] 1 & \text{if } \delta = 1 \end{cases}$$

Further,

1. $\beta^n$ is increasing for each $n$.

2. $\beta^n(0) > \frac{3}{4}$ and $\beta^n(1) = 1$, for all $n$.

3. $\beta^n$ has, at most, two fixed points. $\delta = 1$ is a fixed point of $\beta^n$ for all $n \in \mathbb{N}$ and it may have another fixed point which, if it exists, is higher than $\frac{3}{4}$.

*Proof.* When $\delta = 1$ we have:

$$\beta^n(1) = \frac{3}{2\left(2 - \dfrac{1}{n(n+1)}\displaystyle\sum_{m=1}^{n} m\right)} = \frac{3}{2\left(2 - \dfrac{1}{n(n+1)}\dfrac{n(n+1)}{2}\right)} = 1.$$

Then, suppose $\delta \in (0,1)$, and note that:

$$\sum_{m=1}^{n} m\delta^{n-m} = \delta^n \sum_{m=1}^{n} m\left(\frac{1}{\delta}\right)^m.$$

We use the following identity, that hold for $x \neq 1$:

$$\sum_{m=1}^{n} mx^m = \frac{1 - nx^n + (n-1)x^{n+1}}{(1-x)^2}.$$

Finally, for $\delta \in (0,1)$ we have:

$$\sum_{m=1}^{n} m\delta^{n-m} = \frac{\delta^{n+1} - n\delta^2 + (n-1)\delta}{(1-\delta)^2}.$$

To prove part 1, note that for all $\delta \in [0,1]$:

$$\left(\sum_{m=1}^{n} m\delta^{n-m}\right)' = \sum_{m=1}^{n-1} m(n-m)\delta^{n-m-1} \geq 0.$$

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Then, the expression $\sum_{m=1}^{n} m\delta^{n-m}$ increases with $\delta$ and thus, $\beta^n$ is increasing.

For part 2 notice that $\beta^n(0) = \frac{3(n+1)}{2(2n+1)} > \frac{3}{4}$ for all $n \geq 1$.

To prove part 3, we know that $\delta = 1$ is a fixed point of $\beta^n$. Since $\beta^n(0) > \frac{3}{4}$ and $\beta^n$ is increasing, $\beta^n$ crosses the line $y = x$ at, at most, one point different from $\delta = 1$. Thus, if it exists, the second fixed point is higher than $\frac{3}{4}$.

■

*Lemma 7.* Consider the best-response function $\beta^n$. Then:

1. For each $\delta \in [0, 1]$ it holds that $\beta^n(\delta) \geq \beta^{n+1}(\delta)$.

2. For all $n \leq 10$, $\delta = 1$ is the unique fixed point of $\beta^n$.

3. For all $n > 10$, $\beta^n$ has two and only two fixed points.

*Proof.* (1) We will show that:

$$\frac{1}{n(n+1)}\left[\frac{\delta^{n+1} - n\delta^2 + (n-1)\delta}{(1-\delta)^2}\right] \geq \frac{1}{(n+1)(n+2)}\left[\frac{\delta^{n+2} - (n+1)\delta^2 + n\delta}{(1-\delta)^2}\right].$$

Which is equivalent to:

$$\frac{\delta}{n}(\delta^{n-1} - n) + \frac{n-1}{n} \geq \frac{\delta}{n+2}(\delta^n - (n+1)) + \frac{n}{n+2}.$$

Note that:

$$\frac{\delta}{n} > \frac{\delta}{n+2},$$

$$\delta n - 1 - n > \delta^n - (n+1),$$

and:

$$\frac{n-1}{n} = 1 - \frac{1}{n} \geq 1 - \frac{2}{n+2} = \frac{n}{n+2}.$$

Thus, we finish the proof.

(2) and (3). We have to study the solutions in $[0, 1]$ of the equation:

$$\beta^n(\delta) = \frac{3}{2\left(2 - \dfrac{1}{n(n+1)}\displaystyle\sum_{m=1}^{n} m\delta^{n-m}\right)} = \delta.$$

Since for each $\delta \in [0, 1]$ it holds that $\beta^n(\delta) \geq \beta^{n+1}(\delta)$, if $\beta^n$ has two fixed points for some $n_0$, then $\beta^n$ has two fixed points for all $n$ such that $n \geq n_0$. We know that $\delta = 1$ is one solution of the equation and there may be, at most, one more solution in $[0, 1]$. The equation is equivalent to:

$$p_n(\delta) = \delta^n + \delta^{n-1} + 3\delta^{n-2} + \ldots + (n-1)\delta^2 + (-2n^2 - n)\delta + \frac{3n(n+1)}{2} = 0.$$

As we noted, $\delta = 1$ is a root of $p_n$. We also know that $p_n(0) = \frac{3n(n+1)}{2} > 0$ and that $p_n$ has, at most, one more root. Then, we will prove that for some $n_0$, $p'_{n_0}(1) > 0$, which implies that for all $n \geq n_0$, $p_n$ has two fixed points in $[0, 1]$. Then, compute:

$$p'_n(1) = \left[\sum_{i=1}^{n-1} i(n - i + 1)\right] + (-2n^2 - n) = \frac{n(n+1)(n-10)}{6}.$$

Thus, for all $n$ such that $0 \leq n \leq 10$, $p'_n(1) \leq 0$ and, for all $n > 10$, $p'_n(1) > 0$. This finishes the proof. ■

### 4.6.1 Behavior as $n \to \infty$

*Lemma* 8. For each $\delta \in [0, 1]$,

$$\lim_{n\to\infty} \beta^n(\delta) = \begin{cases} \dfrac{3}{4} & \text{if } \delta \in [0, 1) \\ 1 & \text{if } \delta = 1 \end{cases}.$$

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*Proof.* For $\delta = 1$ we know that $\beta^n(1) = 1$ for all $n$. Assume $\delta < 1$. Then, by Lemma 6, it is enough to show that:

$$\lim_{n \to +\infty} \frac{1}{n(n+1)} \left[ \frac{\delta^{n+1} - n\delta^2 + (n-1)\delta^t}{(1-\delta)^2} \right] = 0.$$

The last expression is equivalent to:

$$\frac{\delta}{(1-\delta)^2} \left[ \frac{(\delta)^n}{n(n+1)} - \frac{\delta}{(n+1)} + \frac{n-1}{n(n+1)} \right].$$

Finally, it is straightforward to show that the limit of the last expression when $n$ tends to infinity is 0. ∎ Note that Lemma 8 implies that the best-response function $\beta^n$ converges to a discontinuous function as $n \to \infty$.

Finally, note that in any symmetric BNE the expected number of hospitals that go early is given by:

$$\sum_{m=0}^{n} m(1 - \delta^*)^{n-m}(\delta^*)^m \binom{n}{m}$$

Where $\delta^*$ is a fixed point of $\beta^n$.

The last expression equals $n\delta^*$. As we noted before, $\beta^n$ has, at most, two fixed points, each one higher than $3/4$. Thus, in any symmetric BNE, the expected number of hospitals that go early is at least $(3/4)n$.

## 4.7 Proof of Theorem 1

When all agents share the prior that different $\delta_h$ are drawn independently from a distribution over $[0, 1]$ with cdf $F$, the best-response function is given by $F(\beta^n(x))$. Since $\beta^n$ is an increasing function and $F(x) \geq x$, we have that $\beta^n(F(x)) \geq \beta^n(x)$. Finally, note that $F(1) = 1$ and that $\beta^n(1) = 1$. Then Theorem 1 follows directly from Theorem 2.

## 4.8 Proof of Theorem 4

In the case where both sides of the market are strategic, the game is analyzed in the same way that we did in the previous sections. The difference is that now the probability that $m$ agents leave early is the probability that the minimum between the hospitals and the doctors that play at $t = 0$, equals $m$. Then, the expected value of $\mathcal{U}_m$, involves the probability distribution of the minimum of two independent binomial random variables.

We introduce some additional notation. Let $x_m$ be the probability that a binomial random variable with parameters $(\delta, n-1)$ equals $m$, and let $h_m$ be the probability that the minimum of two independent such random variables equals $m$. Denote by $G$ the cumulative distribution function of a binomial random variable with parameters $(\delta, n-1)$ and let $\bar{G} = 1 - G$. Therefore, the best-response function is defined by

$$\tilde{\beta}^n(\delta) = \frac{(n+1)^2}{4\left[\sum_{m=0}^{n-1} h_m \mathcal{U}_m\right]}.$$

We use the results of the previous sections to find a lower and upper bound for $\tilde{\beta}^n$. It is straightforward to prove that $\frac{1}{2}\beta^n \leq \tilde{\beta}^n \leq \beta^n$. Then, for all $n > 10$, $\tilde{\beta}^n$ has, at least, one fixed point which lies within the interval $\left(\frac{3}{8}, \frac{3}{4}\right)$. Moreover, as we will prove in the following lemmas, the lower bound can be improved, which allows us to conclude that in the general model, the expected number of agents that go early is at least one half.

We first prove some properties of $\tilde{\beta}^n$. In particular, Lemma 9 shows that: $\delta = 1$ is a fixed point of $\tilde{\beta}^n$ for all $n$, $\tilde{\beta}^n$ is an increasing function of $\delta$, and that $\tilde{\beta}^n(\delta) \geq \tilde{\beta}^{n+1}(\delta)$ for all $n$. Lemma 10 demonstrates that for each $\epsilon > 0$ there exists $n_0$ such that for all $n \geq n_0$ and $\delta \in [0, 1]$ it holds:

$$\frac{\frac{3}{4}}{\left(\frac{3}{2} + \epsilon\right)} \leq \tilde{\beta}^n(\delta) \leq \beta^n(\delta).$$

Then, $\lim_{n \to \infty} \tilde{\beta}^n(\delta) \geq \frac{1}{2}$, and since $\tilde{\beta}$ decreases when $n$ increases, we conclude that for all $n$:

$$\frac{1}{2} \leq \tilde{\beta}^n(\delta) \leq \beta^n(\delta).$$

*Lemma* 9. Consider the best-response function $\tilde{\beta}^n$ as defined before. Then:

1. $\tilde{\beta}^n(1) = 1$ for all $n$.

2. $\tilde{\beta}^n$ is an increasing function of $\delta$

3. For each $\delta \in [0, 1]$, $\tilde{\beta}^n(\delta) \geq \tilde{\beta}^{n+1}(\delta)$, for all $n$.

*Proof.* (1) Since the cumulative distribution function of the minimum of two iid random variables is $1 - (1 - G)^2$, we have:

$$
\begin{aligned}
h_m &= (1 - (1 - G(m))^2) - (1 - (1 - G(m-1))^2) \\
&= (1 - G(m-1))^2 - (1 - G(m))^2 \\
&= 2(G(m) - G(m-1)) + G(m-1)^2 - G(m)^2 \\
&= 2x_m + (G(m-1) - G(m))(G(m) + G(m-1)) \\
&= x_m(2 - G(m-1) - G(m)) \\
&= x_m(\bar{G}(m-1) + \bar{G}(m)).
\end{aligned}
$$

Thus,
$$
\tilde{\beta}^n(\delta) = \frac{(n+1)^2}{4\left[\sum_{m=0}^{n-1}(1-\delta)^{n-1-m}\delta^m \binom{n-1}{n}(\bar{G}(m-1) + \bar{G}(m))\mathcal{U}_m\right]}.
$$

When we compute $\tilde{\beta}^n(1)$ we obtain:
$$
\tilde{\beta}^n(1) = \frac{(n+1)^2}{4[(\bar{G}(n-2) + \bar{G}(n-1))\mathcal{U}_{n-1}]}.
$$

Since $\mathcal{U}_{n-1} = \frac{(n-1)^2}{4}$, and for $\delta = 1$, $\bar{G}(n-2) = 1$ and $\bar{G}(n-1) = 0$, we have that $\tilde{\beta}^n(1) = 1$.

(2) Now, if $\hat{G}$ is the cumulative distribution function of a binomial random variable with parameters $(\hat{\delta}, n-1)$, with $\hat{\delta} > \delta$, we have that $\hat{G}(m) \leq G(m)$ for all $m \in \{0, 1, \ldots, n-1\}$. This implies that $1 - (1 - \hat{G}(m))^2 \leq 1 - (1 - G(m))^2$. Let $\hat{h}_m$ be the probability that the minimum of two independent binomial random variables with parameters $(\hat{\delta}, n-1)$ equals $m$. Then, since $\mathcal{U}_m$ decreases with $m$, we have that

$$\sum_{m=0}^{n-1} \hat{h}_m \mathcal{U}_m \leq \sum_{m=0}^{n-1} h_m \mathcal{U}_m.$$

Therefore, $\tilde{\beta}^n$ is an increasing function of $\delta$.

(3) We know that $\mathcal{U}_m = \frac{(n+1)^2(2(n-m)+1)}{6(n-m+1)}$. Then, the best-response function can be written as:

$$\tilde{\beta}^n(\delta) = \frac{3}{2\left[1 + \sum_{m=0}^{n-1} \frac{n-m}{n-m+1} h_m\right]}.$$

Using a change of variable, $k = n - m$, we obtain:

$$\sum_{m=0}^{n-1} \frac{n-m}{n-m+1} h_m = \sum_{k=1}^{n} \frac{k}{k+1} h_{n-k} = \sum_{k=0}^{n} \frac{k}{k+1} h_{n-k}.$$

Consider two binomial random variables $\tilde{X}_i^n$, $i = 1, 2$. Each random variable $i$ is defined on the same sample space, the space of an infinite number of Bernoulli trials. For $\tilde{X}_i^n$ we count the number of successes in the first $n$ such trials. The sample spaces for $\tilde{X}_1^n$ and $\tilde{X}_2^n$ are independent.

Now, for each $n$ there is also the random variable $\tilde{Y}_i^n$ counting the number of failures. Note that $\tilde{X}_i^n + \tilde{Y}_i^n = n$.

Let $r_k$ be the probability that $\max\{\tilde{Y}_1^n, \tilde{Y}_2^n\} = k$. Observe that $h_{n-k} = r_k$. So we have that:

$$\sum_{m=0}^{n-1} \frac{n-m}{n-m+1} h_m = \sum_{k=0}^{n} \frac{k}{k+1} r_k.$$

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Since we have defined these random variables on the same sample space, it is true that

$$\{\tilde{Y}_i^n \geq x\} \subseteq \{\tilde{Y}_i^{n+1} \geq x\}$$

for any $x$ because any time that we have at least $x$ failures in the first $n$ Bernoulli trials, we have at least $x$ failures in the first $n+1$ Bernoulli trials (past failures cannot be undone).

By the same token:

$$\{\max\{\tilde{Y}_1^n, \tilde{Y}_2^n\} \geq x\} \subseteq \{\max\{\tilde{Y}_1^{n+1}, \tilde{Y}_2^{n+1}\} \geq x\}.$$

So that the probability distribution $(r_k)$ increases in the sense of first-order stochastic dominance (it actually increases in a stronger sense).

The function $k \mapsto k/(k+1)$ is monotone increasing. Thus the sum

$$\sum_{k=0}^{n} \frac{k}{k+1} r_k$$

is increasing in $n$, as it is the expected value of a monotone increasing function, and the probability law is monotone increasing in $n$.

■

*Lemma* 10. Let $\epsilon > 0$. Then, there exists $n_0$ such that for all $n \geq n_0$, the function $\tilde{\beta}^n$ defined previously satisfies:

$$\frac{\frac{3}{4}}{\left(\frac{3}{2} + \epsilon\right)} \leq \tilde{\beta}^n(\delta) \leq \beta^n(\delta).$$

*Proof.* Since the last Lemma we know that:

$$h_m = x_m(\bar{G}(m-1) + \bar{G}(m))$$
$$\leq 2x_m\bar{G}(m-1).$$

Then,

$$\sum_{m=0}^{n-1} h_m \mathcal{U}_m = \sum_{m=0}^{n-1} x_m(\bar{G}(m-1) + \bar{G}(m))\mathcal{U}_m \leq \sum_{m=0}^{n-1} 2x_m\bar{G}(m-1)\mathcal{U}_m.$$

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The median of a binomial distribution with parameter $(n, \delta)$ lies within the interval $[\lfloor n\delta \rfloor, \lceil n\delta \rceil]$. Moreover, if $n\delta$ is an integer, the median is $n\delta$. So, if $n\delta$ is an integer we have that $\bar{G}(n\delta) = Pr[x_m \geq n\delta + 1] \leq \frac{1}{2}$. Otherwise, if $n\delta$ is not an integer, $\bar{G}(\lfloor n\delta \rfloor) = Pr[x_m > \lfloor n\delta \rfloor] = Pr[x_m \geq \lceil n\delta \rceil] \leq \frac{1}{2}$. Thus, if $m \geq \lfloor n\delta \rfloor + 1$, we have that $\bar{G}(m-1) \leq \bar{G}(\lfloor n\delta \rfloor) \leq \frac{1}{2}$. Then:

$$\sum_{m=0}^{n-1} h_m \mathcal{U}_m \leq 2 \left[ \sum_{m=0}^{\lfloor (n-1)\delta \rfloor} \mathcal{U}_m x_m \bar{G}(m-1) + \sum_{m=\lfloor (n-1)\delta \rfloor + 1}^{n-1} \mathcal{U}_m x_m \bar{G}(m-1) \right]$$

$$\leq 2 \left[ \sum_{m=0}^{\lfloor (n-1)\delta \rfloor} \mathcal{U}_m x_m + \frac{1}{2} \sum_{m=\lfloor (n-1)\delta \rfloor + 1}^{n-1} \mathcal{U}_m x_m \right]$$

$$= \sum_{m=0}^{\lfloor (n-1)\delta \rfloor} \mathcal{U}_m x_m + \sum_{m=0}^{n-1} \mathcal{U}_m x_m = g(\delta) + \sum_{m=0}^{\lfloor (n-1)\delta \rfloor} \mathcal{U}_m x_m,$$

where $g(\delta) = \sum_{m=0}^{n-1} \mathcal{U}_m x_m$.

Now, recall that:
$$\mathcal{U}_m = \frac{(n+1)^2 (2(n-m)+1)}{6(n-m+1)}.$$

So we obtain that:

$$\sum_{m=0}^{\lfloor (n-1)\delta \rfloor} \mathcal{U}_m x_m = \frac{(n+1)^2}{6} \sum_{m=0}^{\lfloor (n-1)\delta \rfloor} \frac{(2(n-m)+1)}{n-m+1} x_m$$

$$= \frac{(n+1)^2}{6} \sum_{m=0}^{\lfloor (n-1)\delta \rfloor} \left( 1 + \frac{(n-m)}{n-m+1} \right) x_m$$

$$\leq \frac{(n+1)^2}{6} \sum_{m=0}^{\lfloor (n-1)\delta \rfloor} 2x_m$$

$$\leq \frac{(n+1)^2}{6}.$$

Where, in the last inequality, we use that $G(\lfloor (n-1)\delta \rfloor) \leq \frac{1}{2}$.

Therefore:
$$\frac{\sum_{m=0}^{\lfloor (n-1)\delta \rfloor} \mathcal{U}_m x_m}{g(\delta)} \leq \frac{(n+1)^2/6}{g(\delta)} = \left( \sum_{m=0}^{n-1} \frac{2(n-m)+1}{n-m+1} x_m \right)^{-1}. \tag{2}$$

Now, let $\epsilon > 0$. Choose $\rho_0, \rho_1 \in (0,1)$ such that: [5]

$$\frac{1}{1 + \rho_0 \rho_1} < \frac{1}{2} + \epsilon$$

Let $n$ be large enough such that:

$$Pr \left( \tilde{M} \leq n - \frac{\rho_0}{1 - \rho_0} \right) \geq \rho_1,$$

where $\tilde{M}$ is a binomial random variable with parameters $(n-1, \delta)$.

Clearly, the value of $n$ that satisfies the last inequality depends on $\delta$. Moreover, for higher values of $\delta$, we need to consider higher values of $n$. Then, assume that $\delta \leq \frac{1}{2}$, and take $n$ large enough such that the inequality holds. In the last step of the proof, we extend the result for all values of $\delta$.

Now, $m \leq n - \frac{\rho_0}{1-\rho_0}$ if and only if $\rho_0 \leq (1 - \rho_0)(n - m)$ if and only if

$$\rho_0 \leq \frac{n - m}{n - m + 1}.$$

Note that $\sum_{m=0}^{n-1} \frac{2(n-m)+1}{n-m+1} x_m$ is the expectation of the random variable

$$\left( \frac{2(n - \tilde{M}) + 1}{n - \tilde{M} + 1} \right),$$

then we have:

$$\sum_{m=0}^{n-1} \frac{2(n-m)+1}{n-m+1} x_m = \mathbb{E}_{\tilde{M}} \left( \frac{2(n - \tilde{M}) + 1}{n - \tilde{M} + 1} \right) = \mathbb{E}_{\tilde{M}} 1 + \mathbb{E}_{\tilde{M}} \left( \frac{n - \tilde{M}}{n - \tilde{M} + 1} \right).$$

Now, note that:

---

[5]Note that $\rho_0$ and $\rho_1$ exist since $j(x) = \frac{1}{1+x}$ is a continuous and decreasing function in $[0,1]$ with $j(0) = 1$ and $j(1) = \frac{1}{2}$.

$$\mathbb{E}_{\tilde{M}}\left(\frac{n-\tilde{M}}{n-\tilde{M}+1}\right) = \sum_{m=0}^{n-1}\left(\frac{n-m}{n-m+1}\right)x_m$$

$$\geq \sum_{m=0}^{\lfloor n-\frac{\rho_0}{1-\rho_0}\rfloor}\left(\frac{n-m}{n-m+1}\right)x_m$$

$$\geq \rho_0 \sum_{m=0}^{\lfloor n-\frac{\rho_0}{1-\rho_0}\rfloor} x_m$$

$$= \rho_0 Pr\left(\tilde{M} \leq n-\frac{\rho_0}{1-\rho_0}\right)$$

$$\geq \rho_0\rho_1.$$

Thus:

$$\sum_{m=0}^{n-1}\frac{2(n-m)+1}{n-m+1}x_m \geq 1+\rho_0\rho_1.$$

Now, using Equation (2) and the definition of $\rho_0$ and $\rho_1$ we obtain that:

$$\frac{\sum_{m=0}^{\lfloor(n-1)\delta\rfloor}\mathcal{U}_m x_m}{g(\delta)} \leq \frac{1}{1+\rho_0\rho_1} < \frac{1}{2}+\epsilon.$$

Then:

$$\sum_{m=0}^{\lfloor(n-1)\delta\rfloor}\mathcal{U}_m x_m < (\frac{1}{2}+\epsilon)g(\delta),$$

which implies that:

$$\sum_{m=0}^{n-1}h_m\mathcal{U}_m \leq \left(\frac{3}{2}+\epsilon\right)g(\delta).$$

Finally, note that:

$$\tilde{\beta}^n(\delta) = \frac{(n+1)^2}{4\left[\sum_{m=0}^{n-1}h_m\mathcal{U}_m\right]}$$

$$\geq \frac{(n+1)^2}{4g(\delta)}\frac{1}{\left(\frac{3}{2}+\epsilon\right)}$$

$$= \beta^n(\delta)\frac{1}{\left(\frac{3}{2}+\epsilon\right)}.$$

Therefore, there exists $n_0$ such that for all $n \geq n_0$:

$$\tilde{\beta}^n(\delta) \geq \beta^n(\delta)\frac{1}{\left(\frac{3}{2}+\epsilon\right)} \geq \frac{\frac{3}{4}}{\left(\frac{3}{2}+\epsilon\right)}.$$

for all $\delta \leq \frac{1}{2}$.

Since $\tilde{\beta}^n$ is an increasing function of $\delta$, if $\delta > \frac{1}{2}$:

$$\tilde{\beta}^n(\delta) \geq \tilde{\beta}^n(1/2) \geq \frac{\frac{3}{4}}{\left(\frac{3}{2}+\epsilon\right)}.$$

To prove that $\tilde{\beta}^n(\delta) \leq \beta^n(\delta)$ just note that $1 - (1 - G(m))^2 \geq G(m)$, and since $\mathcal{U}_m$ is decreasing in $m$ we have:

$$\sum_{m=0}^{n-1} \mathcal{U}_m h_m \geq \sum_{m=0}^{n-1} \mathcal{U}_m x_m.$$

Then:

$$\tilde{\beta}^n(\delta) = \frac{(n+1)^2}{4\left[\sum_{m=0}^{n-1}\mathcal{U}_m h_m\right]} \leq \frac{(n+1)^2}{4\left[\sum_{m=0}^{n-1}\mathcal{U}_m x_m\right]} = \beta^n(\delta).$$

Finally, we have that there exists $n_0$ such that for all $n \geq n_0$ and $\delta \in [0, 1]$:

$$\frac{\frac{3}{4}}{\left(\frac{3}{2}+\epsilon\right)} \leq \tilde{\beta}^n(\delta) \leq \beta^n(\delta). \tag{3}$$

■ The lower bond $\left(\frac{3}{2}+\epsilon\right)$ is arbitrarily close to $\frac{3}{2}$. Then, for each $\delta$ we have that:

$$\lim_{n\to\infty}\tilde{\beta}^n(\delta) \geq \frac{1}{2}.$$

Since by Lemma 9, $\tilde{\beta}^n$ decreases when $n$ increases, we have that for all $n$:

$$\frac{1}{2} \leq \tilde{\beta}^n \leq \beta^n.$$

Finally, note that:

1. $\tilde{\beta}^n$ is an increasing function of $\delta$,

2. $\tilde{\beta}^n(1) = 1$,

3. $\beta^n$ has two fixed points if $n > 10$,

4. $\frac{1}{2} \leq \tilde{\beta}^n \leq \beta^n$,

then, $\tilde{\beta}^n$ has exactly two fixed point: $\delta = 1$ and the other between $\frac{1}{2}$ and $\frac{3}{4}$.

Thus, in the general model, the expected number of agents that go early is at least $(\frac{1}{2})n$.

## 4.9    Proof of Theorem 3

Theorem 3 follows from Theorem 4 by observing that $\tilde{\beta}^n(F(\delta)) \geq \tilde{\beta}^n(\delta)$, and by employing the same argument used in Section 4.7.

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